- the graph of g_1 passes through the points (x_0, y_0), (x_1, y_1), and (x_2, y_2);
- the graph of g_2 passes through the points (x_2, y_2), (x_3, y_3), and (x_4, y_4);
- ...
- the graph of $g_{n/2}$ passes through the points (x_{n-2}, y_{n-2}), (x_{n-1}, y_{n-1}), and (x_n, y_n).

Verify that Formula (8) computes the area under a piecewise quadratic function by showing that

$$\sum_{j=1}^{n/2}\left(\int_{x_{2j-2}}^{x_{2j}} g_j(x)\,dx\right) = \frac{1}{3}\left(\frac{b-a}{n}\right)[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n]$$

55. Writing Discuss two different circumstances under which numerical integration is necessary.

56. Writing For the numerical integration methods of this section, better accuracy of an approximation was obtained by increasing the number of subdivisions of the interval. Another strategy is to use the same number of subintervals, but to select subintervals of differing lengths. Discuss a scheme for doing this to approximate $\int_0^4 \sqrt{x}\,dx$ using a trapezoidal approximation with 4 subintervals. Comment on the advantages and disadvantages of your scheme.

✔ QUICK CHECK ANSWERS 7.7

1. (a) $\frac{1}{2}(L_n + R_n)$ (b) $\left(\frac{b-a}{2n}\right)[y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n]$ **2.** $M_n < I < T_n$ **3.** (a) $\frac{2}{3}M_3 + \frac{1}{3}T_3$ (b) $\left(\frac{b-a}{18}\right)(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$ **4.** (a) $\frac{1}{2400}$ (b) $\frac{1}{1200}$ (c) $\frac{1}{1{,}800{,}000}$ **5.** (a) $M_1 = \frac{1}{2}$ (b) $T_1 = \frac{10}{9}$ (c) $S_2 = \frac{19}{27}$

7.8 IMPROPER INTEGRALS

Up to now we have focused on definite integrals with continuous integrands and finite intervals of integration. In this section we will extend the concept of a definite integral to include infinite intervals of integration and integrands that become infinite within the interval of integration.

IMPROPER INTEGRALS

It is assumed in the definition of the definite integral

$$\int_a^b f(x)\,dx$$

that $[a, b]$ is a finite interval and that the limit that defines the integral exists; that is, the function f is integrable. We observed in Theorems 5.5.2 and 5.5.8 that continuous functions are integrable, as are bounded functions with finitely many points of discontinuity. We also observed in Theorem 5.5.8 that functions that are not bounded on the interval of integration are not integrable. Thus, for example, a function with a vertical asymptote within the interval of integration would not be integrable.

Our main objective in this section is to extend the concept of a definite integral to allow for infinite intervals of integration and integrands with vertical asymptotes within the interval of integration. We will call the vertical asymptotes ***infinite discontinuities***, and we will call integrals with infinite intervals of integration or infinite discontinuities within the interval of integration ***improper integrals***. Here are some examples:

- Improper integrals with infinite intervals of integration:

$$\int_1^{+\infty} \frac{dx}{x^2}, \quad \int_{-\infty}^{0} e^x\, dx, \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

- Improper integrals with infinite discontinuities in the interval of integration:

$$\int_{-3}^{3} \frac{dx}{x^2}, \quad \int_1^2 \frac{dx}{x-1}, \quad \int_0^{\pi} \tan x\, dx$$

- Improper integrals with infinite discontinuities and infinite intervals of integration:

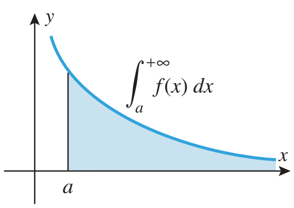
$$\int_0^{+\infty} \frac{dx}{\sqrt{x}}, \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9}, \quad \int_1^{+\infty} \sec x\, dx$$

INTEGRALS OVER INFINITE INTERVALS

To motivate a reasonable definition for improper integrals of the form

$$\int_a^{+\infty} f(x)\, dx$$

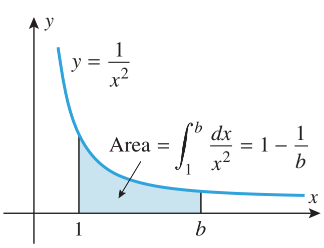
let us begin with the case where f is continuous and nonnegative on $[a, +\infty)$, so we can think of the integral as the area under the curve $y = f(x)$ over the interval $[a, +\infty)$ (Figure 7.8.1). At first, you might be inclined to argue that this area is infinite because the region has infinite extent. However, such an argument would be based on vague intuition rather than precise mathematical logic, since the concept of area has only been defined over intervals of *finite extent*. Thus, before we can make any reasonable statements about the area of the region in Figure 7.8.1, we need to begin by defining what we mean by the area of this region. For that purpose, it will help to focus on a specific example.



▲ **Figure 7.8.1**

Suppose we are interested in the area A of the region that lies below the curve $y = 1/x^2$ and above the interval $[1, +\infty)$ on the x-axis. Instead of trying to find the entire area at once, let us begin by calculating the portion of the area that lies above a finite interval $[1, b]$, where $b > 1$ is arbitrary. That area is

$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x}\bigg]_1^b = 1 - \frac{1}{b}$$



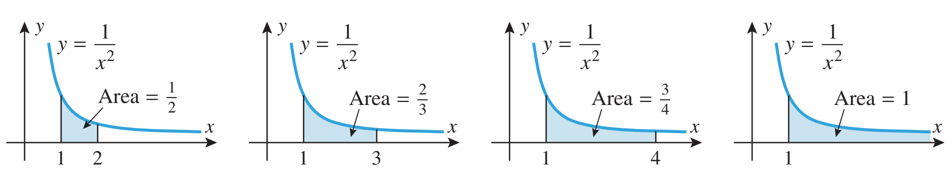
▲ **Figure 7.8.2**

(Figure 7.8.2). If we now allow b to increase so that $b \to +\infty$, then the portion of the area over the interval $[1, b]$ will begin to fill out the area over the entire interval $[1, +\infty)$ (Figure 7.8.3), and hence we can reasonably define the area A under $y = 1/x^2$ over the interval $[1, +\infty)$ to be

$$A = \int_1^{+\infty} \frac{dx}{x^2} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to+\infty} \left(1 - \frac{1}{b}\right) = 1 \tag{1}$$

Thus, the area has a finite value of 1 and is not infinite as we first conjectured.

▶ **Figure 7.8.3**



With the preceding discussion as our guide, we make the following definition (which is applicable to functions with both positive and negative values).

