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4

THE DERIVATIVE IN GRAPHING AND APPLICATIONS

Derivatives can help to find the most cost-effective location for an offshore oil-drilling rig.

In this chapter we will study various applications of the derivative. For example, we will use methods of calculus to analyze functions and their graphs. In the process, we will show how calculus and graphing utilities, working together, can provide most of the important information about the behavior of functions. Another important application of the derivative will be in the solution of optimization problems. For example, if time is the main consideration in a problem, we might be interested in finding the quickest way to perform a task, and if cost is the main consideration, we might be interested in finding the least expensive way to perform a task. Mathematically, optimization problems can be reduced to finding the largest or smallest value of a function on some interval, and determining where the largest or smallest value occurs. Using the derivative, we will develop the mathematical tools necessary for solving such problems. We will also use the derivative to study the motion of a particle moving along a line, and we will show how the derivative can help us to approximate solutions of equations.

4.1 ANALYSIS OF FUNCTIONS I: INCREASE, DECREASE, AND CONCAVITY

Although graphing utilities are useful for determining the general shape of a graph, many problems require more precision than graphing utilities are capable of producing. The purpose of this section is to develop mathematical tools that can be used to determine the exact shape of a graph and the precise locations of its key features.

INCREASING AND DECREASING FUNCTIONS

The terms *increasing*, *decreasing*, and *constant* are used to describe the behavior of a function as we travel left to right along its graph. For example, the function graphed in Figure 4.1.1 can be described as increasing to the left of $x = 0$, decreasing from $x = 0$ to $x = 2$, increasing from $x = 2$ to $x = 4$, and constant to the right of $x = 4$.

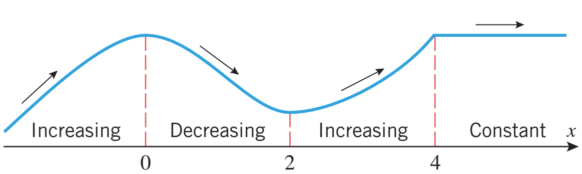


Figure 4.1.1

The following definition, which is illustrated in Figure 4.1.2, expresses these intuitive ideas precisely.

The definitions of "increasing," "decreasing," and "constant" describe the behavior of a function on an *interval* and not at a point. In particular, it is not inconsistent to say that the function in Figure 4.1.1 is decreasing on the interval $[0, 2]$ and increasing on the interval $[2, 4]$.

4.1.1 DEFINITION Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is ***increasing*** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(b) f is ***decreasing*** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

(c) f is ***constant*** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.

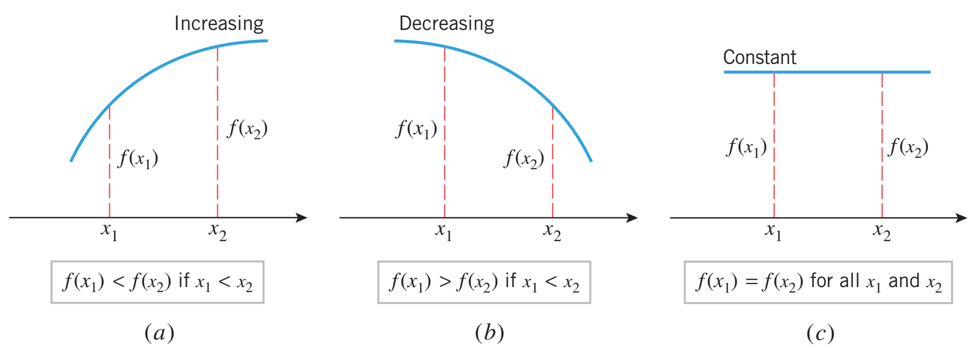


Figure 4.1.2

Figure 4.1.3 suggests that a differentiable function f is increasing on any interval where each tangent line to its graph has positive slope, is decreasing on any interval where each tangent line to its graph has negative slope, and is constant on any interval where each tangent line to its graph has zero slope. This intuitive observation suggests the following important theorem that will be proved in Section 4.8.

