LIM	AP CALCULUS BC	
3	Topic: 10.1	Defining Convergent and Divergent Infinite Series
Learning Objective LIM-7.A: Determine whether a series converges or diverges.		

Sequences

Consider the following *list* of numbers:

$$\{1,4,7,10,13,16,\ldots\}$$

How would you suppose this list is generated?
This "list" is an example of a **sequence** where each number in the sequence is called a **term** of the sequence.
We can denote a sequence in a variety of ways – three of the most popular are:

$$\{a_1,a_2,a_3,a_4,\ldots,a_n,\ldots\},\{a_n\}_{n=1}^{\infty},\text{and}\{a_n\}$$

The subscript that appears with this sequence is called an **index** and it indicates the order of the terms in the sequence. A sequence will often start with an index of $n = 1$ but can sometimes begin with $n = 0$.

There are two ways to define a sequence as illustrated in the table below.

Defining a Sequence	
Recursive Definition (a.k.a. Implicit Definition)	**Explicit Definition**
Note: Each term in the sequence is three more than the preceding term. Therefore, $a_1=1$ $a_2=a_1+3$ $a_3=a_2+3$ $\vdots$ $a_{n+1}=a_n+3$ Putting this all together, formally we should say: $a_1=1$ and $a_{n+1}=a_n+3$ where $n=1, 2, 3, \ldots$	**Note:** Each term in the sequence can be represented by ordered pairs using the convention: (order of the term, actual value of the term) Therefore, we can rename those terms with $(1,1)$, $(2,3)$, $(3,7)$, $(4,10)$, $\ldots$ and use a known relationship between the ordered pairs. Here, we have a linear relationship. $slope=\frac{4-1}{2-1}=\frac{7-4}{3-2}=\frac{10-7}{4-3}=\ldots=3$ Using the *point-slope* formula with the first coordinate point $a_n-1=3(n-1)$ or $a_n=3n-2$
Benefit: The relationship is typically easy to see.	**Benefit:** The explicit definition is extremely helpful in finding term deep into the sequence like a_{50}.
Drawback: The recursive definition is not very helpful if we want to find a term deep into the sequence like a_{50}.	**Drawback:** The relationship can be difficult to see if the sequence is not so "well-behaved", i.e., linear, or geometric.

Example 1: Explicit Formulas
Use the explicit formula given for each sequence, $\{a_n\}_{n=1}^{\infty}$, to write out the first four terms of each sequence.

a. $a_n=\frac{1}{2^n}$

b. $a_n=\frac{(-1)^n n}{n^2+1}$

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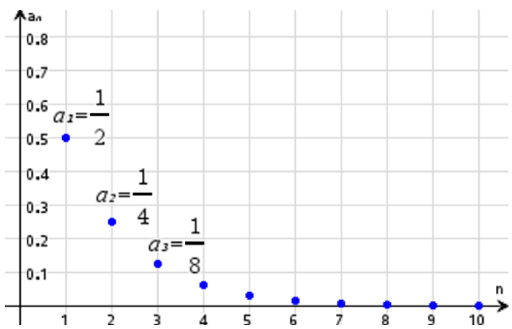
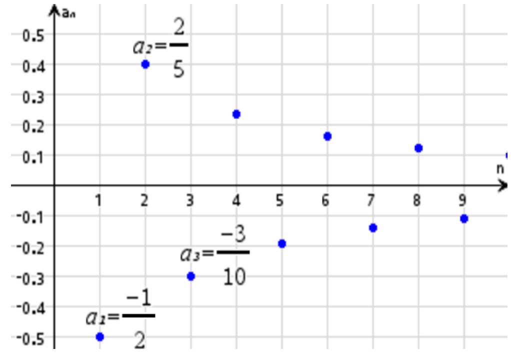
Scan the QR Code above to watch a video covering Example 1

Graphically, the two sequences in Example 1 would look like the following.





Example 2: Working with Sequences
For each of the following sequences, find

i.) the next two terms of the sequence.

ii.) a recursive definition that generates the sequence.

iii.) an explicit formula for the *n*th term of the sequence.

a. $\{a_n\} = \{-2, 5, 12, 19, \ldots\}$

b. $\{b_n\} = \{3, 6, 12, 24, 48, \ldots\}$

Scan the QR Code above to watch a video covering Example 2

Limit of a Sequence

DEFINITION OF THE LIMIT OF A SEQUENCE

Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if the terms of the sequence approach a unique number, L, as n increases.

If the limit L of a sequence exists, then the sequence **converges** to L.
If the limit to a sequence does not exist, then the sequence **diverges**.

THEOREM 10.1: LIMIT OF A SEQUENCE

Let L be a real number. Let f be a function of a real variable such that $\lim_{x\to\infty} f(x) = L$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then $\lim_{n\to\infty} a_n = L$

Example 3: Limit of a Sequence

For each of the following sequences, write out the first four terms. If you believe the sequence converges, make a conjecture about its limit. If the sequence appears to diverge, explain why. Assume each sequence has a beginning index of $n = 1$ and is defined infinitely.

a. $\{a_n\} = \left\{\frac{(-1)^n}{n^2+1}\right\}$	**b.** $\{b_n\} = \{cos(n\pi)\}$
c. $\{c_n\} = \left\{\frac{4n^3}{n^3+1}\right\}$	**d.** $d_{n+1} = -2d_n$

Example 4: A Bouncing Basketball

A basketball is tossed straight up in the air and reaches a high point before falling to the floor. Each time the ball bounces on the floor, it rebounds to 0.6 of its previous height. Let h_n be the high point after the nth bounce, with the initial height being 20 feet.

a. Find a recursive definition and an explicit formula for the sequence $\{h_n\}$.

b. What is the high point after the 10^{th} bounce?

c. What conjecture can you make about the limit of the sequence $\{h_n\}$?