If f is nonnegative over the interval $[a, +\infty)$, then the improper integral in Definition 7.8.1 can be interpreted to be the area under the graph of f over the interval $[a, +\infty)$. If the integral converges, then the area is finite and equal to the value of the integral, and if the integral diverges, then the area is regarded to be infinite.

7.8.1 DEFINITION The ***improper integral of f over the interval*** $[a, +\infty)$ is defined to be

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx$$

In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

Example 1 Evaluate

$$\text{(a)} \int_1^{+\infty} \frac{dx}{x^3} \qquad \text{(b)} \int_1^{+\infty} \frac{dx}{x}$$

Solution (a). Following the definition, we replace the infinite upper limit by a finite upper limit b, and then take the limit of the resulting integral. This yields

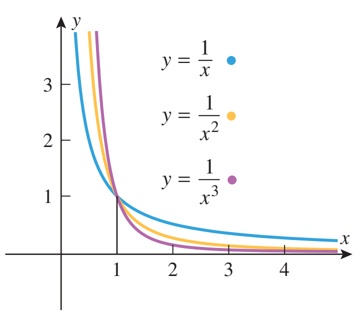
$$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x^3} = \lim_{b\to+\infty} \left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to+\infty} \left(\frac{1}{2} - \frac{1}{2b^2}\right) = \frac{1}{2}$$

Since the limit is finite, the integral converges and its value is $1/2$.

Solution (b).

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x} = \lim_{b\to+\infty} \left[\ln x\right]_1^b = \lim_{b\to+\infty} \ln b = +\infty$$

In this case the integral diverges and hence has no value. ◀



▲ **Figure 7.8.4**

Because the functions $1/x^3$, $1/x^2$, and $1/x$ are nonnegative over the interval $[1, +\infty)$, it follows from (1) and the last example that over this interval the area under $y = 1/x^3$ is $\frac{1}{2}$, the area under $y = 1/x^2$ is 1, and the area under $y = 1/x$ is infinite. However, on the surface the graphs of the three functions seem very much alike (Figure 7.8.4), and there is nothing to suggest why one of the areas should be infinite and the other two finite. One explanation is that $1/x^3$ and $1/x^2$ approach zero more rapidly than $1/x$ as $x \to +\infty$, so that the area over the interval $[1, b]$ accumulates less rapidly under the curves $y = 1/x^3$ and $y = 1/x^2$ than under $y = 1/x$ as $b \to +\infty$, and the difference is just enough that the first two areas are finite and the third is infinite.

Example 2 For what values of p does the integral $\displaystyle\int_1^{+\infty} \frac{dx}{x^p}$ converge?

Solution. We know from the preceding example that the integral diverges if $p = 1$, so let us assume that $p \neq 1$. In this case we have

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b\to+\infty} \int_1^b x^{-p}\,dx = \lim_{b\to+\infty} \left.\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to+\infty} \left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

If $p > 1$, then the exponent $1 - p$ is negative and $b^{1-p} \to 0$ as $b \to +\infty$; and if $p < 1$, then the exponent $1 - p$ is positive and $b^{1-p} \to +\infty$ as $b \to +\infty$. Thus, the integral converges if $p > 1$ and diverges otherwise. In the convergent case the value of the integral is

$$\int_1^{+\infty} \frac{dx}{x^p} = \left[0 - \frac{1}{1-p}\right] = \frac{1}{p-1} \qquad (p > 1) \text{ ◀}$$

The following theorem summarizes this result.

7.8.2 THEOREM

$$\int_1^{+\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \text{if} \quad p > 1 \\ \text{diverges} & \text{if} \quad p \le 1 \end{cases}$$

▶ **Example 3** Evaluate $\displaystyle\int_0^{+\infty} (1-x)e^{-x}\,dx$.

Solution. We begin by evaluating the indefinite integral using integration by parts. Setting $u = 1 - x$ and $dv = e^{-x}\,dx$ yields

$$\int (1-x)e^{-x}\,dx = -e^{-x}(1-x) - \int e^{-x}\,dx = -e^{-x} + xe^{-x} + e^{-x} + C = xe^{-x} + C$$

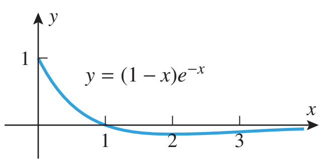
Thus,

$$\int_0^{+\infty} (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \int_0^b (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \Big[xe^{-x}\Big]_0^b = \lim_{b\to+\infty} \frac{b}{e^b}$$

The limit is an indeterminate form of type ∞/∞, so we will apply L'Hôpital's rule by differentiating the numerator and denominator with respect to b. This yields

$$\int_0^{+\infty} (1-x)e^{-x}\,dx = \lim_{b\to+\infty} \frac{1}{e^b} = 0$$

We can interpret this to mean that the net signed area between the graph of $y = (1-x)e^{-x}$ and the interval $[0, +\infty)$ is 0 (Figure 7.8.5). ◀



The net signed area between the graph and the interval $[0, +\infty)$ is zero.

▲ **Figure 7.8.5**

7.8.3 DEFINITION The ***improper integral of f over the interval*** $(-\infty, b]$ is defined to be

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx \tag{2}$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not.

The ***improper integral of f over the interval*** $(-\infty, +\infty)$ is defined as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{+\infty} f(x)\,dx \tag{3}$$

where c is any real number. The improper integral is said to ***converge*** if *both* terms converge and ***diverge*** if *either* term diverges.

If f is nonnegative over the interval $(-\infty, +\infty)$, then the improper integral

$$\int_{-\infty}^{+\infty} f(x)\,dx$$

can be interpreted to be the area under the graph of f over the interval $(-\infty, +\infty)$. The area is finite and equal to the value of the integral if the integral converges and is infinite if it diverges.

▶ **Example 4** Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

Solution. We will evaluate the integral by choosing $c = 0$ in (3). With this value for c we obtain

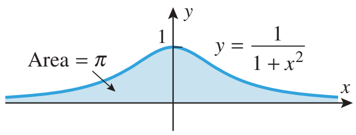
$$\int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to+\infty} \Big[\tan^{-1} x\Big]_0^b = \lim_{b\to+\infty} (\tan^{-1} b) = \frac{\pi}{2}$$

$$\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty} \Big[\tan^{-1} x\Big]_a^0 = \lim_{a\to-\infty} (-\tan^{-1} a) = \frac{\pi}{2}$$

Although we usually choose $c = 0$ in (3), the choice does not matter because it can be proved that neither the convergence nor the value of the integral is affected by the choice of c.

