Notes 11. 3 The Cross Product of Two Vectors

- Many applications in Physics, engineering and Geometry involve finding a vector in space the is **orthogonal** to two given vectors.
- A **cross product** would yield such a **vector**. (A dot product yields a scalar)
- **Cross products** are not commutative
- The definition of a Cross Product only applies to three-dimensional vectors.

Definition of Cross Product of Two Vectors in Space

Let

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

be vectors in space. The **cross product** of **u** and **v** is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

- Here is a convenient way to memorize the formula

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Put **u** in Row 2.
Put **v** in Row 3.

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Ex. 1

a.) Find $\boldsymbol{u} \text{ x } \boldsymbol{v}$, if $\boldsymbol{u} = i + 2j + k$ and $\boldsymbol{v} = 3i + j + 2k$

b.) Show the cross-product vector is orthogonal to both original vectors.

Geometric Properties of the Cross Product

Let **u** and **v** be nonzero vectors in space, and let θ be the angle between **u** and **v**.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**.
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$.
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if **u** and **v** are scalar multiples.
4. $\|\mathbf{u} \times \mathbf{v}\|$ = area of parallelogram having **u** and **v** as adjacent sides.

Ex. 2 Find a unit vector that is orthogonal to both $\boldsymbol{u} = 3i - 4j + k$ and $\boldsymbol{v} = -3i + 6j$

Ex. 3 If ABCD is a quadrilateral with vertices: $A(5,2,0)$, $B(2,6,1)$, $C(2,4,7)$, $D(5,0,6)$

a.) Plot ABCD

b.) Show ABCD is a parallelogram

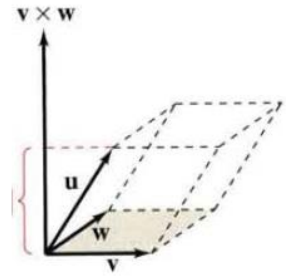
c.) Find the area of ABCD

d.) Determine if ABCD is a rectangle

The Triple Scalar Product

The **triple scalar product** of **u**, **v**, and **w** is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$



Area of base $= \|\mathbf{v} \times \mathbf{w}\|$
Volume of parallelepiped $= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$

Parallelepiped – a solid body of which each face is a parallelogram

If $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$, do not lie in the same plane the Triple Scalar Product can be used to find the volume of the parallelepiped if the vectors are adjacent edges. See figure above.

Ex. 4 Find the volume of the parallelepiped $\boldsymbol{u} = i + 2j - k$ and $\boldsymbol{v} = -i + 2j + 2k$ and $\boldsymbol{w} = 2i + k$