20. $\int \sec(\sin\theta)\tan(\sin\theta)\cos\theta\,d\theta$

21. $\int \frac{\operatorname{csch}^2(2/x)}{x^2}\,dx$

22. $\int \frac{dx}{\sqrt{x^2-4}}$

23. $\int \frac{e^{-x}}{4-e^{-2x}}\,dx$

24. $\int \frac{\cos(\ln x)}{x}\,dx$

25. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

26. $\int \frac{\sinh(x^{-1/2})}{x^{3/2}}\,dx$

27. $\int \frac{x}{\csc(x^2)}\,dx$

28. $\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$

29. $\int x4^{-x^2}\,dx$

30. $\int 2^{\pi x}\,dx$

FOCUS ON CONCEPTS

31. (a) Evaluate the integral $\int \sin x\cos x\,dx$ using the substitution $u=\sin x$.
(b) Evaluate the integral $\int \sin x\cos x\,dx$ using the identity $\sin 2x = 2\sin x\cos x$.
(c) Explain why your answers to parts (a) and (b) are consistent.

32. (a) Derive the identity

$$\frac{\operatorname{sech}^2 x}{1+\tanh^2 x} = \operatorname{sech} 2x$$

(b) Use the result in part (a) to evaluate $\int \operatorname{sech} x\,dx$.
(c) Derive the identity

$$\operatorname{sech} x = \frac{2e^x}{e^{2x}+1}$$

(d) Use the result in part (c) to evaluate $\int \operatorname{sech} x\,dx$.
(e) Explain why your answers to parts (b) and (d) are consistent.

33. (a) Derive the identity

$$\frac{\sec^2 x}{\tan x} = \frac{1}{\sin x\cos x}$$

(b) Use the identity $\sin 2x = 2\sin x\cos x$ along with the result in part (a) to evaluate $\int \csc x\,dx$.
(c) Use the identity $\cos x = \sin[(\pi/2)-x]$ along with your answer to part (a) to evaluate $\int \sec x\,dx$.

✔QUICK CHECK ANSWERS 7.1

1. (a) $x+\ln|x|+C$ (b) $x+\ln|x+1|+C$ (c) $\ln(x^2+1)+\tan^{-1}x+C$ (d) $\frac{x^5}{5}+C$ **2.** (a) $-\cos x+C$ (b) $\tan x+C$ (c) $-\cot x+C$ (d) $\ln(1+\sin x)+C$ **3.** (a) $\frac{2}{3}(x-1)^{3/2}+C$ (b) $\frac{1}{2}e^{2x+1}+C$ (c) $\frac{1}{2}\sin^2 x+C$ (d) $\frac{1}{4}\tanh x+C$

7.2 INTEGRATION BY PARTS

In this section we will discuss an integration technique that is essentially an antiderivative formulation of the formula for differentiating a product of two functions.

THE PRODUCT RULE AND INTEGRATION BY PARTS

Our primary goal in this section is to develop a general method for attacking integrals of the form

$$\int f(x)g(x)\,dx$$

As a first step, let $G(x)$ be *any* antiderivative of $g(x)$. In this case $G'(x)=g(x)$, so the product rule for differentiating $f(x)G(x)$ can be expressed as

$$\frac{d}{dx}[f(x)G(x)] = f(x)G'(x)+f'(x)G(x) = f(x)g(x)+f'(x)G(x) \tag{1}$$

This implies that $f(x)G(x)$ is an antiderivative of the function on the right side of (1), so we can express (1) in integral form as

$$\int [f(x)g(x)+f'(x)G(x)]\,dx = f(x)G(x)$$

or, equivalently, as

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx \tag{2}$$

This formula allows us to integrate $f(x)g(x)$ by integrating $f'(x)G(x)$ instead, and in many cases the net effect is to replace a difficult integration with an easier one. The application of this formula is called ***integration by parts***.