Thus, the integral converges and its value is

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$



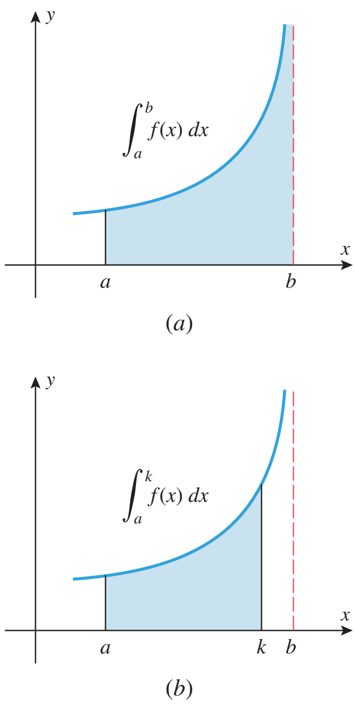
▲ Figure 7.8.6

Since the integrand is nonnegative on the interval $(-\infty, +\infty)$, the integral represents the area of the region shown in Figure 7.8.6. ◀

INTEGRALS WHOSE INTEGRANDS HAVE INFINITE DISCONTINUITIES

Next we will consider improper integrals whose integrands have infinite discontinuities. We will start with the case where the interval of integration is a finite interval $[a, b]$ and the infinite discontinuity occurs at the right-hand endpoint.

To motivate an appropriate definition for such an integral let us consider the case where f is nonnegative on $[a, b]$, so we can interpret the improper integral $\int_a^b f(x)\,dx$ as the area of the region in Figure 7.8.7*a*. The problem of finding the area of this region is complicated by the fact that it extends indefinitely in the positive y-direction. However, instead of trying to find the entire area at once, we can proceed indirectly by calculating the portion of the area over the interval $[a, k]$, where $a \le k < b$, and then letting k approach b to fill out the area of the entire region (Figure 7.8.7*b*). Motivated by this idea, we make the following definition.



▲ Figure 7.8.7

7.8.4 DEFINITION If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then the ***improper integral of f over the interval* [a, b]** is defined as

$$\int_a^b f(x)\,dx = \lim_{k \to b^-} \int_a^k f(x)\,dx \tag{4}$$

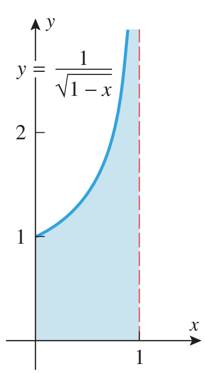
In the case where the indicated limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

▶ **Example 5** Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

Solution. The integral is improper because the integrand approaches $+\infty$ as x approaches the upper limit 1 from the left (Figure 7.8.8). From (4),

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^-} \int_0^k \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^-} \Big[-2\sqrt{1-x}\Big]_0^k$$

$$= \lim_{k \to 1^-} \Big[-2\sqrt{1-k} + 2\Big] = 2 \;◀$$



▲ Figure 7.8.8

Improper integrals with an infinite discontinuity at the left-hand endpoint or inside the interval of integration are defined as follows.

7.8.5 DEFINITION If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the ***improper integral of f over the interval* [a, b]** is defined as

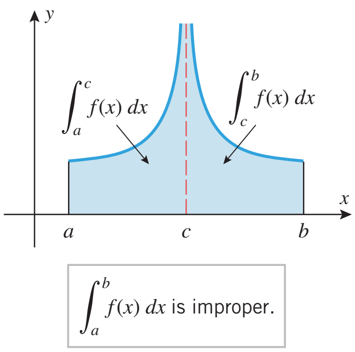
$$\int_a^b f(x)\,dx = \lim_{k\to a^+} \int_k^b f(x)\,dx \tag{5}$$

The integral is said to ***converge*** if the indicated limit exists and ***diverge*** if it does not.

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b), then the ***improper integral of f over the interval* [a, b]** is defined as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{6}$$

where the two integrals on the right side are themselves improper. The improper integral on the left side is said to ***converge*** if *both* terms on the right side converge and ***diverge*** if *either* term on the right side diverges (Figure 7.8.9).



▲ **Figure 7.8.9**

▶ **Example 6** Evaluate

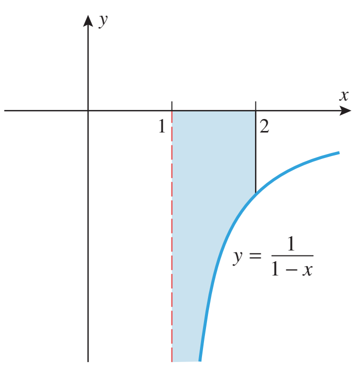
$$\text{(a)}\ \int_1^2 \frac{dx}{1-x} \qquad \text{(b)}\ \int_1^4 \frac{dx}{(x-2)^{2/3}}$$

Solution (a). The integral is improper because the integrand approaches $-\infty$ as x approaches the lower limit 1 from the right (Figure 7.8.10). From Definition 7.8.5 we obtain

$$\int_1^2 \frac{dx}{1-x} = \lim_{k\to 1^+} \int_k^2 \frac{dx}{1-x} = \lim_{k\to 1^+} \Big[-\ln|1-x|\Big]_k^2$$

$$= \lim_{k\to 1^+} \big[-\ln|-1| + \ln|1-k|\big] = \lim_{k\to 1^+} \ln|1-k| = -\infty$$

so the integral diverges.



▲ **Figure 7.8.10**

Solution (b). The integral is improper because the integrand approaches $+\infty$ at $x = 2$, which is inside the interval of integration. From Definition 7.8.5 we obtain

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}} \tag{7}$$

and we must investigate the convergence of both improper integrals on the right. Since

$$\int_1^2 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-} \int_1^k \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-} \Big[3(k-2)^{1/3} - 3(1-2)^{1/3}\Big] = 3$$

$$\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^+} \int_k^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^+} \Big[3(4-2)^{1/3} - 3(k-2)^{1/3}\Big] = 3\sqrt[3]{2}$$

we have from (7) that

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = 3 + 3\sqrt[3]{2} \ ◀$$

WARNING It is sometimes tempting to apply the Fundamental Theorem of Calculus directly to an improper integral without taking the appropriate limits. To illustrate what can go wrong with this procedure, suppose we fail to recognize that the integral

$$\int_0^2 \frac{dx}{(x-1)^2} \tag{8}$$

is improper and mistakenly evaluate this integral as

$$-\frac{1}{x-1}\bigg]_0^2 = -1 - (1) = -2$$

This result is clearly incorrect because the integrand is never negative and hence the integral cannot be negative! To evaluate (8) correctly we should first write

$$\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

and then treat each term as an improper integral. For the first term,

$$\int_0^1 \frac{dx}{(x-1)^2} = \lim_{k\to 1^-} \int_0^k \frac{dx}{(x-1)^2} = \lim_{k\to 1^-} \left[-\frac{1}{k-1} - 1\right] = +\infty$$

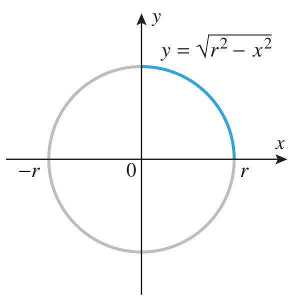
so (8) diverges.