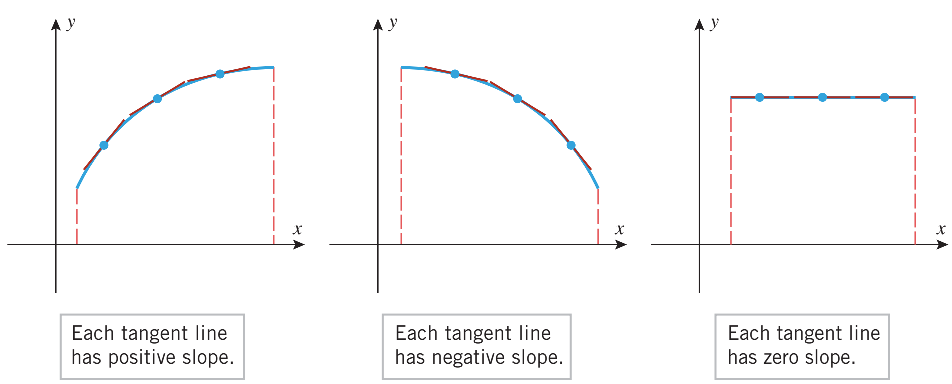


Figure 4.1.3

4.1.2 THEOREM *Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).*

(a) If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.

(c) If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

Observe that the derivative conditions in Theorem 4.1.2 are only required to hold *inside* the interval $[a, b]$, even though the conclusions apply to the entire interval.

Although stated for closed intervals, Theorem 4.1.2 is applicable on any interval on which f is continuous. For example, if f is continuous on $[a, +\infty)$ and $f'(x) > 0$ on $(a, +\infty)$, then f is increasing on $[a, +\infty)$; and if f is continuous on $(-\infty, +\infty)$ and $f'(x) < 0$ on $(-\infty, +\infty)$, then f is decreasing on $(-\infty, +\infty)$.

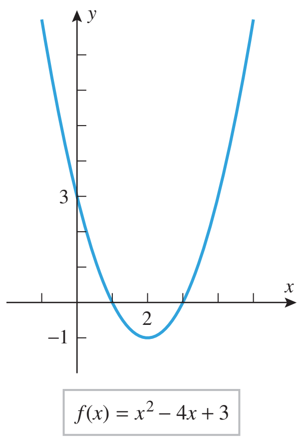


Figure 4.1.4

Example 1 Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals on which it is decreasing.

Solution. The graph of f in Figure 4.1.4 suggests that f is decreasing for $x \leq 2$ and increasing for $x \geq 2$. To confirm this, we analyze the sign of f'. The derivative of f is

$$f'(x) = 2x - 4 = 2(x - 2)$$

It follows that

$$\begin{aligned} f'(x) < 0 \quad &\text{if} \quad x < 2 \\ f'(x) > 0 \quad &\text{if} \quad 2 < x \end{aligned}$$

Since f is continuous everywhere, it follows from the comment after Theorem 4.1.2 that

$$\begin{aligned} &f \text{ is decreasing on } (-\infty, 2] \\ &f \text{ is increasing on } [2, +\infty) \end{aligned}$$

These conclusions are consistent with the graph of f in Figure 4.1.4. ◀

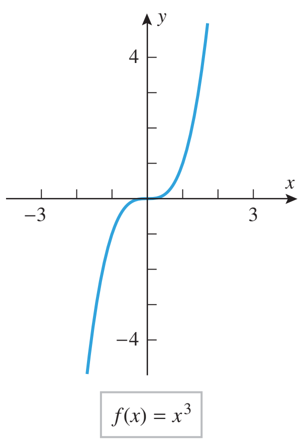


Figure 4.1.5

Example 2 Find the intervals on which $f(x) = x^3$ is increasing and the intervals on which it is decreasing.

Solution. The graph of f in Figure 4.1.5 suggests that f is increasing over the entire x-axis. To confirm this, we differentiate f to obtain $f'(x) = 3x^2$. Thus,

$$\begin{aligned} f'(x) > 0 \quad &\text{if} \quad x < 0 \\ f'(x) > 0 \quad &\text{if} \quad 0 < x \end{aligned}$$

Since f is continuous everywhere,

$$\begin{aligned} &f \text{ is increasing on } (-\infty, 0] \\ &f \text{ is increasing on } [0, +\infty) \end{aligned}$$

Since f is increasing on the adjacent intervals $(-\infty, 0]$ and $[0, +\infty)$, it follows that f is increasing on their union $(-\infty, +\infty)$ (see Exercise 59). ◀