In practice, we usually rewrite (2) by letting

$$u = f(x), \quad du = f'(x)\,dx$$
$$v = G(x), \quad dv = G'(x)\,dx = g(x)\,dx$$

This yields the following alternative form for (2):

$$\int u\,dv = uv - \int v\,du \tag{3}$$

Note that in Example 1 we omitted the constant of integration in calculating v from dv. Had we included a constant of integration, it would have eventually dropped out. This is always the case in integration by parts [Exercise 68(b)], so it is common to omit the constant at this stage of the computation. However, there are certain cases in which making a clever choice of a constant of integration to include with v can simplify the computation of $\int v\,du$ (Exercises 69–71).

Example 1 Use integration by parts to evaluate $\int x\cos x\,dx$.

Solution. We will apply Formula (3). The first step is to make a choice for u and dv to put the given integral in the form $\int u\,dv$. We will let

$$u = x \quad \text{and} \quad dv = \cos x\,dx$$

(Other possibilities will be considered later.) The second step is to compute du from u and v from dv. This yields

$$du = dx \quad \text{and} \quad v = \int dv = \int \cos x\,dx = \sin x$$

The third step is to apply Formula (3). This yields

$$\int \underbrace{x}_{u}\,\underbrace{\cos x\,dx}_{dv} = \underbrace{x}_{u}\,\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\,\underbrace{dx}_{du}$$
$$= x\sin x - (-\cos x) + C = x\sin x + \cos x + C$$ ◄

GUIDELINES FOR INTEGRATION BY PARTS

The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate than the original. In general, there are no hard and fast rules for doing this; it is mainly a matter of experience that comes from lots of practice. A strategy that often works is to choose u and dv so that u becomes "simpler" when differentiated, while leaving a dv that can be readily integrated to obtain v. Thus, for the integral $\int x\cos x\,dx$ in Example 1, both goals were achieved by letting $u = x$ and $dv = \cos x\,dx$. In contrast, $u = \cos x$ would not have been a good first choice in that example, since $du/dx = -\sin x$ is no simpler than u. Indeed, had we chosen

$$u = \cos x \qquad dv = x\,dx$$
$$du = -\sin x\,dx \qquad v = \int x\,dx = \frac{x^2}{2}$$

then we would have obtained

$$\int x\cos x\,dx = \frac{x^2}{2}\cos x - \int \frac{x^2}{2}(-\sin x)\,dx = \frac{x^2}{2}\cos x + \frac{1}{2}\int x^2\sin x\,dx$$

For this choice of u and dv, the new integral is actually more complicated than the original.

The LIATE method is discussed in the article "A Technique for Integration by Parts," *American Mathematical Monthly*, Vol. 90, 1983, pp. 210–211, by Herbert Kasube.

There is another useful strategy for choosing u and dv that can be applied when the integrand is a product of two functions from *different* categories in the list

Logarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential

In this case you will often be successful if you take u to be the function whose category occurs earlier in the list and take dv to be the rest of the integrand. The acronym LIATE will help you to remember the order. The method does not work all the time, but it works often enough to be useful.

Note, for example, that the integrand in Example 1 consists of the product of the *algebraic* function x and the *trigonometric* function $\cos x$. Thus, the LIATE method suggests that we should let $u = x$ and $dv = \cos x\,dx$, which proved to be a successful choice.

▶ **Example 2** Evaluate $\int xe^x\,dx$.

Solution. In this case the integrand is the product of the algebraic function x with the exponential function e^x. According to LIATE we should let

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

so that

$$du = dx \quad \text{and} \quad v = \int e^x\,dx = e^x$$

Thus, from (3)

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C \blacktriangleleft$$

▶ **Example 3** Evaluate $\int \ln x\,dx$.

Solution. One choice is to let $u = 1$ and $dv = \ln x\,dx$. But with this choice finding v is equivalent to evaluating $\int \ln x\,dx$ and we have gained nothing. Therefore, the only reasonable choice is to let

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = \int dx = x$$

With this choice it follows from (3) that

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int dx = x\ln x - x + C \blacktriangleleft$$

REPEATED INTEGRATION BY PARTS

It is sometimes necessary to use integration by parts more than once in the same problem.