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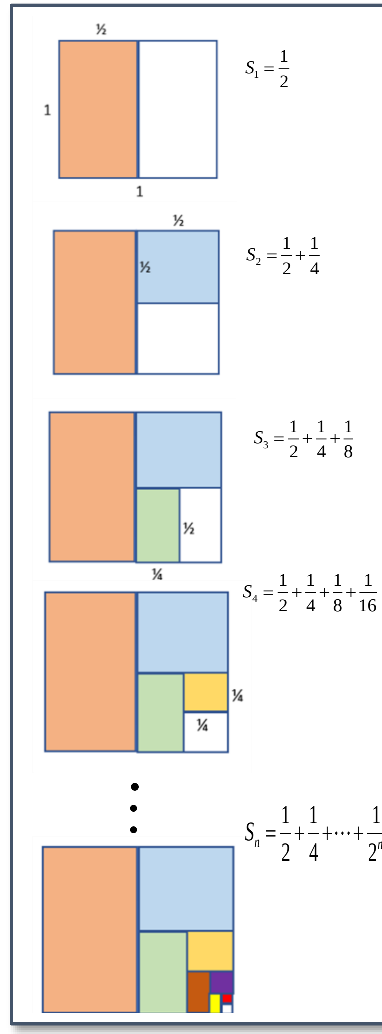
Scan the QR Code above to watch a video covering Example 4

Infinite Series and the Sequence of Partial Sums

An infinite series can be viewed as a sum of an infinite set of numbers and looks like $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$,
where the terms of the series, $a_1, a_2, \ldots$, are real numbers.

How is it possible, though, to add up an infinite set of numbers?



Consider a square with sides of length 1 that is subdivided as shown in the images to the right. We can let S_n be the area of the colored region in the nth figure of the progression.
The area of the colored region in the first figure is

$$S_1 = 1 \cdot \frac{1}{2} = \frac{1}{2} \qquad \frac{1}{2} = \frac{2^1 - 1}{2^1}$$

The area of the colored region in the second figure is S_1 plus the area of the smaller blue square, which is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Therefore,

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \qquad \frac{3}{4} = \frac{2^2 - 1}{2^2}$$

The area of the colored region in the third figure is S_2 plus the area of the smaller green rectangle, which is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. Therefore,

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \qquad \frac{7}{8} = \frac{2^3 - 1}{2^3}$$

If we continue in this manner, we discover that

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

If we continue this process indefinitely, the area of the colored region, S_n approaches entire area of the square, which is 1. Therefore, it is possible to say

$$\lim_{n\to\infty} S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

DEFINITION OF INFINITE SERIES

Given a sequence of numbers $\{a_1, a_2, a_3, \ldots\}$, the sum

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

Is called an **infinite series**. Its **sequence of partial sums**, $\{S_n\}$, has the terms

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_3 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \text{ for } n = 1, 2, 3, \ldots$$

If the sequence of partial sums, $\{S_n\}$, has a limit L, the infinite series **converges** to that limit, and we can write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

Example 5: **Sequence of Partial Sums**

Consider the infinite series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

a. Find the first four terms of the sequence of partial sums.

b. Find an expression for S_n and make a conjecture about the value of the series.

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Scan the QR Code above to watch a video covering Example 5

GRAPHING PARTIAL SUMS TI - 84

**** Set Calculator to Sequence Mode**

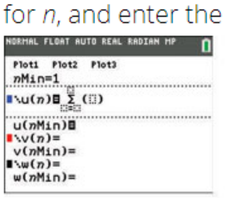
Here are the steps to graph the partial sums of an infinite series:

1. Press [2nd][ZOOM], highlight TIME, and press [ENTER].

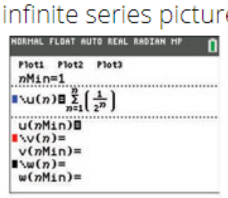
2. Press [Y=] to access the Y= editor.

3. Enter a value for *n*Min.
   *n*Min is the value where *n* starts counting.

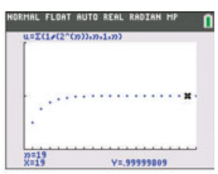
4. Press [ALPHA][WINDOW][2] to use the summation template to enter *u*(*n*).
   See the first screen. Press
   [X,T,Θ,*n*]
   for *n*, and enter the infinite series pictured in the second screen.



Summation template



Enter series



Press GRAPH

5. Press [WINDOW] and adjust the variables.
   Here are the variables changed: nMax = 20, Xmin = 0, Xmax = 20, Ymin = 0, Ymax = 2.

6. Press [GRAPH].

7. Press [TRACE] and use the right-arrow key to find the partial sums.
   See the third screen.

Example 6: A Sequence Versus a Series

Consider the following sequence defined by $a_n = \frac{n}{2n+1}$ and its corresponding series defined by $S_n = \sum_{n=1}^{\infty} \frac{n}{2n+1}$.

Graph each on your calcutor.

Find each of the following limits.

Scan the QR Code above to watch a video covering Example 6

a. $\lim_{n\to\infty} a_n$

b. $\lim_{n\to\infty} S_n$