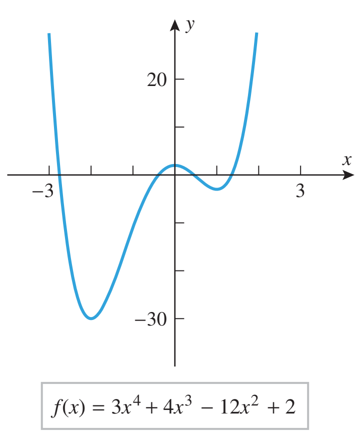


Figure 4.1.6

Example 3

(a) Use the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ in Figure 4.1.6 to make a conjecture about the intervals on which f is increasing or decreasing.

(b) Use Theorem 4.1.2 to determine whether your conjecture is correct.

Solution (a). The graph suggests that the function f is decreasing if $x \leq -2$, increasing if $-2 \leq x \leq 0$, decreasing if $0 \leq x \leq 1$, and increasing if $x \geq 1$.

Solution (b). Differentiating f we obtain

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x + 2)(x - 1)$$

The sign analysis of f' in Table 4.1.1 can be obtained using the method of test points discussed in Web Appendix E. The conclusions in Table 4.1.1 confirm the conjecture in part (a). ◀

Table 4.1.1

INTERVAL	$(12x)(x+2)(x-1)$	$f'(x)$	CONCLUSION
$x < -2$	$(-)(-)(-)$	$-$	f is decreasing on $(-\infty, -2]$
$-2 < x < 0$	$(-)(+)(-)$	$+$	f is increasing on $[-2, 0]$
$0 < x < 1$	$(+)(+)(-)$	$-$	f is decreasing on $[0, 1]$
$1 < x$	$(+)(+)(+)$	$+$	f is increasing on $[1, +\infty)$

CONCAVITY

Although the sign of the derivative of f reveals where the graph of f is increasing or decreasing, it does not reveal the direction of *curvature*. For example, the graph is increasing on both sides of the point in Figure 4.1.7, but on the left side it has an upward curvature ("holds water") and on the right side it has a downward curvature ("spills water"). On intervals where the graph of f has upward curvature we say that f is *concave up*, and on intervals where the graph has downward curvature we say that f is *concave down*.

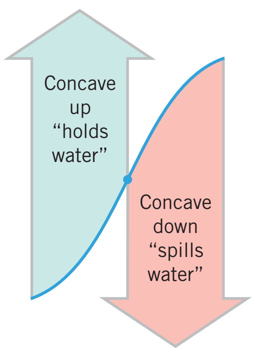


Figure 4.1.7

Figure 4.1.8 suggests two ways to characterize the concavity of a differentiable function f on an open interval:

- f is concave up on an open interval if its tangent lines have increasing slopes on that interval and is concave down if they have decreasing slopes.
- f is concave up on an open interval if its graph lies above its tangent lines on that interval and is concave down if it lies below its tangent lines.

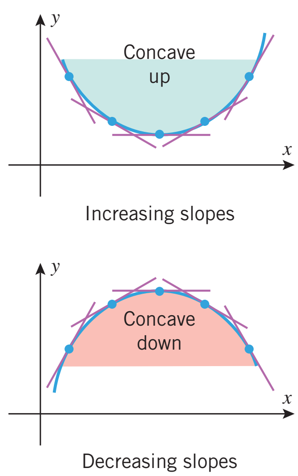


Figure 4.1.8

Our formal definition for "concave up" and "concave down" corresponds to the first of these characterizations.

4.1.3 DEFINITION If f is differentiable on an open interval, then f is said to be ***concave up*** on the open interval if f' is increasing on that interval, and f is said to be ***concave down*** on the open interval if f' is decreasing on that interval.

Since the slopes of the tangent lines to the graph of a differentiable function f are the values of its derivative f', it follows from Theorem 4.1.2 (applied to f' rather than f) that f' will be increasing on intervals where f'' is positive and that f' will be decreasing on intervals where f'' is negative. Thus, we have the following theorem.