▶ **Example 4** Evaluate $\int x^2e^{-x}\,dx$.

Solution. Let

$$u = x^2, \quad dv = e^{-x}\,dx, \quad du = 2x\,dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that from (3)

$$\begin{aligned}\int x^2e^{-x}\,dx &= \int u\,dv = uv - \int v\,du \\ &= x^2(-e^{-x}) - \int -e^{-x}(2x)\,dx \\ &= -x^2e^{-x} + 2\int xe^{-x}\,dx \end{aligned} \tag{4}$$

The last integral is similar to the original except that we have replaced x^2 by x. Another integration by parts applied to $\int xe^{-x}\,dx$ will complete the problem. We let

$$u = x, \quad dv = e^{-x}\,dx, \quad du = dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Finally, substituting this into the last line of (4) yields

$$\begin{aligned}\int x^2e^{-x}\,dx &= -x^2e^{-x} + 2\int xe^{-x}\,dx = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + C \\ &= -(x^2 + 2x + 2)e^{-x} + C \end{aligned}$$ ◀

The LIATE method suggests that integrals of the form

$$\int e^{ax}\sin bx\,dx \quad \text{and} \quad \int e^{ax}\cos bx\,dx$$

can be evaluated by letting $u = \sin bx$ or $u = \cos bx$ and $dv = e^{ax}\,dx$. However, this will require a technique that deserves special attention.

▶ **Example 5** Evaluate $\int e^x \cos x\,dx$.

Solution. Let

$$u = \cos x, \quad dv = e^x\,dx, \quad du = -\sin x\,dx, \quad v = \int e^x\,dx = e^x$$

Thus,

$$\int e^x\cos x\,dx = \int u\,dv = uv - \int v\,du = e^x\cos x + \int e^x \sin x\,dx \tag{5}$$

Since the integral $\int e^x \sin x\,dx$ is similar in form to the original integral $\int e^x\cos x\,dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let

$$u = \sin x, \quad dv = e^x\,dx, \quad du = \cos x\,dx, \quad v = \int e^x\,dx = e^x$$

Thus,

$$\int e^x\sin x\,dx = \int u\,dv = uv - \int v\,du = e^x\sin x - \int e^x\cos x\,dx$$

Together with Equation (5) this yields

$$\int e^x\cos x\,dx = e^x\cos x + e^x\sin x - \int e^x\cos x\,dx \tag{6}$$

which is an equation we can solve for the unknown integral. We obtain

$$2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x$$

and hence

$$\int e^x \cos x\,dx = \tfrac{1}{2}e^x \cos x + \tfrac{1}{2}e^x \sin x + C \blacktriangleleft$$

A TABULAR METHOD FOR REPEATED INTEGRATION BY PARTS

Integrals of the form

$$\int p(x)f(x)\,dx$$

where $p(x)$ is a polynomial, can sometimes be evaluated using repeated integration by parts in which u is taken to be $p(x)$ or one of its derivatives at each stage. Since du is computed by differentiating u, the repeated differentiation of $p(x)$ will eventually produce 0, at which point you may be left with a simplified integration problem. A convenient method for organizing the computations into two columns is called ***tabular integration by parts***.

More information on tabular integration by parts can be found in the articles "Tabular Integration by Parts," *College Mathematics Journal*, Vol. 21, 1990, pp. 307–311, by David Horowitz and "More on Tabular Integration by Parts," *College Mathematics Journal*, Vol. 22, 1991, pp. 407–410, by Leonard Gillman.