ARC LENGTH AND SURFACE AREA USING IMPROPER INTEGRALS

In Definitions 6.4.2 and 6.5.2 for arc length and surface area we required the function f to be smooth (continuous first derivative) to ensure the integrability in the resulting formula. However, smoothness is overly restrictive since some of the most basic formulas in geometry involve functions that are not smooth but lead to convergent improper integrals. Accordingly, let us agree to extend the definitions of arc length and surface area to allow functions that are not smooth, but for which the resulting integral in the formula converges.

▶ **Example 7** Derive the formula for the circumference of a circle of radius r.

Solution. For convenience, let us assume that the circle is centered at the origin, in which case its equation is $x^2 + y^2 = r^2$. We will find the arc length of the portion of the circle that lies in the first quadrant and then multiply by 4 to obtain the total circumference (Figure 7.8.11).



▲ Figure 7.8.11

Since the equation of the upper semicircle is $y = \sqrt{r^2 - x^2}$, it follows from Formula (4) of Section 6.4 that the circumference C is

$$\begin{aligned} C &= 4\int_0^r \sqrt{1 + (dy/dx)^2}\, dx = 4\int_0^r \sqrt{1 + \left(-\frac{x}{\sqrt{r^2 - x^2}}\right)^2}\, dx \\ &= 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} \end{aligned}$$

This integral is improper because of the infinite discontinuity at $x = r$, and hence we evaluate it by writing

$$\begin{aligned} C &= 4r \lim_{k\to r^-} \int_0^k \frac{dx}{\sqrt{r^2 - x^2}} \\ &= 4r \lim_{k\to r^-} \left[\sin^{-1}\left(\frac{x}{r}\right)\right]_0^k \\ &= 4r \lim_{k\to r^-} \left[\sin^{-1}\left(\frac{k}{r}\right) - \sin^{-1} 0\right] \\ &= 4r[\sin^{-1} 1 - \sin^{-1} 0] = 4r\left(\frac{\pi}{2} - 0\right) = 2\pi r \end{aligned}$$ ◀

Formula (77) in the Endpaper Integral Table

✔QUICK CHECK EXERCISES 7.8

(See page 557 for answers.)

1. In each part, determine whether the integral is improper, and if so, explain why. Do not evaluate the integrals.

(a) $\int_{\pi/4}^{3\pi/4} \cot x\, dx$ (b) $\int_{\pi/4}^{\pi} \cot x\, dx$

(c) $\int_{0}^{+\infty} \frac{1}{x^2+1}\, dx$ (d) $\int_{1}^{+\infty} \frac{1}{x^2-1}\, dx$

2. Express each improper integral in Quick Check Exercise 1 in terms of one or more appropriate limits. Do not evaluate the limits.

3. The improper integral

$$\int_1^{+\infty} x^{-p}\, dx$$

converges to ________ provided ________.

4. Evaluate the integrals that converge.

(a) $\int_0^{+\infty} e^{-x}\, dx$ (b) $\int_0^{+\infty} e^{x}\, dx$

(c) $\int_0^{1} \frac{1}{x^3}\, dx$ (d) $\int_0^{1} \frac{1}{\sqrt[3]{x^2}}\, dx$

EXERCISE SET 7.8 Graphing Utility C CAS

1. In each part, determine whether the integral is improper, and if so, explain why.

(a) $\int_1^5 \frac{dx}{x-3}$ (b) $\int_1^5 \frac{dx}{x+3}$ (c) $\int_0^1 \ln x\, dx$

(d) $\int_1^{+\infty} e^{-x}\, dx$ (e) $\int_{-\infty}^{+\infty} \frac{dx}{\sqrt[3]{x-1}}$ (f) $\int_0^{\pi/4} \tan x\, dx$

2. In each part, determine all values of p for which the integral is improper.

(a) $\int_0^1 \frac{dx}{x^p}$ (b) $\int_1^2 \frac{dx}{x-p}$ (c) $\int_0^1 e^{-px}\, dx$

3–32 Evaluate the integrals that converge.

3. $\int_0^{+\infty} e^{-2x}\, dx$

4. $\int_{-1}^{+\infty} \frac{x}{1+x^2}\, dx$

5. $\int_3^{+\infty} \frac{2}{x^2-1}\, dx$

6. $\int_0^{+\infty} xe^{-x^2}\, dx$

7. $\int_e^{+\infty} \frac{1}{x\ln^3 x}\, dx$

8. $\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}}\, dx$

9. $\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$

10. $\int_{-\infty}^{3} \frac{dx}{x^2+9}$

11. $\int_{-\infty}^{0} e^{3x}\, dx$

12. $\int_{-\infty}^{0} \frac{e^x\, dx}{3-2e^x}$

13. $\int_{-\infty}^{+\infty} x\, dx$

14. $\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\, dx$

15. $\int_{-\infty}^{+\infty} \frac{x}{(x^2+3)^2}\, dx$

16. $\int_{-\infty}^{+\infty} \frac{e^{-t}}{1+e^{-2t}}\, dt$

17. $\int_0^4 \frac{dx}{(x-4)^2}$

18. $\int_0^8 \frac{dx}{\sqrt[3]{x}}$

19. $\int_0^{\pi/2} \tan x\, dx$

20. $\int_0^4 \frac{dx}{\sqrt{4-x}}$

21. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

22. $\int_{-3}^{1} \frac{x\, dx}{\sqrt{9-x^2}}$

23. $\int_{\pi/3}^{\pi/2} \frac{\sin x}{\sqrt{1-2\cos x}}\, dx$

24. $\int_0^{\pi/4} \frac{\sec^2 x}{1-\tan x}\, dx$

25. $\int_0^3 \frac{dx}{x-2}$

26. $\int_{-2}^{2} \frac{dx}{x^2}$

27. $\int_{-1}^{8} x^{-1/3}\, dx$

28. $\int_0^1 \frac{dx}{(x-1)^{2/3}}$

29. $\int_0^{+\infty} \frac{1}{x^2}\, dx$

30. $\int_1^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$

31. $\int_0^1 \frac{dx}{\sqrt{x}(x+1)}$

32. $\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)}$

33–36 True–False Determine whether the statement is true or false. Explain your answer.