4.1.4 THEOREM *Let f be twice differentiable on an open interval.*

(*a*) *If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.*

(*b*) *If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.*

Example 4 Figure 4.1.4 suggests that the function $f(x) = x^2 - 4x + 3$ is concave up on the interval $(-\infty, +\infty)$. This is consistent with Theorem 4.1.4, since $f'(x) = 2x - 4$ and $f''(x) = 2$, so

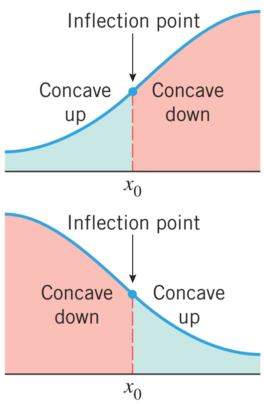
$$f''(x) > 0 \quad \text{on the interval } (-\infty, +\infty)$$

Also, Figure 4.1.5 suggests that $f(x) = x^3$ is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, +\infty)$. This agrees with Theorem 4.1.4, since $f'(x) = 3x^2$ and $f''(x) = 6x$, so

$$f''(x) < 0 \quad \text{if } x < 0 \quad \text{and} \quad f''(x) > 0 \quad \text{if } x > 0 \blacktriangleleft$$

INFLECTION POINTS

We see from Example 4 and Figure 4.1.5 that the graph of $f(x) = x^3$ changes from concave down to concave up at $x = 0$. Points where a curve changes from concave up to concave down or vice versa are of special interest, so there is some terminology associated with them.



▶ **Figure 4.1.9**

4.1.5 DEFINITION If f is continuous on an open interval containing a value x_0, and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an ***inflection point at*** $\boldsymbol{x_0}$, and we call the point $(x_0, f(x_0))$ on the graph of f an ***inflection point*** of f (Figure 4.1.9).

▶ **Example 5** Figure 4.1.10 shows the graph of the function $f(x) = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.

Solution. Calculating the first two derivatives of f we obtain

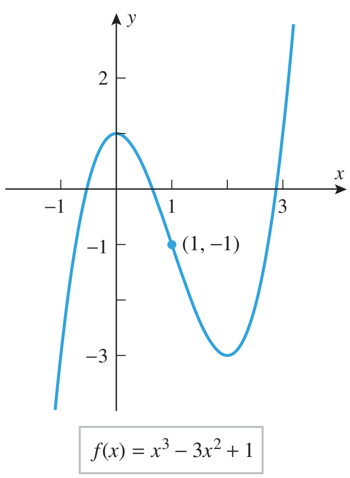
$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

$$f''(x) = 6x - 6 = 6(x - 1)$$

The sign analysis of these derivatives is shown in the following tables:

INTERVAL	$(3x)(x-2)$	$f'(x)$	CONCLUSION
$x < 0$	$(-)(-)$	$+$	f is increasing on $(-\infty, 0]$
$0 < x < 2$	$(+)(-)$	$-$	f is decreasing on $[0, 2]$
$x > 2$	$(+)(+)$	$+$	f is increasing on $[2, +\infty)$

INTERVAL	$6(x-1)$	$f''(x)$	CONCLUSION
$x < 1$	$(-)$	$-$	f is concave down on $(-\infty, 1)$
$x > 1$	$(+)$	$+$	f is concave up on $(1, +\infty)$



▶ **Figure 4.1.10**

The second table shows that there is an inflection point at $x = 1$, since f changes from concave down to concave up at that point. The inflection point is $(1, f(1)) = (1, -1)$. All of these conclusions are consistent with the graph of f. ◀

One can correctly guess from Figure 4.1.10 that the function $f(x) = x^3 - 3x^2 + 1$ has an inflection point at $x = 1$ without actually computing derivatives. However, sometimes changes in concavity are so subtle that calculus is essential to confirm their existence and identify their location. Here is an example.

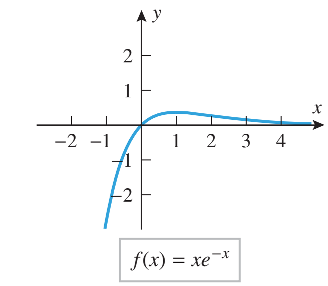


Figure 4.1.11

Example 6 Figure 4.1.11 suggests that the function $f(x) = xe^{-x}$ has an inflection point but its exact location is not evident from the graph in this figure. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down. Locate all inflection points.

