35. Find the area of the surface generated when the curve in Exercise 34 is revolved about the x-axis.

36. Find the volume of the solid generated when the region enclosed by $x = y(1 - y^2)^{1/4}$, $y = 0$, $y = 1$, and $x = 0$ is revolved about the y-axis.

37–48 Evaluate the integral.

37. $\displaystyle\int \frac{dx}{x^2 - 4x + 5}$

38. $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$

39. $\displaystyle\int \frac{dx}{\sqrt{3 + 2x - x^2}}$

40. $\displaystyle\int \frac{dx}{16x^2 + 16x + 5}$

41. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 6x + 10}}$

42. $\displaystyle\int \frac{x}{x^2 + 2x + 2}\,dx$

43. $\displaystyle\int \sqrt{3 - 2x - x^2}\,dx$

44. $\displaystyle\int \frac{e^x}{\sqrt{1 + e^x + e^{2x}}}\,dx$

45. $\displaystyle\int \frac{dx}{2x^2 + 4x + 7}$

46. $\displaystyle\int \frac{2x + 3}{4x^2 + 4x + 5}\,dx$

47. $\displaystyle\int_1^2 \frac{dx}{\sqrt{4x - x^2}}$

48. $\displaystyle\int_0^4 \sqrt{x(4 - x)}\,dx$

C **49–50** There is a good chance that your CAS will not be able to evaluate these integrals as stated. If this is so, make a substitution that converts the integral into one that your CAS can evaluate.

49. $\displaystyle\int \cos x \sin x \sqrt{1 - \sin^4 x}\,dx$

50. $\displaystyle\int (x \cos x + \sin x)\sqrt{1 + x^2 \sin^2 x}\,dx$

51. (a) Use the ***hyperbolic substitution*** $x = 3 \sinh u$, the identity $\cosh^2 u - \sinh^2 u = 1$, and Theorem 6.9.4 to evaluate

$$\int \frac{dx}{\sqrt{x^2 + 9}}$$

(b) Evaluate the integral in part (a) using a trigonometric substitution and show that the result agrees with that obtained in part (a).

52. Use the hyperbolic substitutionn $x = \cosh u$, the identity $\sinh^2 u = \frac{1}{2}(\cosh 2u - 1)$, and the results referenced in Exercise 51 to evaluate

$$\int \sqrt{x^2 - 1}\,dx, \quad x \geq 1$$

53. Writing The trigonometric substitution $x = a \sin\theta$, $-\pi/2 \leq \theta \leq \pi/2$, is suggested for an integral whose integrand involves $\sqrt{a^2 - x^2}$. Discuss the implications of restricting θ to $\pi/2 \leq \theta \leq 3\pi/2$, and explain why the restriction $-\pi/2 \leq \theta \leq \pi/2$ should be preferred.

54. Writing The trigonometric substitution $x = a \cos\theta$ could also be used for an integral whose integrand involves $\sqrt{a^2 - x^2}$. Determine an appropriate restriction for θ with the substitution $x = a \cos\theta$, and discuss how to apply this substitution in appropriate integrals. Illustrate your discussion by evaluating the integral in Example 1 using a substitution of this type.

✔ QUICK CHECK ANSWERS 7.4

1. (a) $x = a \sin\theta$ (b) $x = a \tan\theta$ (c) $x = a \sec\theta$ **2.** (a) $\dfrac{\sqrt{x^2 - 4}}{x}$ (b) $\dfrac{2}{x}$ (c) $\dfrac{\sqrt{x^2 - 4}}{2}$ **3.** (a) $x = 3 \tan\theta$ (b) $x = 3 \sin\theta$ (c) $x = \frac{1}{3} \sin\theta$ (d) $x = 3 \sec\theta$ (e) $x = \sqrt{3} \tan\theta$ (f) $x = \frac{1}{9} \tan\theta$ **4.** (a) $x - 1$ (b) $x - 3$ (c) $x + 2$

7.5 INTEGRATING RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

Recall that a rational function is a ratio of two polynomials. In this section we will give a general method for integrating rational functions that is based on the idea of decomposing a rational function into a sum of simple rational functions that can be integrated by the methods studied in earlier sections.