33. $\int_1^{+\infty} x^{-4/3}\, dx$ converges to 3.

34. If f is continuous on $[a, +\infty]$ and $\lim_{x\to+\infty} f(x) = 1$, then $\int_a^{+\infty} f(x)\, dx$ converges.

35. $\int_1^2 \frac{1}{x(x-3)}\, dx$ is an improper integral.

36. $\int_{-1}^{1} \frac{1}{x^3}\, dx = 0$

37–40 Make the u-substitution and evaluate the resulting definite integral.

37. $\int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx;\ u = \sqrt{x}$ [*Note:* $u\to+\infty$ as $x\to+\infty$.]

38. $\int_{12}^{+\infty} \frac{dx}{\sqrt{x}(x+4)};\ u = \sqrt{x}$ [*Note:* $u\to+\infty$ as $x\to+\infty$.]

39. $\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-x}}}\, dx;\ u = 1-e^{-x}$
[*Note:* $u\to 1$ as $x\to+\infty$.]

40. $\displaystyle\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx;\ u = e^{-x}$

C **41–42** Express the improper integral as a limit, and then evaluate that limit with a CAS. Confirm the answer by evaluating the integral directly with the CAS. ■

41. $\displaystyle\int_0^{+\infty} e^{-x}\cos x\,dx$ **42.** $\displaystyle\int_0^{+\infty} xe^{-3x}\,dx$

C **43.** In each part, try to evaluate the integral exactly with a CAS. If your result is not a simple numerical answer, then use the CAS to find a numerical approximation of the integral.

(a) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^8+x+1}\,dx$ (b) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{1+x^3}}\,dx$

(c) $\displaystyle\int_1^{+\infty} \frac{\ln x}{e^x}\,dx$ (d) $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$

C **44.** In each part, confirm the result with a CAS.

(a) $\displaystyle\int_0^{+\infty} \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$ (b) $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$

(c) $\displaystyle\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}$

45. Find the length of the curve $y = (4 - x^{2/3})^{3/2}$ over the interval $[0, 8]$.

46. Find the length of the curve $y = \sqrt{4 - x^2}$ over the interval $[0, 2]$.

47–48 Use L'Hôpital's rule to help evaluate the improper integral. ■

47. $\displaystyle\int_0^1 \ln x\,dx$ **48.** $\displaystyle\int_1^{+\infty} \frac{\ln x}{x^2}\,dx$

49. Find the area of the region between the x-axis and the curve $y = e^{-3x}$ for $x \geq 0$.

50. Find the area of the region between the x-axis and the curve $y = 8/(x^2 - 4)$ for $x \geq 4$.

51. Suppose that the region between the x-axis and the curve $y = e^{-x}$ for $x \geq 0$ is revolved about the x-axis.
(a) Find the volume of the solid that is generated.
(b) Find the surface area of the solid.

FOCUS ON CONCEPTS

52. Suppose that f and g are continuous functions and that

$$0 \leq f(x) \leq g(x)$$

if $x \geq a$. Give a reasonable informal argument using areas to explain why the following results are true.
(a) If $\int_a^{+\infty} f(x)\,dx$ diverges, then $\int_a^{+\infty} g(x)\,dx$ diverges.
(b) If $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges and $\int_a^{+\infty} f(x)\,dx \leq \int_a^{+\infty} g(x)\,dx$.
[*Note:* The results in this exercise are sometimes called ***comparison tests*** for improper integrals.]

53–56 Use the results in Exercise 52. ■

53. (a) Confirm graphically and algebraically that

$$e^{-x^2} \leq e^{-x} \quad (x \geq 1)$$

(b) Evaluate the integral

$$\int_1^{+\infty} e^{-x}\,dx$$

(c) What does the result obtained in part (b) tell you about the integral

$$\int_1^{+\infty} e^{-x^2}\,dx?$$

54. (a) Confirm graphically and algebraically that

$$\frac{1}{2x+1} \leq \frac{e^x}{2x+1} \quad (x \geq 0)$$

(b) Evaluate the integral

$$\int_0^{+\infty} \frac{dx}{2x+1}$$

(c) What does the result obtained in part (b) tell you about the integral

$$\int_0^{+\infty} \frac{e^x}{2x+1}\,dx?$$

55. Let R be the region to the right of $x = 1$ that is bounded by the x-axis and the curve $y = 1/x$. When this region is revolved about the x-axis it generates a solid whose surface is known as ***Gabriel's Horn*** (for reasons that should be clear from the accompanying figure). Show that the solid has a finite volume but its surface has an infinite area. [*Note:* It has been suggested that if one could saturate the interior of the solid with paint and allow it to seep through to the surface, then one could paint an infinite surface with a finite amount of paint! What do you think?]

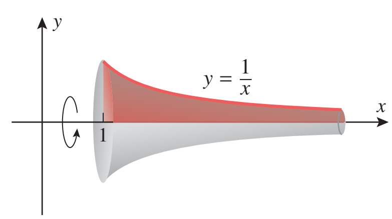


Figure Ex-55

56. In each part, use Exercise 52 to determine whether the integral converges or diverges. If it converges, then use part (b) of that exercise to find an upper bound on the value of the integral.

(a) $\displaystyle\int_2^{+\infty} \frac{\sqrt{x^3+1}}{x}\,dx$ (b) $\displaystyle\int_2^{+\infty} \frac{x}{x^5+1}\,dx$

(c) $\displaystyle\int_0^{+\infty} \frac{xe^x}{2x+1}\,dx$

FOCUS ON CONCEPTS

57. Sketch the region whose area is

$$\int_0^{+\infty} \frac{dx}{1+x^2}$$

and use your sketch to show that

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \int_0^1 \sqrt{\frac{1-y}{y}}\,dy$$

58. (a) Give a reasonable informal argument, based on areas, that explains why the integrals

$$\int_0^{+\infty} \sin x\,dx \quad \text{and} \quad \int_0^{+\infty} \cos x\,dx$$

diverge.

(b) Show that $\displaystyle\int_0^{+\infty} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ diverges.

59. In electromagnetic theory, the magnetic potential at a point on the axis of a circular coil is given by

$$u = \frac{2\pi NIr}{k}\int_a^{+\infty} \frac{dx}{(r^2+x^2)^{3/2}}$$

where N, I, r, k, and a are constants. Find u.