Solution. Calculating the first two derivatives of f we obtain (verify)

$$f'(x) = (1 - x)e^{-x}$$
$$f''(x) = (x - 2)e^{-x}$$

Keeping in mind that e^{-x} is positive for all x, the sign analysis of these derivatives is easily determined:

INTERVAL	$(1-x)(e^{-x})$	$f'(x)$	CONCLUSION
$x < 1$	$(+)(+)$	$+$	f is increasing on $(-\infty, 1]$
$x > 1$	$(-)(+)$	$-$	f is decreasing on $[1, +\infty)$

INTERVAL	$(x-2)(e^{-x})$	$f''(x)$	CONCLUSION
$x < 2$	$(-)(+)$	$-$	f is concave down on $(-\infty, 2)$
$x > 2$	$(+)(+)$	$+$	f is concave up on $(2, +\infty)$

The second table shows that there is an inflection point at $x = 2$, since f changes from concave down to concave up at that point. All of these conclusions are consistent with the graph of f. ◀

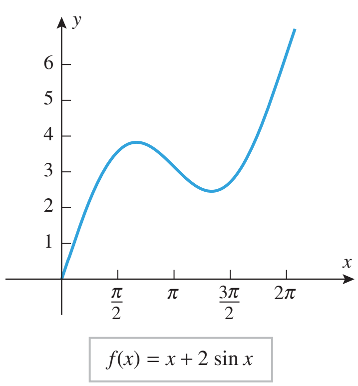


Figure 4.1.12

Example 7 Figure 4.1.12 shows the graph of the function $f(x) = x + 2\sin x$ over the interval $[0, 2\pi]$. Use the first and second derivatives of f to determine where f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.

Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 1 + 2\cos x$$
$$f''(x) = -2\sin x$$

Since f' is a continuous function, it changes sign on the interval $(0, 2\pi)$ only at points where $f'(x) = 0$ (why?). These values are solutions of the equation

$$1 + 2\cos x = 0 \quad \text{or equivalently} \quad \cos x = -\tfrac{1}{2}$$

There are two solutions of this equation in the interval $(0, 2\pi)$, namely, $x = 2\pi/3$ and $x = 4\pi/3$ (verify). Similarly, f'' is a continuous function, so its sign changes in the interval $(0, 2\pi)$ will occur only at values of x for which $f''(x) = 0$. These values are solutions of the equation

$$-2\sin x = 0$$

There is one solution of this equation in the interval $(0, 2\pi)$, namely, $x = \pi$. With the help of these "sign transition points" we obtain the sign analysis shown in the following tables:

INTERVAL	$f'(x) = 1 + 2\cos x$	CONCLUSION
$0 < x < 2\pi/3$	+	f is increasing on $[0, 2\pi/3]$
$2\pi/3 < x < 4\pi/3$	–	f is decreasing on $[2\pi/3, 4\pi/3]$
$4\pi/3 < x < 2\pi$	+	f is increasing on $[4\pi/3, 2\pi]$

INTERVAL	$f''(x) = -2\sin x$	CONCLUSION
$0 < x < \pi$	–	f is concave down on $(0, \pi)$
$\pi < x < 2\pi$	+	f is concave up on $(\pi, 2\pi)$

The signs in the two tables of Example 7 can be obtained either using the method of test points or using the unit circle definition of the sine and cosine functions.

The second table shows that there is an inflection point at $x = \pi$, since f changes from concave down to concave up at that point. All of these conclusions are consistent with the graph of f. ◀

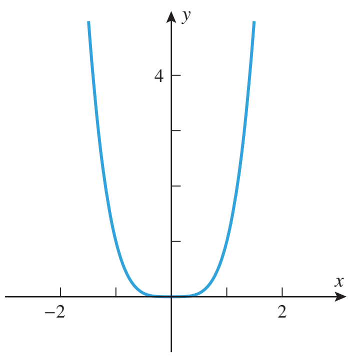
In the preceding examples the inflection points of f occurred wherever $f''(x) = 0$. However, this is not always the case. Here is a specific example.

▶ Example 8 Find the inflection points, if any, of $f(x) = x^4$.

Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$

Since $f''(x)$ is positive for $x < 0$ and for $x > 0$, the function f is concave up on the interval $(-\infty, 0)$ and on the interval $(0, +\infty)$. Thus, there is no change in concavity and hence no inflection point at $x = 0$, even though $f''(0) = 0$ (Figure 4.1.13). ◀



▶ Figure 4.1.13

We will see later that if a function f has an inflection point at $x = x_0$ and $f''(x_0)$ exists, then $f''(x_0) = 0$. Also, we will see in Section 4.3 that an inflection point may also occur where $f''(x)$ is not defined.