Tabular Integration by Parts

Step 1. Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate $f(x)$ repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

This process is illustrated in Figure 7.2.1 for the integral $\int (x^2 - x)\cos x\,dx$.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
$x^2 - x$	$+$	$\cos x$
$2x - 1$	$-$	$\sin x$
2	$+$	$-\cos x$
0		$-\sin x$

$$\begin{aligned}\int (x^2 - x)\cos x\,dx &= (x^2 - x)\sin x + (2x - 1)\cos x - 2\sin x + C\\ &= (x^2 - x - 2)\sin x + (2x - 1)\cos x + C\end{aligned}$$

▶ **Figure 7.2.1**

▶ **Example 6** In Example 11 of Section 5.3 we evaluated $\int x^2\sqrt{x-1}\,dx$ using u-substitution. Evaluate this integral using tabular integration by parts.

Solution.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/2}$
0		$\frac{8}{105}(x-1)^{7/2}$

The result obtained in Example 6 looks quite different from that obtained in Example 11 of Section 5.3. Show that the two answers are equivalent.

Thus, it follows that

$$\int x^2\sqrt{x-1}\,dx = \tfrac{2}{3}x^2(x-1)^{3/2} - \tfrac{8}{15}x(x-1)^{5/2} + \tfrac{16}{105}(x-1)^{7/2} + C \blacktriangleleft$$

INTEGRATION BY PARTS FOR DEFINITE INTEGRALS

For definite integrals the formula corresponding to (3) is

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du \tag{7}$$

REMARK It is important to keep in mind that the variables u and v in this formula are functions of x and that the limits of integration in (7) are limits on the variable x. Sometimes it is helpful to emphasize this by writing (7) as

$$\int_{x=a}^b u\,dv = uv\Big]_{x=a}^b - \int_{x=a}^b v\,du \tag{8}$$

The next example illustrates how integration by parts can be used to integrate the inverse trigonometric functions.

▶ Example 7 Evaluate $\int_0^1 \tan^{-1} x\,dx$.

Solution. Let

$$u = \tan^{-1} x, \quad dv = dx, \quad du = \frac{1}{1+x^2}\,dx, \quad v = x$$

Thus,

$$\int_0^1 \tan^{-1} x\,dx = \int_0^1 u\,dv = uv\Big]_0^1 - \int_0^1 v\,du$$

$$= x\tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

The limits of integration refer to x; that is, $x = 0$ and $x = 1$.

But

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{1}{2}\ln 2$$

so

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1} x\Big]_0^1 - \frac{1}{2}\ln 2 = \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2} \blacktriangleleft$$

REDUCTION FORMULAS

Integration by parts can be used to derive ***reduction formulas*** for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a *lower* power of that function. For example, if n is a positive integer and $n \geq 2$, then integration by parts can be used to obtain the reduction formulas

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \tag{9}$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \tag{10}$$

To illustrate how such formulas can be obtained, let us derive (10). We begin by writing $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$ and letting

$$\begin{aligned} u &= \cos^{n-1} x & dv &= \cos x\,dx \\ du &= (n-1)\cos^{n-2} x(-\sin x)\,dx & v &= \sin x \\ &= -(n-1)\cos^{n-2} x\sin x\,dx \end{aligned}$$

so that

$$\begin{aligned} \int \cos^n x\,dx = \int \cos^{n-1} x\cos x\,dx &= \int u\,dv = uv - \int v\,du \\ &= \cos^{n-1} x\sin x + (n-1)\int \sin^2 x\cos^{n-2} x\,dx \\ &= \cos^{n-1} x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx \\ &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx \end{aligned}$$

Moving the last term on the right to the left side yields

$$n\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx$$

from which (10) follows. The derivation of reduction formula (9) is similar (Exercise 63).

Reduction formulas (9) and (10) reduce the exponent of sine (or cosine) by 2. Thus, if the formulas are applied repeatedly, the exponent can eventually be reduced to 0 if n is even or 1 if n is odd, at which point the integration can be completed. We will discuss this method in more detail in the next section, but for now, here is an example that illustrates how reduction formulas work.

Example 8 Evaluate $\int \cos^4 x\,dx$.