C **60.** The *average speed*, $\bar{v}$, of the molecules of an ideal gas is given by

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\int_0^{+\infty} v^3 e^{-Mv^2/(2RT)}\,dv$$

and the *root-mean-square speed*, v_{rms}, by

$$v_{\text{rms}}^2 = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\int_0^{+\infty} v^4 e^{-Mv^2/(2RT)}\,dv$$

where v is the molecular speed, T is the gas temperature, M is the molecular weight of the gas, and R is the gas constant.

(a) Use a CAS to show that

$$\int_0^{+\infty} x^3 e^{-a^2x^2}\,dx = \frac{1}{2a^4}, \quad a > 0$$

and use this result to show that $\bar{v} = \sqrt{8RT/(\pi M)}$.

(b) Use a CAS to show that

$$\int_0^{+\infty} x^4 e^{-a^2x^2}\,dx = \frac{3\sqrt{\pi}}{8a^5}, \quad a > 0$$

and use this result to show that $v_{\text{rms}} = \sqrt{3RT/M}$.

61. In Exercise 25 of Section 6.6, we determined the work required to lift a 6000 lb satellite to an orbital position that is 1000 mi above the Earth's surface. The ideas discussed in that exercise will be needed here.

(a) Find a definite integral that represents the work required to lift a 6000 lb satellite to a position b miles above the Earth's surface.

(b) Find a definite integral that represents the work required to lift a 6000 lb satellite an "infinite distance" above the Earth's surface. Evaluate the integral. [*Note:* The result obtained here is sometimes called the work required to "escape" the Earth's gravity.]

62–63 A ***transform*** is a formula that converts or "transforms" one function into another. Transforms are used in applications to convert a difficult problem into an easier problem whose solution can then be used to solve the original difficult problem. The ***Laplace transform*** of a function $f(t)$, which plays an important role in the study of differential equations, is denoted by $\mathcal{L}\{f(t)\}$ and is defined by

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\,dt$$

In this formula s is treated as a constant in the integration process; thus, the Laplace transform has the effect of transforming $f(t)$ into a function of s. Use this formula in these exercises. ■

62. Show that

(a) $\mathcal{L}\{1\} = \dfrac{1}{s},\ s > 0$ (b) $\mathcal{L}\{e^{2t}\} = \dfrac{1}{s-2},\ s > 2$

(c) $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1},\ s > 0$

(d) $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1},\ s > 0.$

63. In each part, find the Laplace transform.

(a) $f(t) = t,\ s > 0$ (b) $f(t) = t^2,\ s > 0$

(c) $f(t) = \begin{cases} 0, & t < 3 \\ 1, & t \ge 3 \end{cases},\ s > 0$

C **64.** Later in the text, we will show that

$$\int_0^{+\infty} e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}$$

Confirm that this is reasonable by using a CAS or a calculator with a numerical integration capability.

65. Use the result in Exercise 64 to show that

(a) $\displaystyle\int_{-\infty}^{+\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}},\ a > 0$

(b) $\displaystyle\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-x^2/2\sigma^2}\,dx = 1,\ \sigma > 0.$

66–67 A convergent improper integral over an infinite interval can be approximated by first replacing the infinite limit(s) of integration by finite limit(s), then using a numerical integration technique, such as Simpson's rule, to approximate the integral with finite limit(s). This technique is illustrated in these exercises. ■

66. Suppose that the integral in Exercise 64 is approximated by first writing it as

$$\int_0^{+\infty} e^{-x^2}\,dx = \int_0^K e^{-x^2}\,dx + \int_K^{+\infty} e^{-x^2}\,dx$$

then dropping the second term, and then applying Simpson's rule to the integral

$$\int_0^K e^{-x^2}\,dx$$

The resulting approximation has two sources of error: the error from Simpson's rule and the error

$$E = \int_K^{+\infty} e^{-x^2}\,dx$$

(cont.)

that results from discarding the second term. We call E the ***truncation error***.

(a) Approximate the integral in Exercise 64 by applying Simpson's rule with $n = 10$ subdivisions to the integral
$$\int_0^3 e^{-x^2}\,dx$$
Round your answer to four decimal places and compare it to $\frac{1}{2}\sqrt{\pi}$ rounded to four decimal places.

(b) Use the result that you obtained in Exercise 52 and the fact that $e^{-x^2} \le \frac{1}{3}xe^{-x^2}$ for $x \ge 3$ to show that the truncation error for the approximation in part (a) satisfies $0 < E < 2.1 \times 10^{-5}$.

67. (a) It can be shown that
$$\int_0^{+\infty} \frac{1}{x^6+1}\,dx = \frac{\pi}{3}$$
Approximate this integral by applying Simpson's rule with $n = 20$ subdivisions to the integral
$$\int_0^4 \frac{1}{x^6+1}\,dx$$
Round your answer to three decimal places and compare it to $\pi/3$ rounded to three decimal places.

(b) Use the result that you obtained in Exercise 52 and the fact that $1/(x^6+1) < 1/x^6$ for $x \ge 4$ to show that the truncation error for the approximation in part (a) satisfies $0 < E < 2 \times 10^{-4}$.

68. For what values of p does $\displaystyle\int_0^{+\infty} e^{px}\,dx$ converge?

69. Show that $\displaystyle\int_0^1 dx/x^p$ converges if $p < 1$ and diverges if $p \ge 1$.

C **70.** It is sometimes possible to convert an improper integral into a "proper" integral having the same value by making an appropriate substitution. Evaluate the following integral by making the indicated substitution, and investigate what happens if you evaluate the integral directly using a CAS.
$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx;\ u = \sqrt{1-x}$$

71–72 Transform the given improper integral into a proper integral by making the stated u-substitution; then approximate the proper integral by Simpson's rule with $n = 10$ subdivisions. Round your answer to three decimal places. ■

71. $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx;\ u = \sqrt{x}$

72. $\displaystyle\int_0^1 \frac{\sin x}{\sqrt{1-x}}\,dx;\ u = \sqrt{1-x}$

73. Writing What is "improper" about an integral over an infinite interval? Explain why Definition 5.5.1 for $\int_a^b f(x)\,dx$ fails for $\int_a^{+\infty} f(x)\,dx$. Discuss a strategy for assigning a value to $\int_a^{+\infty} f(x)\,dx$.

74. Writing What is "improper" about a definite integral over an interval on which the integrand has an infinite discontinuity? Explain why Definition 5.5.1 for $\int_a^b f(x)\,dx$ fails if the graph of f has a vertical asymptote at $x = a$. Discuss a strategy for assigning a value to $\int_a^b f(x)\,dx$ in this circumstance.

