Topic 3.3 – Derivatives of Inverse Functions

In 1 – 6, find the derivative of f^{-1} for the function at the specified value of x using what we referred to as Method 2. No Calculator

Using Implicit

Using Formula

1. $f(x) = x^3 + 2x - 1$ at $x = 2$

$x = y^3 + 2y - 1$

$1 = 3y^2 \cdot \frac{dy}{dx} + 2 \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{3y^2 + 2}$

$(f^{-1})'(2) = \left. \frac{1}{3y^2 + 2} \right|_{y=1} = \frac{1}{5}$

$2 = y^3 + 2y - 1$

By trial and error, $y = 1$

$f'(x) = 3x^2 + 2x$

$2 = x^3 + 2x - 1$

$3 = x^3 + 2x \rightarrow x = 1$

$f'(1) = 3(1)^2 + 2(1) = 5$

$\boxed{[f^{-1}(2)]'} = \frac{1}{5}$

2. $f(x) = 2x^5 + x^3 + 1$ at $x = 4$

$x = 2y^5 + y^3 + 1$

$1 = 10y^4 \cdot \frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{10y^4 + 3y^2}$

$(f^{-1})'(4) = \left. \frac{1}{10y^4 + 3y^2} \right|_{y=1} = \frac{1}{13}$

$4 = 2y^5 + y^3 + 1$

By trial and error, $y = 1$

$f'(x) = 10x^4 + 3x^2$

$4 = 2x^5 + x^3 + 1$

$3 = 2x^5 + x^3 \Rightarrow x = 1$

$f'(1) = 10(1)^4 + 3(1)^2 = 13$

$[f^{-1}(4)]' = \frac{1}{13}$

3. $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ at $x = \frac{1}{2}$

$x = \sin y$

$1 = \cos y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$(f^{-1})'\left(\frac{1}{2}\right) = \left. \frac{1}{\cos y} \right|_{y=\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

$\frac{1}{2} = \sin y$

$y = \frac{\pi}{6}$

$f'(x) = \cos x$

$\frac{1}{2} = \sin x \Rightarrow x = \frac{\pi}{6}$

$f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\boxed{\left[f^{-1}\left(\frac{1}{2}\right)\right]'} = \frac{2}{\sqrt{3}}$

Using Implicit

4. $f(x) = \cos 2x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ at $x = 1$

$x = \cos 2y$

$1 = -2\sin 2y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{-2\sin 2y}$

$(f^{-1})'(1) = \left.\frac{1}{-2\sin 2y}\right|_{y=0} = \text{undefined}$

$1 = \cos 2y$

$2y = \cos^{-1}(1)$

$2y = 0 \rightarrow y = 0$

5. $f(x) = x^3 - \frac{4}{x}, x > 0$ at $x = 6$

$x = y^3 - \frac{4}{y}$

$1 = 3y^2 \cdot \frac{dy}{dx} - 4\left(-\frac{1}{y^2}\right) \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{3y^2 + \frac{4}{y^2}}$

$(f^{-1})'(6) = \left.\frac{1}{3y^2 + \frac{4}{y^2}}\right|_{y=2} = \frac{1}{12+1} = \frac{1}{13}$

$6 = y^3 - \frac{4}{y}$

By trial and error

6. $f(x) = x\,\ln(3x - 5)$ at $x = 0$

$x = y\ln(3y - 5)$

$1 = \frac{dy}{dx} \cdot \ln(3y-5) + y \cdot \frac{3}{3y-5} \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\ln(3y-5) + \frac{3y}{3y-5}}$

$(f^{-1})'(0) = \left.\frac{1}{\ln(3y-5) + \frac{3y}{3y-5}}\right|_{y=2} = \frac{1}{\ln(1) + \frac{6}{1}} = \frac{1}{6}$

$0 = y\ln(3y - 5)$

$\ln(3y - 5) = 0$ (Note: $y \neq 0$)