Solution. From (10) with $n = 4$

$$\begin{aligned} \int \cos^4 x\,dx &= \tfrac{1}{4}\cos^3 x\sin x + \tfrac{3}{4}\int \cos^2 x\,dx \\ &= \tfrac{1}{4}\cos^3 x\sin x + \tfrac{3}{4}\left(\tfrac{1}{2}\cos x\sin x + \tfrac{1}{2}\int dx\right) \\ &= \tfrac{1}{4}\cos^3 x\sin x + \tfrac{3}{8}\cos x\sin x + \tfrac{3}{8}x + C \end{aligned}$$

Now apply (10) with $n = 2$.

✔ QUICK CHECK EXERCISES 7.2 (See page 500 for answers.)

1. (a) If $G'(x) = g(x)$, then
$$\int f(x)g(x)\,dx = f(x)G(x) - \_\_\_\_\_\_\_$$
(b) If $u = f(x)$ and $v = G(x)$, then the formula in part (a) can be written in the form $\int u\,dv = \_\_\_\_\_\_\_$.

2. Find an appropriate choice of u and dv for integration by parts of each integral. Do not evaluate the integral.
(a) $\int x \ln x\,dx$; $u = \_\_\_\_\_\_\_$, $dv = \_\_\_\_\_\_\_$
(b) $\int (x-2)\sin x\,dx$; $u = \_\_\_\_\_\_\_$, $dv = \_\_\_\_\_\_\_$
(c) $\int \sin^{-1} x\,dx$; $u = \_\_\_\_\_\_\_$, $dv = \_\_\_\_\_\_\_$
(d) $\int \dfrac{x}{\sqrt{x-1}}\,dx$; $u = \_\_\_\_\_\_\_$, $dv = \_\_\_\_\_\_\_$

3. Use integration by parts to evaluate the integral.
(a) $\int xe^{2x}\,dx$ (b) $\int \ln(x-1)\,dx$
(c) $\int_0^{\pi/6} x \sin 3x\,dx$

4. Use a reduction formula to evaluate $\int \sin^3 x\,dx$.

EXERCISE SET 7.2

1–38 Evaluate the integral. ■

1. $\int xe^{-2x}\,dx$
2. $\int xe^{3x}\,dx$
3. $\int x^2e^{x}\,dx$
4. $\int x^2e^{-2x}\,dx$
5. $\int x\sin 3x\,dx$
6. $\int x\cos 2x\,dx$
7. $\int x^2\cos x\,dx$
8. $\int x^2\sin x\,dx$
9. $\int x\ln x\,dx$
10. $\int \sqrt{x}\ln x\,dx$
11. $\int (\ln x)^2\,dx$
12. $\int \dfrac{\ln x}{\sqrt{x}}\,dx$
13. $\int \ln(3x-2)\,dx$
14. $\int \ln(x^2+4)\,dx$
15. $\int \sin^{-1} x\,dx$
16. $\int \cos^{-1}(2x)\,dx$
17. $\int \tan^{-1}(3x)\,dx$
18. $\int x\tan^{-1} x\,dx$
19. $\int e^x\sin x\,dx$
20. $\int e^{3x}\cos 2x\,dx$
21. $\int \sin(\ln x)\,dx$
22. $\int \cos(\ln x)\,dx$
23. $\int x\sec^2 x\,dx$
24. $\int x\tan^2 x\,dx$
25. $\int x^3e^{x^2}\,dx$
26. $\int \dfrac{xe^x}{(x+1)^2}\,dx$
27. $\int_0^2 xe^{2x}\,dx$
28. $\int_0^1 xe^{-5x}\,dx$
29. $\int_1^e x^2\ln x\,dx$
30. $\int_{\sqrt{e}}^{e} \dfrac{\ln x}{x^2}\,dx$
31. $\int_{-1}^{1} \ln(x+2)\,dx$
32. $\int_0^{\sqrt{3}/2} \sin^{-1} x\,dx$
33. $\int_2^4 \sec^{-1}\sqrt{\theta}\,d\theta$
34. $\int_1^2 x\sec^{-1} x\,dx$
35. $\int_0^{\pi} x\sin 2x\,dx$
36. $\int_0^{\pi} (x + x\cos x)\,dx$
37. $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$
38. $\int_0^2 \ln(x^2+1)\,dx$