PARTIAL FRACTIONS

In algebra, one learns to combine two or more fractions into a single fraction by finding a common denominator. For example,

$$\frac{2}{x - 4} + \frac{3}{x + 1} = \frac{2(x + 1) + 3(x - 4)}{(x - 4)(x + 1)} = \frac{5x - 10}{x^2 - 3x - 4} \tag{1}$$

However, for purposes of integration, the left side of (1) is preferable to the right side since each of the terms is easy to integrate:

$$\int \frac{5x-10}{x^2-3x-4}\,dx = \int \frac{2}{x-4}\,dx + \int \frac{3}{x+1}\,dx = 2\ln|x-4| + 3\ln|x+1| + C$$

Thus, it is desirable to have some method that will enable us to obtain the left side of (1), starting with the right side. To illustrate how this can be done, we begin by noting that on the left side the numerators are constants and the denominators are the factors of the denominator on the right side. Thus, to find the left side of (1), starting from the right side, we could factor the denominator of the right side and look for constants A and B such that

$$\frac{5x-10}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1} \tag{2}$$

One way to find the constants A and B is to multiply (2) through by $(x-4)(x+1)$ to clear fractions. This yields

$$5x - 10 = A(x+1) + B(x-4) \tag{3}$$

This relationship holds for all x, so it holds in particular if $x = 4$ or $x = -1$. Substituting $x = 4$ in (3) makes the second term on the right drop out and yields the equation $10 = 5A$ or $A = 2$; and substituting $x = -1$ in (3) makes the first term on the right drop out and yields the equation $-15 = -5B$ or $B = 3$. Substituting these values in (2) we obtain

$$\frac{5x-10}{(x-4)(x+1)} = \frac{2}{x-4} + \frac{3}{x+1} \tag{4}$$

which agrees with (1).

A second method for finding the constants A and B is to multiply out the right side of (3) and collect like powers of x to obtain

$$5x - 10 = (A+B)x + (A-4B)$$

Since the polynomials on the two sides are identical, their corresponding coefficients must be the same. Equating the corresponding coefficients on the two sides yields the following system of equations in the unknowns A and B:

$$\begin{aligned} A + B &= 5 \\ A - 4B &= -10 \end{aligned}$$

Solving this system yields $A = 2$ and $B = 3$ as before (verify).

The terms on the right side of (4) are called ***partial fractions*** of the expression on the left side because they each constitute *part* of that expression. To find those partial fractions we first had to make a guess about their form, and then we had to find the unknown constants. Our next objective is to extend this idea to general rational functions. For this purpose, suppose that $P(x)/Q(x)$ is a ***proper rational function***, by which we mean that the degree of the numerator is less than the degree of the denominator. There is a theorem in advanced algebra which states that every proper rational function can be expressed as a sum

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x)$$

where $F_1(x), F_2(x), \ldots, F_n(x)$ are rational functions of the form

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^k}$$

in which the denominators are factors of $Q(x)$. The sum is called the ***partial fraction decomposition*** of $P(x)/Q(x)$, and the terms are called ***partial fractions***. As in our opening example, there are two parts to finding a partial fraction decomposition: determining the exact form of the decomposition and finding the unknown constants.

FINDING THE FORM OF A PARTIAL FRACTION DECOMPOSITION

The first step in finding the form of the partial fraction decomposition of a proper rational function $P(x)/Q(x)$ is to factor $Q(x)$ completely into linear and irreducible quadratic factors, and then collect all repeated factors so that $Q(x)$ is expressed as a product of *distinct* factors of the form

$$(ax+b)^m \quad \text{and} \quad (ax^2+bx+c)^m$$

From these factors we can determine the form of the partial fraction decomposition using two rules that we will now discuss.

LINEAR FACTORS

If all of the factors of $Q(x)$ are linear, then the partial fraction decomposition of $P(x)/Q(x)$ can be determined by using the following rule:

LINEAR FACTOR RULE For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \ldots, A_m$ are constants to be determined. In the case where $m=1$, only the first term in the sum appears.

Example 1 Evaluate $\displaystyle\int \frac{dx}{x^2+x-2}$.

Solution. The integrand is a proper rational function that can be written as

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)}$$

The factors $x-1$ and $x+2$ are both linear and appear to the first power, so each contributes one term to the partial fraction decomposition by the linear factor rule. Thus, the decomposition has the form

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \tag{5}$$

where A and B are constants to be determined. Multiplying this expression through by $(x-1)(x+2)$ yields

$$1 = A(x+2) + B(x-1) \tag{6}$$

As discussed earlier, there are two methods for finding A and B: we can substitute values of x that are chosen to make terms on the right drop out, or we can multiply out on the right and equate corresponding coefficients on the two sides to obtain a system of equations that can be solved for A and B. We will use the first approach.