$1 = 3y - 5 \rightarrow y = 2$

Using formula

$f'(x) = -2\sin(2x)$

$1 = \cos(2x) \Rightarrow 2x = 0$
$x = 0$

$f'(0) = -2\sin(2\cdot 0) = 0$

$[f^{-1}(1)]' = \text{undefined}$

$f'(x) = 3x^2 + \frac{4}{x^2}$

$6 = x^3 - \frac{4}{x} \Rightarrow 6x = x^4 - 4$
$\boxed{x = 2}$

$f'(2) = 3(2)^2 + \frac{4}{2^2} = 13$

$[f^{-1}(6)]' = \frac{1}{13}$

$f'(x) = x \cdot \frac{1 \cdot 3}{3x-5} + \ln(3x-5)$

$0 = x \cdot \ln(3x-5)$
$\ln(3x-5) = 0$
means $3x - 5 = 1$
or $\boxed{x = 2}$

$f'(2) = 2 \cdot \frac{3}{6-5} + \ln(6-5)$
$= 6 + 0$
$= 6$

$[f^{-1}(0)]' = \frac{1}{6}$

7. Find $(f^{-1})'(-3)$ if $f(x) = \sqrt[3]{3x-5}$

$x = \sqrt[3]{3y-5}$ $f'(x) = \frac{1}{3}(3y-5)^{-2/3} \cdot 3$

$-3 = \sqrt[3]{3y-5}$ $f'(x) = \frac{1}{\sqrt[3]{(3y-5)^2}}$

$-27 = 3y-5$

$y = \frac{-22}{3}$ $f'\left(-\frac{22}{3}\right) = \frac{1}{\sqrt[3]{(-22-5)^2}} = \frac{1}{9}$

$(f^{-1})'(-3) = \frac{1}{f'\left(-\frac{22}{3}\right)} = \frac{1}{\frac{1}{9}} = 9$

$f(x) = (3x-5)^{1/3}$

$f'(x) = \frac{1}{3}(3x-5)^{-2/3} \cdot 3 = (3x-5)^{-2/3}$

$-3 = \sqrt[3]{3x-5} \Rightarrow -27 = 3x-5$

$x = -22/3$

$f'\left(\frac{-22}{3}\right) = \left(3 \cdot -\frac{22}{3} - 5\right)^{-2/3}$

$= (-27)^{-2/3}$

$= \left(\frac{-1}{27}\right)^{2/3} \Rightarrow \left(\frac{-1}{3}\right)^2 = \frac{1}{9}$

$[f^{-1}(-3)]' = 9$

8. Find $(f^{-1})'(-3)$ given the table of values of a ***strictly monotonic function*** (Google the definition) and its derivative.

x	$f(x)$	$f'(x)$
−3	5	6
8	−3	−2

$f^{-1}(-3) = 8$

$(f^{-1})'(-3) = \frac{1}{f'(8)} = \frac{1}{-2} = -\frac{1}{2}$

9. Selected values of a strictly monotonic function and its derivative are shown on the table below. Let $f(x)$ be a function such that $f(x) = h^{-1}(x)$. Find $f'(-1)$.

x	−1	0	2	4
$h(x)$	−5	−1	4	7
$h'(x)$	3	$\frac{1}{2}$	$\frac{1}{6}$	5

$h^{-1}(-1) = 0$

$f'(-1) = (h^{-1})'(-1) = \frac{1}{h'(0)} = \frac{1}{\frac{1}{2}} = 2$

10. Selected values of a strictly monotonic function and its derivative are shown on the table below.

Let $f(x)$ be a function such that $f(x) = h^{-1}(x)$. Find $f'(4)$

x	−1	0	2	4
$h(x)$	−5	−1	4	7
$h'(x)$	3	$\frac{1}{2}$	$\frac{1}{6}$	5

$h^{-1}(4) = 2$

$f'(4) = (h^{-1})'(4) = \frac{1}{h'(2)} = \frac{1}{\frac{1}{6}} = 6$