Topic 6.4 & 6.7 The Fundamental Theorem of Calculus and Accumulation Functions Involving Tables

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

Let f be continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$. (t is called a dummy variable)

2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

The second part of this theorem has a particular useful application in determining a functions value after it has undergone some amount of change. This is outlined in the theorem below.

The Net Change Theorem

The definite integral of a rate of change, F', is the net change in the original function F. That is,

$\int_a^b F'(x)dx = F(b) - F(a)$. Written another way, $F(b) = F(a) + \int_a^b F'(x)dx$.

End Amount — Start Amount — Net Change

Topic 8.1 Find the Average Value of a Function on an Interval

If $f(x)$ is integrable on the closed interval $[a,b]$, then the **average value** of $f(x)$ on that interval is $\frac{1}{b-a}\int_a^b f(x)dx$ or the **Mean Value Theorem for Integrals** $\int_a^b f(x)dx = f(c)(b-a)$.

Topic 8.2 Connecting position, velocity and acceleration of Functions using Integrals

$s(t) = \int v(t)\,dt$ and $v(t) = \int a(t)\,dt$

$$\begin{bmatrix}\text{displacement} \\ \text{over the time} \\ \text{interval } [t_0, t_1]\end{bmatrix} = \int_{t_0}^{t_1} v(t)\,dt = \int_{t_0}^{t_1} s'(t)\,dt = s(t_1) - s(t_0)$$

$$\begin{bmatrix}\text{distance traveled} \\ \text{during time} \\ \text{interval } [t_0, t_1]\end{bmatrix} = \int_{t_0}^{t_1} |v(t)|\,dt$$

The examples that follow on the next couple pages have been found on the AP Calculus exam in the Free Response section every year except 2019. They are commonly referred to as **"Table Problems." You may notice that these **full** FRQs ask you to find information that pertain to concepts that are taught in Unit 6 and in prior units. Having confidence on solving problems like these can have a significant impact your AP exam score.

Example 1: A graphing calculator may be used.

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

Example 2: No calculator is allowed.

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

Example 3: No calculator is allowed.

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.