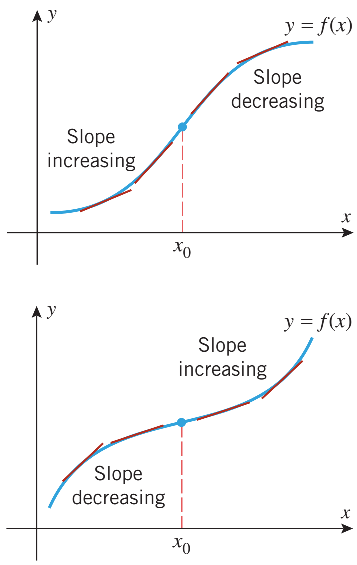
Give an argument to show that the function $f(x) = x^4$ graphed in Figure 4.1.13 is concave up on the interval $(-\infty, +\infty)$.

INFLECTION POINTS IN APPLICATIONS

Inflection points of a function f are those points on the graph of $y = f(x)$ where the slopes of the tangent lines change from increasing to decreasing or vice versa (Figure 4.1.14). Since the slope of the tangent line at a point on the graph of $y = f(x)$ can be interpreted as the rate of change of y with respect to x at that point, we can interpret inflection points in the following way:

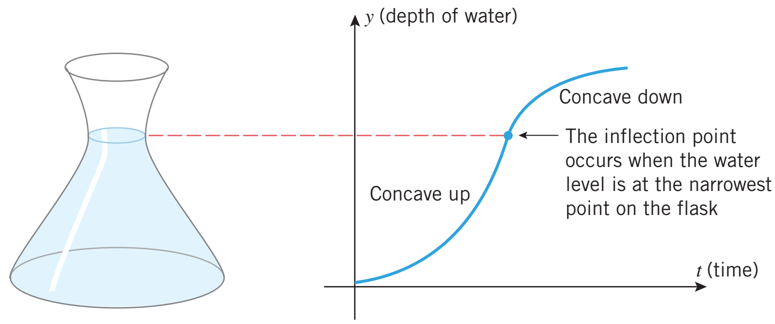
Inflection points mark the places on the curve $y = f(x)$ where the rate of change of y with respect to x changes from increasing to decreasing, or vice versa.

This is a subtle idea, since we are dealing with a change in a rate of change. It can help with your understanding of this idea to realize that inflection points may have interpretations in more familiar contexts. For example, consider the statement "Oil prices rose sharply during the first half of the year but have since begun to level off." If the price of oil is plotted as a function of time of year, this statement suggests the existence of an inflection point

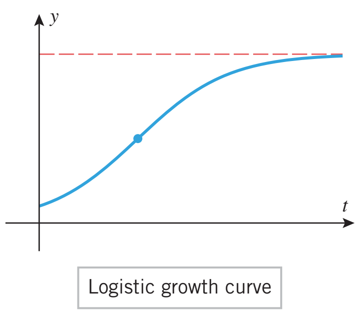


▶ **Figure 4.1.14**

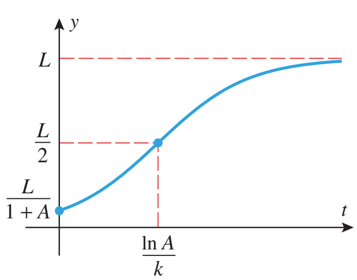
on the graph near the end of June. (Why?) To give a more visual example, consider the flask shown in Figure 4.1.15. Suppose that water is added to the flask so that the volume increases at a constant rate with respect to the time t, and let us examine the rate at which the water level y rises with respect to t. Initially, the level y will rise at a slow rate because of the wide base. However, as the diameter of the flask narrows, the rate at which the level y rises will increase until the level is at the narrow point in the neck. From that point on the rate at which the level rises will decrease as the diameter gets wider and wider. Thus, the narrow point in the neck is the point at which the rate of change of y with respect to t changes from increasing to decreasing.