Setting $x=1$ makes the second term in (6) drop out and yields $1=3A$ or $A=\frac{1}{3}$; and setting $x=-2$ makes the first term in (6) drop out and yields $1=-3B$ or $B=-\frac{1}{3}$. Substituting these values in (5) yields the partial fraction decomposition

$$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}}{x+2}$$

The integration can now be completed as follows:

$$\int \frac{dx}{(x-1)(x+2)} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2}$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C \blacktriangleleft$$

If the factors of $Q(x)$ are linear and none are repeated, as in the last example, then the recommended method for finding the constants in the partial fraction decomposition is to substitute appropriate values of x to make terms drop out. However, if some of the linear factors are repeated, then it will not be possible to find all of the constants in this way. In this case the recommended procedure is to find as many constants as possible by substitution and then find the rest by equating coefficients. This is illustrated in the next example.

▶ **Example 2** Evaluate $\int \frac{2x+4}{x^3-2x^2}\,dx$.

Solution. The integrand can be rewritten as

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

Although x^2 is a quadratic factor, it is *not* irreducible since $x^2 = xx$. Thus, by the linear factor rule, x^2 introduces two terms (since $m = 2$) of the form

$$\frac{A}{x} + \frac{B}{x^2}$$

and the factor $x - 2$ introduces one term (since $m = 1$) of the form

$$\frac{C}{x-2}$$

so the partial fraction decomposition is

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \tag{7}$$

Multiplying by $x^2(x-2)$ yields

$$2x + 4 = Ax(x-2) + B(x-2) + Cx^2 \tag{8}$$

which, after multiplying out and collecting like powers of x, becomes

$$2x + 4 = (A + C)x^2 + (-2A + B)x - 2B \tag{9}$$

Setting $x = 0$ in (8) makes the first and third terms drop out and yields $B = -2$, and setting $x = 2$ in (8) makes the first and second terms drop out and yields $C = 2$ (verify). However, there is no substitution in (8) that produces A directly, so we look to Equation (9) to find this value. This can be done by equating the coefficients of x^2 on the two sides to obtain

$$A + C = 0 \quad \text{or} \quad A = -C = -2$$

Substituting the values $A = -2$, $B = -2$, and $C = 2$ in (7) yields the partial fraction decomposition

$$\frac{2x+4}{x^2(x-2)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

Thus,

$$\int \frac{2x+4}{x^2(x-2)}\,dx = -2\int \frac{dx}{x} - 2\int \frac{dx}{x^2} + 2\int \frac{dx}{x-2}$$

$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C = 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C \blacktriangleleft$$

QUADRATIC FACTORS

If some of the factors of $Q(x)$ are irreducible quadratics, then the contribution of those factors to the partial fraction decomposition of $P(x)/Q(x)$ can be determined from the following rule:

QUADRATIC FACTOR RULE For each factor of the form $(ax^2 + bx + c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example 3 Evaluate $\displaystyle\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\, dx$.

Solution. The denominator in the integrand can be factored by grouping:

$$3x^3 - x^2 + 3x - 1 = x^2(3x - 1) + (3x - 1) = (3x - 1)(x^2 + 1)$$

By the linear factor rule, the factor $3x - 1$ introduces one term, namely,

$$\frac{A}{3x - 1}$$

and by the quadratic factor rule, the factor $x^2 + 1$ introduces one term, namely,

$$\frac{Bx + C}{x^2 + 1}$$

Thus, the partial fraction decomposition is

$$\frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)} = \frac{A}{3x - 1} + \frac{Bx + C}{x^2 + 1} \tag{10}$$

Multiplying by $(3x - 1)(x^2 + 1)$ yields

$$x^2 + x - 2 = A(x^2 + 1) + (Bx + C)(3x - 1) \tag{11}$$

We could find A by substituting $x = \frac{1}{3}$ to make the last term drop out, and then find the rest of the constants by equating corresponding coefficients. However, in this case it is just as easy to find *all* of the constants by equating coefficients and solving the resulting system. For this purpose we multiply out the right side of (11) and collect like terms:

$$x^2 + x - 2 = (A + 3B)x^2 + (-B + 3C)x + (A - C)$$

Equating corresponding coefficients gives

$$\begin{aligned} A + 3B \qquad &= 1 \\ - B + 3C &= 1 \\ A \qquad - C &= -2 \end{aligned}$$