✔ QUICK CHECK ANSWERS 7.8

1. (a) proper (b) improper, since $\cot x$ has an infinite discontinuity at $x = \pi$ (c) improper, since there is an infinite interval of integration (d) improper, since there is an infinite interval of integration and the integrand has an infinite discontinuity at $x = 1$

2. (b) $\displaystyle\lim_{b\to\pi^-}\int_{\pi/4}^b \cot x\,dx$ (c) $\displaystyle\lim_{b\to+\infty}\int_0^b \frac{1}{x^2+1}\,dx$ (d) $\displaystyle\lim_{a\to 1^+}\int_a^2 \frac{1}{x^2-1}\,dx + \lim_{b\to+\infty}\int_2^b \frac{1}{x^2-1}\,dx$ **3.** $\dfrac{1}{p-1};\ p > 1$

4. (a) 1 (b) diverges (c) diverges (d) 3

CHAPTER 7 REVIEW EXERCISES

1–6 Evaluate the given integral with the aid of an appropriate u-substitution. ■

1. $\displaystyle\int \sqrt{4+9x}\,dx$

2. $\displaystyle\int \frac{1}{\sec \pi x}\,dx$

3. $\displaystyle\int \sqrt{\cos x}\sin x\,dx$

4. $\displaystyle\int \frac{dx}{x\ln x}$

5. $\displaystyle\int x\tan^2(x^2)\sec^2(x^2)\,dx$

6. $\displaystyle\int_0^9 \frac{\sqrt{x}}{x+9}\,dx$

7. (a) Evaluate the integral
$$\int \frac{1}{\sqrt{2x-x^2}}\,dx$$
three ways: using the substitution $u = \sqrt{x}$, using the substitution $u = \sqrt{2-x}$, and completing the square.

(b) Show that the answers in part (a) are equivalent.

8. Evaluate the integral $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$
(a) using integration by parts
(b) using the substitution $u = \sqrt{x^2+1}$.

9–12 Use integration by parts to evaluate the integral. ■

9. $\int xe^{-x}\,dx$ **10.** $\int x \sin 2x\,dx$

11. $\int \ln(2x+3)\,dx$ **12.** $\int_0^{1/2} \tan^{-1}(2x)\,dx$

13. Evaluate $\int 8x^4 \cos 2x\,dx$ using tabular integration by parts.

14. A particle moving along the x-axis has velocity function $v(t) = t^2 e^{-t}$. How far does the particle travel from time $t = 0$ to $t = 5$?

15–20 Evaluate the integral. ■

15. $\int \sin^2 5\theta\,d\theta$ **16.** $\int \sin^3 2x \cos^2 2x\,dx$

17. $\int \sin x \cos 2x\,dx$ **18.** $\int_0^{\pi/6} \sin 2x \cos 4x\,dx$

19. $\int \sin^4 2x\,dx$ **20.** $\int x \cos^5(x^2)\,dx$

21–26 Evaluate the integral by making an appropriate trigonometric substitution. ■

21. $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ **22.** $\int \frac{dx}{x^2\sqrt{16-x^2}}$

23. $\int \frac{dx}{\sqrt{x^2-1}}$ **24.** $\int \frac{x^2}{\sqrt{x^2-25}}\,dx$

25. $\int \frac{x^2}{\sqrt{9+x^2}}\,dx$ **26.** $\int \frac{\sqrt{1+4x^2}}{x}\,dx$

27–32 Evaluate the integral using the method of partial fractions. ■

27. $\int \frac{dx}{x^2+3x-4}$ **28.** $\int \frac{dx}{x^2+8x+7}$

29. $\int \frac{x^2+2}{x+2}\,dx$ **30.** $\int \frac{x^2+x-16}{(x-1)(x-3)^2}\,dx$

31. $\int \frac{x^2}{(x+2)^3}\,dx$ **32.** $\int \frac{dx}{x^3+x}$

33. Consider the integral $\int \frac{1}{x^3-x}\,dx$.
(a) Evaluate the integral using the substitution $x = \sec\theta$. For what values of x is your result valid?
(b) Evaluate the integral using the substitution $x = \sin\theta$. For what values of x is your result valid?
(c) Evaluate the integral using the method of partial fractions. For what values of x is your result valid?

34. Find the area of the region that is enclosed by the curves $y = (x-3)/(x^3+x^2)$, $y = 0$, $x = 1$, and $x = 2$.

35–40 Use the Endpaper Integral Table to evaluate the integral. ■

35. $\int \sin 7x \cos 9x\,dx$ **36.** $\int (x^3 - x^2)e^{-x}\,dx$

37. $\int x\sqrt{x-x^2}\,dx$ **38.** $\int \frac{dx}{x\sqrt{4x+3}}$

39. $\int \tan^2 2x\,dx$ **40.** $\int \frac{3x-1}{2+x^2}\,dx$

41–42 Approximate the integral using (a) the midpoint approximation M_{10}, (b) the trapezoidal approximation T_{10}, and (c) Simpson's rule approximation S_{20}. In each case, find the exact value of the integral and approximate the absolute error. Express your answers to at least four decimal places. ■

41. $\int_1^3 \frac{1}{\sqrt{x+1}}\,dx$ **42.** $\int_{-1}^1 \frac{1}{1+x^2}\,dx$

43–44 Use inequalities (12), (13), and (14) of Section 7.7 to find upper bounds on the errors in parts (a), (b), or (c) of the indicated exercise. ■

43. Exercise 41 **44.** Exercise 42

45–46 Use inequalities (12), (13), and (14) of Section 7.7 to find a number n of subintervals for (a) the midpoint approximation M_n, (b) the trapezoidal approximation T_n, and (c) Simpson's rule approximation S_n to ensure the absolute error will be less than 10^{-4}. ■

45. Exercise 41 **46.** Exercise 42

47–50 Evaluate the integral if it converges. ■

47. $\int_0^{+\infty} e^{-x}\,dx$ **48.** $\int_{-\infty}^2 \frac{dx}{x^2+4}$

49. $\int_0^9 \frac{dx}{\sqrt{9-x}}$ **50.** $\int_0^1 \frac{1}{2x-1}\,dx$

51. Find the area that is enclosed between the x-axis and the curve $y = (\ln x - 1)/x^2$ for $x \ge e$.

52. Find the volume of the solid that is generated when the region between the x-axis and the curve $y = e^{-x}$ for $x \ge 0$ is revolved about the y-axis.

