Topic 10.8 Ratio Test for Convergence

Limits with Factorials

What is the $\lim_{n\to\infty} \frac{e^n}{n^2+4}$? How would you approach solving for this limit? What do you know about the two functions offered in this limit that drives your thinking? Can you deduce what the limit is without mathematically justifying the solution?

What is the $\lim_{n\to\infty} \frac{e^n}{n!}$? With your knowledge of functions, rates of change, and function races, what do your instincts tell you about this limit? Can this limit be "L'Hospitalized?"

Who wins in our race "to infinity and beyond?"

When dealing with a factorial, we can usually determine who "wins" to infinity. The issue is, how do you justify the result properly?

For our purposes, there are three types of function families that we will "race" in this lesson:

- Polynomials ($n^{to\ a\ positive\ integer}$)
- Exponentials (e^n)
- Factorials ($n!$)

When racing a polynomial and an exponential, we can use L'Hospital's Rule to help us solve the limit. When a factorial is involved, a common (but not guaranteed) strategy is to use the ratio test to evaluate the limit. This test compares the ratio of the ***N+1st*** terms to the ***Nth*** terms. As we go out "to infinity and beyond," many (but not all) of the limits behave like geometric functions. L'Hospital's Rule is not an option since factorials are not differentiable functions.

The Ratio Test

THEOREM: RATIO TEST

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

2. $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$.

3. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

***Series that grow very rapidly such as factorial and/or exponentials work especially well with the ratio test.

Example Determine if the series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n^2 3^{n+1}}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{(n+1)!}{3^n}$

(c) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$