To solve this system, subtract the third equation from the first to eliminate A. Then use the resulting equation together with the second equation to solve for B and C. Finally, determine A from the first or third equation. This yields (verify)

$$A = -\frac{7}{5}, \quad B = \frac{4}{5}, \quad C = \frac{3}{5}$$

Thus, (10) becomes

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{-\frac{7}{5}}{3x-1} + \frac{\frac{4}{5}x+\frac{3}{5}}{x^2+1}$$

and

$$\begin{aligned}\int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx &= -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{dx}{x^2+1} \\ &= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln(x^2+1) + \frac{3}{5}\tan^{-1}x + C \end{aligned}$$ ◀

TECHNOLOGY MASTERY

Computer algebra systems have built-in capabilities for finding partial fraction decompositions. If you have a CAS, use it to find the decompositions in Examples 1, 2, and 3.

▶ **Example 4** Evaluate $\displaystyle\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx$.

Solution. Observe that the integrand is a proper rational function since the numerator has degree 4 and the denominator has degree 5. Thus, the method of partial fractions is applicable. By the linear factor rule, the factor $x+2$ introduces the single term

$$\frac{A}{x+2}$$

and by the quadratic factor rule, the factor $(x^2+3)^2$ introduces two terms (since $m=2$):

$$\frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

Thus, the partial fraction decomposition of the integrand is

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2} \tag{12}$$

Multiplying by $(x+2)(x^2+3)^2$ yields

$$\begin{aligned}&3x^4+4x^3+16x^2+20x+9 \\ &\qquad = A(x^2+3)^2 + (Bx+C)(x^2+3)(x+2) + (Dx+E)(x+2)\end{aligned} \tag{13}$$

which, after multiplying out and collecting like powers of x, becomes

$$\begin{aligned}&3x^4+4x^3+16x^2+20x+9 \\ &\qquad = (A+B)x^4 + (2B+C)x^3 + (6A+3B+2C+D)x^2 \\ &\qquad\qquad + (6B+3C+2D+E)x + (9A+6C+2E)\end{aligned} \tag{14}$$

Equating corresponding coefficients in (14) yields the following system of five linear equations in five unknowns:

$$\begin{aligned} A+B &= 3 \\ 2B+C &= 4 \\ 6A+3B+2C+D &= 16 \\ 6B+3C+2D+E &= 20 \\ 9A+6C+2E &= 9 \end{aligned} \tag{15}$$

Efficient methods for solving systems of linear equations such as this are studied in a branch of mathematics called ***linear algebra***; those methods are outside the scope of this text. However, as a practical matter most linear systems of any size are solved by computer, and most computer algebra systems have commands that in many cases can solve linear systems exactly. In this particular case we can simplify the work by first substituting $x=-2$

in (13), which yields $A = 1$. Substituting this known value of A in (15) yields the simpler system

$$\begin{aligned} B &= 2 \\ 2B + C &= 4 \\ 3B + 2C + D &= 10 \\ 6B + 3C + 2D + E &= 20 \\ 6C + 2E &= 0 \end{aligned} \tag{16}$$

This system can be solved by starting at the top and working down, first substituting $B = 2$ in the second equation to get $C = 0$, then substituting the known values of B and C in the third equation to get $D = 4$, and so forth. This yields

$$A = 1, \quad B = 2, \quad C = 0, \quad D = 4, \quad E = 0$$

Thus, (12) becomes

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

and so

$$\begin{aligned} \int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2}\, dx &= \int \frac{dx}{x+2} + \int \frac{2x}{x^2+3}\, dx + 4\int \frac{x}{(x^2+3)^2}\, dx \\ &= \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3} + C \end{aligned}$$ ◀

INTEGRATING IMPROPER RATIONAL FUNCTIONS

Although the method of partial fractions only applies to proper rational functions, an improper rational function can be integrated by performing a long division and expressing the function as the quotient plus the remainder over the divisor. The remainder over the divisor will be a proper rational function, which can then be decomposed into partial fractions. This idea is illustrated in the following example.