53. Find a positive value of a that satisfies the equation

$$\int_0^{+\infty} \frac{1}{x^2+a^2}\,dx = 1$$

54. Consider the following methods for evaluating integrals: u-substitution, integration by parts, partial fractions, reduction formulas, and trigonometric substitutions. In each part, state the approach that you would try first to evaluate the integral. If none of them seems appropriate, then say so. You need not evaluate the integral.

(a) $\int x \sin x\,dx$ (b) $\int \cos x \sin x\,dx$

(cont.)

(c) $\int \tan^7 x\,dx$

(d) $\int \tan^7 x \sec^2 x\,dx$

(e) $\int \frac{3x^2}{x^3+1}\,dx$

(f) $\int \frac{3x^2}{(x+1)^3}\,dx$

(g) $\int \tan^{-1} x\,dx$

(h) $\int \sqrt{4-x^2}\,dx$

(i) $\int x\sqrt{4-x^2}\,dx$

55–74 Evaluate the integral. ■

55. $\int \frac{dx}{(3+x^2)^{3/2}}$

56. $\int x\cos 3x\,dx$

57. $\int_0^{\pi/4} \tan^7 \theta\,d\theta$

58. $\int \frac{\cos\theta}{\sin^2\theta - 6\sin\theta + 12}\,d\theta$

59. $\int \sin^2 2x \cos^3 2x\,dx$

60. $\int_0^4 \frac{1}{(x-3)^2}\,dx$

61. $\int e^{2x}\cos 3x\,dx$

62. $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\,dx$

63. $\int \frac{dx}{(x-1)(x+2)(x-3)}$

64. $\int_0^{1/3} \frac{dx}{(4-9x^2)^2}$

65. $\int_4^8 \frac{\sqrt{x-4}}{x}\,dx$

66. $\int_0^{\ln 2} \sqrt{e^x-1}\,dx$

67. $\int \frac{1}{\sqrt{e^x+1}}\,dx$

68. $\int \frac{dx}{x(x^2+x+1)}$

69. $\int_0^{1/2} \sin^{-1} x\,dx$

70. $\int \tan^5 4x \sec^4 4x\,dx$

71. $\int \frac{x+3}{\sqrt{x^2+2x+2}}\,dx$

72. $\int \frac{\sec^2\theta}{\tan^3\theta - \tan^2\theta}\,d\theta$

73. $\int_a^{+\infty} \frac{x}{(x^2+1)^2}\,dx$

74. $\int_0^{+\infty} \frac{dx}{a^2+b^2x^2}, \quad a, b > 0$

CHAPTER 7 MAKING CONNECTIONS

C CAS

1. Recall from Theorem 3.3.1 and the discussion preceding it that if $f'(x) > 0$, then the function f is increasing and has an inverse function. Parts (a), (b), and (c) of this problem show that if this condition is satisfied and if f' is continuous, then a definite integral of f^{-1} can be expressed in terms of a definite integral of f.

(a) Use integration by parts to show that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_a^b xf'(x)\,dx$$

(b) Use the result in part (a) to show that if $y = f(x)$, then

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$

(c) Show that if we let $\alpha = f(a)$ and $\beta = f(b)$, then the result in part (b) can be written as

$$\int_\alpha^\beta f^{-1}(x)\,dx = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha) - \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x)\,dx$$

2. In each part, use the result in Exercise 1 to obtain the equation, and then confirm that the equation is correct by performing the integrations.

(a) $\int_0^{1/2} \sin^{-1} x\,dx = \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \int_0^{\pi/6} \sin x\,dx$

(b) $\int_e^{e^2} \ln x\,dx = (2e^2 - e) - \int_1^2 e^x\,dx$

3. The ***Gamma function***, $\Gamma(x)$, is defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,dt$$

It can be shown that this improper integral converges if and only if $x > 0$.

(a) Find $\Gamma(1)$.

(b) Prove: $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. [*Hint:* Use integration by parts.]

(c) Use the results in parts (a) and (b) to find $\Gamma(2)$, $\Gamma(3)$, and $\Gamma(4)$; and then make a conjecture about $\Gamma(n)$ for positive integer values of n.

(d) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. [*Hint:* See Exercise 64 of Section 7.8.]

(e) Use the results obtained in parts (b) and (d) to show that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$.

4. Refer to the Gamma function defined in Exercise 3 to show that

(a) $\int_0^1 (\ln x)^n\,dx = (-1)^n\Gamma(n+1), \quad n > 0$

[*Hint:* Let $t = -\ln x$.]

(b) $\int_0^{+\infty} e^{-x^n}\,dx = \Gamma\left(\frac{n+1}{n}\right), \quad n > 0.$

[*Hint:* Let $t = x^n$. Use the result in Exercise 3(b).]

C **5.** A ***simple pendulum*** consists of a mass that swings in a vertical plane at the end of a massless rod of length L, as shown in the accompanying figure. Suppose that a simple pendulum is displaced through an angle θ_0 and released from rest. It can be shown that in the absence of friction, the time T required for the pendulum to make one complete back-and-forth swing, called the ***period***, is given by

$$T = \sqrt{\frac{8L}{g}} \int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\, d\theta \qquad (1)$$

where $\theta = \theta(t)$ is the angle the pendulum makes with the vertical at time t. The improper integral in (1) is difficult to evaluate numerically. By a substitution outlined below it can be shown that the period can be expressed as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2\phi}}\, d\phi \qquad (2)$$

where $k = \sin(\theta_0/2)$. The integral in (2) is called a ***complete elliptic integral of the first kind*** and is more easily evaluated by numerical methods.

(a) Obtain (2) from (1) by substituting

$$\cos\theta = 1 - 2\sin^2(\theta/2)$$
$$\cos\theta_0 = 1 - 2\sin^2(\theta_0/2)$$
$$k = \sin(\theta_0/2)$$

and then making the change of variable

$$\sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} = \frac{\sin(\theta/2)}{k}$$

(b) Use (2) and the numerical integration capability of your CAS to estimate the period of a simple pendulum for which $L = 1.5$ ft, $\theta_0 = 20°$, and $g = 32$ ft/s^2.

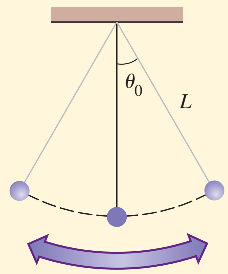


Figure Ex-5

EXPANDING THE CALCULUS HORIZON

To learn how numerical integration can be applied to the cost analysis of an engineering project, see the module entitled **Railroad Design** at:

www.wiley.com/college/anton