39–42 True–False Determine whether the statement is true or false. Explain your answer. ■

39. The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate than the original.

40. Applying the LIATE strategy to evaluate $\int x^3 \ln x\,dx$, we should choose $u = x^3$ and $dv = \ln x\,dx$.

41. To evaluate $\int \ln e^x\,dx$ using integration by parts, choose $dv = e^x\,dx$.

42. Tabular integration by parts is useful for integrals of the form $\int p(x)f(x)\,dx$, where $p(x)$ is a polynomial and $f(x)$ can be repeatedly integrated.

43–44 Evaluate the integral by making a u-substitution and then integrating by parts. ■

43. $\int e^{\sqrt{x}}\,dx$
44. $\int \cos\sqrt{x}\,dx$

45. Prove that tabular integration by parts gives the correct answer for
$$\int p(x)f(x)\,dx$$
where $p(x)$ is any *quadratic polynomial* and $f(x)$ is any function that can be repeatedly integrated.

46. The computations of any integral evaluated by repeated integration by parts can be organized using tabular integration by parts. Use this organization to evaluate $\int e^x\cos x\,dx$ in

two ways: first by repeated differentiation of $\cos x$ (compare Example 5), and then by repeated differentiation of e^x.

47–52 Evaluate the integral using tabular integration by parts.

47. $\int (3x^2 - x + 2)e^{-x}\,dx$

48. $\int (x^2 + x + 1)\sin x\,dx$

49. $\int 4x^4 \sin 2x\,dx$

50. $\int x^3\sqrt{2x+1}\,dx$

51. $\int e^{ax}\sin bx\,dx$

52. $\int e^{-3\theta}\sin 5\theta\,d\theta$

53. Consider the integral $\int \sin x \cos x\,dx$.
(a) Evaluate the integral two ways: first using integration by parts, and then using the substitution $u = \sin x$.
(b) Show that the results of part (a) are equivalent.
(c) Which of the two methods do you prefer? Discuss the reasons for your preference.

54. Evaluate the integral

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$$

using
(a) integration by parts
(b) the substitution $u = \sqrt{x^2+1}$.

55. (a) Find the area of the region enclosed by $y = \ln x$, the line $x = e$, and the x-axis.
(b) Find the volume of the solid generated when the region in part (a) is revolved about the x-axis.

56. Find the area of the region between $y = x\sin x$ and $y = x$ for $0 \le x \le \pi/2$.

57. Find the volume of the solid generated when the region between $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$ is revolved about the y-axis.

58. Find the volume of the solid generated when the region enclosed between $y = \cos x$ and $y = 0$ for $0 \le x \le \pi/2$ is revolved about the y-axis.

59. A particle moving along the x-axis has velocity function $v(t) = t^3 \sin t$. How far does the particle travel from time $t = 0$ to $t = \pi$?

60. The study of sawtooth waves in electrical engineering leads to integrals of the form

$$\int_{-\pi/\omega}^{\pi/\omega} t\sin(k\omega t)\,dt$$

where k is an integer and ω is a nonzero constant. Evaluate the integral.

61. Use reduction formula (9) to evaluate
(a) $\int \sin^4 x\,dx$ (b) $\int_0^{\pi/2} \sin^5 x\,dx$.

62. Use reduction formula (10) to evaluate
(a) $\int \cos^5 x\,dx$ (b) $\int_0^{\pi/2} \cos^6 x\,dx$.

63. Derive reduction formula (9).

64. In each part, use integration by parts or other methods to derive the reduction formula.

(a) $\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$

(b) $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(c) $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

65–66 Use the reduction formulas in Exercise 64 to evaluate the integrals.