▶ **Example 5** Evaluate $\displaystyle\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\, dx$.

Solution. The integrand is an improper rational function since the numerator has degree 4 and the denominator has degree 2. Thus, we first perform the long division

$$\begin{array}{r|l} & \quad\; 3x^2 \qquad\quad + 1 \\ x^2 + x - 2 & 3x^4 + 3x^3 - 5x^2 + x - 1 \\ & \underline{3x^4 + 3x^3 - 6x^2} \\ & \qquad\qquad\quad\; x^2 + x - 1 \\ & \qquad\qquad\quad\; \underline{x^2 + x - 2} \\ & \qquad\qquad\qquad\qquad\;\; 1 \end{array}$$

It follows that the integrand can be expressed as

$$\frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} = (3x^2 + 1) + \frac{1}{x^2 + x - 2}$$

and hence

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\, dx = \int (3x^2 + 1)\, dx + \int \frac{dx}{x^2 + x - 2}$$

The second integral on the right now involves a proper rational function and can thus be evaluated by a partial fraction decomposition. Using the result of Example 1 we obtain

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\,dx = x^3 + x + \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C \blacktriangleleft$$

CONCLUDING REMARKS

There are some cases in which the method of partial fractions is inappropriate. For example, it would be inefficient to use partial fractions to perform the integration

$$\int \frac{3x^2 + 2}{x^3 + 2x - 8}\,dx = \ln|x^3 + 2x - 8| + C$$

since the substitution $u = x^3 + 2x - 8$ is more direct. Similarly, the integration

$$\int \frac{2x-1}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} = \ln(x^2+1) - \tan^{-1} x + C$$

requires only a little algebra since the integrand is already in partial fraction form.

✔QUICK CHECK EXERCISES 7.5 *(See page 523 for answers.)*

1. A partial fraction is a rational function of the form ________ or of the form ________.
2. (a) What is a proper rational function?
   (b) What condition must the degree of the numerator and the degree of the denominator of a rational function satisfy for the method of partial fractions to be applicable directly?
   (c) If the condition in part (b) is not satisfied, what must you do if you want to use partial fractions?
3. Suppose that the function $f(x) = P(x)/Q(x)$ is a proper rational function.
   (a) For each factor of $Q(x)$ of the form $(ax+b)^m$, the partial fraction decomposition of f contains the following sum of m partial fractions: ________
   (b) For each factor of $Q(x)$ of the form $(ax^2+bx+c)^m$, where ax^2+bx+c is an irreducible quadratic, the partial fraction decomposition of f contains the following sum of m partial fractions: ________
4. Complete the partial fraction decomposition.
   (a) $\dfrac{-3}{(x+1)(2x-1)} = \dfrac{A}{x+1} - \dfrac{2}{2x-1}$
   (b) $\dfrac{2x^2-3x}{(x^2+1)(3x+2)} = \dfrac{B}{3x+2} - \dfrac{1}{x^2+1}$
5. Evaluate the integral.
   (a) $\displaystyle\int \frac{3}{(x+1)(1-2x)}\,dx$ (b) $\displaystyle\int \frac{2x^2-3x}{(x^2+1)(3x+2)}\,dx$

EXERCISE SET 7.5 C CAS

1–8 Write out the form of the partial fraction decomposition. (Do not find the numerical values of the coefficients.)

1. $\dfrac{3x-1}{(x-3)(x+4)}$
2. $\dfrac{5}{x(x^2-4)}$
3. $\dfrac{2x-3}{x^3-x^2}$
4. $\dfrac{x^2}{(x+2)^3}$
5. $\dfrac{1-x^2}{x^3(x^2+2)}$
6. $\dfrac{3x}{(x-1)(x^2+6)}$
7. $\dfrac{4x^3-x}{(x^2+5)^2}$
8. $\dfrac{1-3x^4}{(x-2)(x^2+1)^2}$

9–34 Evaluate the integral.

9. $\displaystyle\int \frac{dx}{x^2-3x-4}$
10. $\displaystyle\int \frac{dx}{x^2-6x-7}$
11. $\displaystyle\int \frac{11x+17}{2x^2+7x-4}\,dx$
12. $\displaystyle\int \frac{5x-5}{3x^2-8x-3}\,dx$
13. $\displaystyle\int \frac{2x^2-9x-9}{x^3-9x}\,dx$
14. $\displaystyle\int \frac{dx}{x(x^2-1)}$
15. $\displaystyle\int \frac{x^2-8}{x+3}\,dx$
16. $\displaystyle\int \frac{x^2+1}{x-1}\,dx$
17. $\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx$
18. $\displaystyle\int \frac{x^2}{x^2-3x+2}\,dx$
19. $\displaystyle\int \frac{2x-3}{x^2-3x-10}\,dx$
20. $\displaystyle\int \frac{3x+1}{3x^2+2x-1}\,dx$
21. $\displaystyle\int \frac{x^5+x^2+2}{x^3-x}\,dx$
22. $\displaystyle\int \frac{x^5-4x^3+1}{x^3-4x}\,dx$
23. $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}\,dx$
24. $\displaystyle\int \frac{3x^2-x+1}{x^3-x^2}\,dx$

25. $\int \frac{2x^2 - 10x + 4}{(x+1)(x-3)^2}\,dx$

26. $\int \frac{2x^2 - 2x - 1}{x^3 - x^2}\,dx$

27. $\int \frac{x^2}{(x+1)^3}\,dx$

28. $\int \frac{2x^2 + 3x + 3}{(x+1)^3}\,dx$

29. $\int \frac{2x^2 - 1}{(4x-1)(x^2+1)}\,dx$

30. $\int \frac{dx}{x^3 + 2x}$

31. $\int \frac{x^3 + 3x^2 + x + 9}{(x^2+1)(x^2+3)}\,dx$

32. $\int \frac{x^3 + x^2 + x + 2}{(x^2+1)(x^2+2)}\,dx$

33. $\int \frac{x^3 - 2x^2 + 2x - 2}{x^2 + 1}\,dx$

34. $\int \frac{x^4 + 6x^3 + 10x^2 + x}{x^2 + 6x + 10}\,dx$

35–38 True–False Determine whether the statement is true or false. Explain your answer. ■

35. The technique of partial fractions is used for integrals whose integrands are ratios of polynomials.

36. The integrand in

$$\int \frac{3x^4 + 5}{(x^2+1)^2}\,dx$$

is a proper rational function.

37. The partial fraction decomposition of

$$\frac{2x+3}{x^2} \quad \text{is} \quad \frac{2}{x} + \frac{3}{x^2}$$

38. If $f(x) = P(x)/(x+5)^3$ is a proper rational function, then the partial fraction decomposition of $f(x)$ has terms with constant numerators and denominators $(x+5)$, $(x+5)^2$, and $(x+5)^3$.

39–42 Evaluate the integral by making a substitution that converts the integrand to a rational function. ■

39. $\int \frac{\cos\theta}{\sin^2\theta + 4\sin\theta - 5}\,d\theta$

40. $\int \frac{e^t}{e^{2t} - 4}\,dt$

41. $\int \frac{e^{3x}}{e^{2x} + 4}\,dx$

42. $\int \frac{5 + 2\ln x}{x(1 + \ln x)^2}\,dx$

43. Find the volume of the solid generated when the region enclosed by $y = x^2/(9 - x^2)$, $y = 0$, $x = 0$, and $x = 2$ is revolved about the x-axis.

44. Find the area of the region under the curve $y = 1/(1 + e^x)$, over the interval $[-\ln 5, \ln 5]$. [*Hint:* Make a substitution that converts the integrand to a rational function.]

C **45–46** Use a CAS to evaluate the integral in two ways: (i) integrate directly; (ii) use the CAS to find the partial fraction decomposition and integrate the decomposition. Integrate by hand to check the results. ■

45. $\int \frac{x^2 + 1}{(x^2 + 2x + 3)^2}\,dx$

46. $\int \frac{x^5 + x^4 + 4x^3 + 4x^2 + 4x + 4}{(x^2+2)^3}\,dx$

C **47–48** Integrate by hand and check your answers using a CAS. ■

47. $\int \frac{dx}{x^4 - 3x^3 - 7x^2 + 27x - 18}$

48. $\int \frac{dx}{16x^3 - 4x^2 + 4x - 1}$

FOCUS ON CONCEPTS

49. Show that

$$\int_0^1 \frac{x}{x^4 + 1}\,dx = \frac{\pi}{8}$$

50. Use partial fractions to derive the integration formula

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

51. Suppose that $ax^2 + bx + c$ is a quadratic polynomial and that the integration

$$\int \frac{1}{ax^2 + bx + c}\,dx$$

produces a function with no inverse tangent terms. What does this tell you about the roots of the polynomial?