65. (a) $\int \tan^4 x\,dx$ (b) $\int \sec^4 x\,dx$ (c) $\int x^3 e^x\,dx$

66. (a) $\int x^2 e^{3x}\,dx$ (b) $\int_0^1 xe^{-\sqrt{x}}\,dx$
[*Hint:* First make a substitution.]

67. Let f be a function whose second derivative is continuous on $[-1, 1]$. Show that

$$\int_{-1}^{1} xf''(x)\,dx = f'(1) + f'(-1) - f(1) + f(-1)$$

FOCUS ON CONCEPTS

68. (a) In the integral $\int x\cos x\,dx$, let

$$u = x, \quad dv = \cos x\,dx,$$
$$du = dx, \quad v = \sin x + C_1$$

Show that the constant C_1 cancels out, thus giving the same solution obtained by omitting C_1.
(b) Show that in general

$$uv - \int v\,du = u(v + C_1) - \int (v + C_1)\,du$$

thereby justifying the omission of the constant of integration when calculating v in integration by parts.

69. Evaluate $\int \ln(x+1)\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = 1$ when going from dv to v.

70. Evaluate $\int \ln(3x-2)\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = -\frac{2}{3}$ when going from dv to v. Compare your solution with your answer to Exercise 13.

71. Evaluate $\int x\tan^{-1} x\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = \frac{1}{2}$ when going from dv to v.

72. What equation results if integration by parts is applied to the integral

$$\int \frac{1}{x\ln x}\,dx$$

with the choices

$$u = \frac{1}{\ln x} \quad \text{and} \quad dv = \frac{1}{x}\,dx?$$

In what sense is this equation true? In what sense is it false?

73. Writing Explain how the product rule for derivatives and the technique of integration by parts are related.

74. Writing For what sort of problems are the integration techniques of substitution and integration by parts "competing" techniques? Describe situations, with examples, where each of these techniques would be preferred over the other.

✔ QUICK CHECK ANSWERS 7.2

1. (a) $\int f'(x)G(x)\,dx$ (b) $uv - \int v\,du$ **2.** (a) $\ln x;\ x\,dx$ (b) $x-2;\ \sin x\,dx$ (c) $\sin^{-1}x;\ dx$ (d) $x;\ \dfrac{1}{\sqrt{x-1}}\,dx$
3. (a) $\left(\dfrac{x}{2}-\dfrac{1}{4}\right)e^{2x}+C$ (b) $(x-1)\ln(x-1)-x+C$ (c) $\frac{1}{9}$ **4.** $-\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x + C$

7.3 INTEGRATING TRIGONOMETRIC FUNCTIONS

In the last section we derived reduction formulas for integrating positive integer powers of sine, cosine, tangent, and secant. In this section we will show how to work with those reduction formulas, and we will discuss methods for integrating other kinds of integrals that involve trigonometric functions.

INTEGRATING POWERS OF SINE AND COSINE

We begin by recalling two reduction formulas from the preceding section.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx \tag{1}$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx \tag{2}$$

In the case where $n=2$, these formulas yield

$$\int \sin^2 x\,dx = -\tfrac{1}{2}\sin x\cos x + \tfrac{1}{2}\int dx = \tfrac{1}{2}x - \tfrac{1}{2}\sin x\cos x + C \tag{3}$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}\cos x\sin x + \tfrac{1}{2}\int dx = \tfrac{1}{2}x + \tfrac{1}{2}\sin x\cos x + C \tag{4}$$

Alternative forms of these integration formulas can be derived from the trigonometric identities

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \text{and} \quad \cos^2 x = \tfrac{1}{2}(1+\cos 2x) \tag{5–6}$$

which follow from the double-angle formulas

$$\cos 2x = 1 - 2\sin^2 x \quad \text{and} \quad \cos 2x = 2\cos^2 x - 1$$

These identities yield

$$\int \sin^2 x\,dx = \tfrac{1}{2}\int (1-\cos 2x)\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C \tag{7}$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}\int (1+\cos 2x)\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C \tag{8}$$