52. Suppose that $ax^2 + bx + c$ is a quadratic polynomial and that the integration

$$\int \frac{1}{ax^2 + bx + c}\,dx$$

produces a function with neither logarithmic nor inverse tangent terms. What does this tell you about the roots of the polynomial?

53. Does there exist a quadratic polynomial $ax^2 + bx + c$ such that the integration

$$\int \frac{x}{ax^2 + bx + c}\,dx$$

produces a function with no logarithmic terms? If so, give an example; if not, explain why no such polynomial can exist.

54. Writing Suppose that $P(x)$ is a cubic polynomial. State the general form of the partial fraction decomposition for

$$f(x) = \frac{P(x)}{(x+5)^4}$$

and state the implications of this decomposition for evaluating the integral $\int f(x)\,dx$.

55. Writing Consider the functions

$$f(x) = \frac{1}{x^2 - 4} \quad \text{and} \quad g(x) = \frac{x}{x^2 - 4}$$

Each of the integrals $\int f(x)\,dx$ and $\int g(x)\,dx$ can be evaluated using partial fractions and using at least one other integration technique. Demonstrate two different techniques for evaluating each of these integrals, and then discuss the considerations that would determine which technique you would use.

✔ QUICK CHECK ANSWERS 7.5

1. $\dfrac{A}{(ax+b)^k}$; $\dfrac{Ax+B}{(ax^2+bx+c)^k}$ **2.** (a) A proper rational function is a rational function in which the degree of the numerator is less than the degree of the denominator. (b) The degree of the numerator must be less than the degree of the denominator. (c) Divide the denominator into the numerator, which results in the sum of a polynomial and a proper rational function.
3. (a) $\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_m}{(ax+b)^m}$ (b) $\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_mx+B_m}{(ax^2+bx+c)^m}$
4. (a) $A=1$ (b) $B=2$
5. (a) $\displaystyle\int \frac{3}{(x+1)(1-2x)}\,dx = \ln\left|\frac{x+1}{1-2x}\right| + C$ (b) $\displaystyle\int \frac{2x^2-3x}{(x^2+1)(3x+2)}\,dx = \frac{2}{3}\ln|3x+2| - \tan^{-1}x + C$

7.6 USING COMPUTER ALGEBRA SYSTEMS AND TABLES OF INTEGRALS

In this section we will discuss how to integrate using tables, and we will see some special substitutions to try when an integral doesn't match any of the forms in an integral table. In particular, we will discuss a method for integrating rational functions of $\sin x$ *and* $\cos x$. *We will also address some of the issues that relate to using computer algebra systems for integration. Readers who are not using computer algebra systems can skip that material.*

INTEGRAL TABLES

Tables of integrals are useful for eliminating tedious hand computation. The endpapers of this text contain a relatively brief table of integrals that we will refer to as the ***Endpaper Integral Table***; more comprehensive tables are published in standard reference books such as the *CRC Standard Mathematical Tables and Formulae*, CRC Press, Inc., 2002.

All integral tables have their own scheme for classifying integrals according to the form of the integrand. For example, the Endpaper Integral Table classifies the integrals into 15 categories; *Basic Functions*, *Reciprocals of Basic Functions*, *Powers of Trigonometric Functions*, *Products of Trigonometric Functions*, and so forth. The first step in working with tables is to read through the classifications so that you understand the classification scheme and know where to look in the table for integrals of different types.

PERFECT MATCHES

If you are lucky, the integral you are attempting to evaluate will match up perfectly with one of the forms in the table. However, when looking for matches you may have to make an adjustment for the variable of integration. For example, the integral

$$\int x^2 \sin x\, dx$$

is a perfect match with Formula (46) in the Endpaper Integral Table, except for the letter used for the variable of integration. Thus, to apply Formula (46) to the given integral we need to change the variable of integration in the formula from u to x. With that minor modification we obtain

$$\int x^2 \sin x\, dx = 2x\sin x + (2-x^2)\cos x + C$$

Here are some more examples of perfect matches.