▶ **Figure 4.1.15**



▲ **Figure 4.1.16**



▲ **Figure 4.1.17**

LOGISTIC CURVES

When a population grows in an environment in which space or food is limited, the graph of population versus time is typically an S-shaped curve of the form shown in Figure 4.1.16. The scenario described by this curve is a population that grows slowly at first and then more and more rapidly as the number of individuals producing offspring increases. However, at a certain point in time (where the inflection point occurs) the environmental factors begin to show their effect, and the growth rate begins a steady decline. Over an extended period of time the population approaches a limiting value that represents the upper limit on the number of individuals that the available space or food can sustain. Population growth curves of this type are called ***logistic growth curves***.

▶ **Example 9** We will see in a later chapter that logistic growth curves arise from equations of the form

$$y = \frac{L}{1 + Ae^{-kt}} \tag{1}$$

where y is the population at time t ($t \geq 0$) and A, k, and L are positive constants. Show that Figure 4.1.17 correctly describes the graph of this equation when $A > 1$.

Solution. It follows from (1) that at time $t = 0$ the value of y is

$$y = \frac{L}{1 + A}$$

and it follows from (1) and the fact that $0 < e^{-kt} \leq 1$ for $t \geq 0$ that

$$\frac{L}{1 + A} \leq y < L \tag{2}$$

(verify). This is consistent with the graph in Figure 4.1.17. The horizontal asymptote at $y = L$ is confirmed by the limit

$$\lim_{t \to +\infty} y = \lim_{t \to +\infty} \frac{L}{1 + Ae^{-kt}} = \frac{L}{1 + 0} = L \tag{3}$$

Physically, Formulas (2) and (3) tell us that L is an upper limit on the population and that the population approaches this limit over time. Again, this is consistent with the graph in Figure 4.1.17.

To investigate intervals of increase and decrease, concavity, and inflection points, we need the first and second derivatives of y with respect to t. By multiplying both sides of Equation (1) by $e^{kt}(1 + Ae^{-kt})$, we can rewrite (1) as

$$ye^{kt} + Ay = Le^{kt}$$

Using implicit differentiation, we can derive that

$$\frac{dy}{dt} = \frac{k}{L}y(L - y) \tag{4}$$

$$\frac{d^2y}{dt^2} = \frac{k^2}{L^2}y(L - y)(L - 2y) \tag{5}$$

(Exercise 70). Since $k > 0$, $y > 0$, and $L - y > 0$, it follows from (4) that $dy/dt > 0$ for all t. Thus, y is always increasing, which is consistent with Figure 4.1.17.

Since $y > 0$ and $L - y > 0$, it follows from (5) that

$$\frac{d^2y}{dt^2} > 0 \quad \text{if} \quad L - 2y > 0$$

$$\frac{d^2y}{dt^2} < 0 \quad \text{if} \quad L - 2y < 0$$

Thus, the graph of y versus t is concave up if $y < L/2$, concave down if $y > L/2$, and has an inflection point where $y = L/2$, all of which is consistent with Figure 4.1.17.

Finally, we leave it for you to solve the equation

$$\frac{L}{2} = \frac{L}{1 + Ae^{-kt}}$$

for t to show that the inflection point occurs at

$$t = \frac{1}{k}\ln A = \frac{\ln A}{k} \tag{6}$$

✔ QUICK CHECK EXERCISES 4.1

(See page 244 for answers.)

1. (a) A function f is increasing on (a, b) if ________ whenever $a < x_1 < x_2 < b$.
(b) A function f is decreasing on (a, b) if ________ whenever $a < x_1 < x_2 < b$.
(c) A function f is concave up on (a, b) if f' is ________ on (a, b).
(d) If $f''(a)$ exists and f has an inflection point at $x = a$, then $f''(a)$ ________.

2. Let $f(x) = 0.1(x^3 - 3x^2 - 9x)$. Then

$$f'(x) = 0.1(3x^2 - 6x - 9) = 0.3(x + 1)(x - 3)$$

$$f''(x) = 0.6(x - 1)$$

(a) Solutions to $f'(x) = 0$ are $x =$ ________.
(b) The function f is increasing on the interval(s) ________.
(c) The function f is concave down on the interval(s) ________.
(d) ________ is an inflection point on the graph of f.

3. Suppose that $f(x)$ has derivative $f'(x) = (x - 4)^2e^{-x/2}$. Then $f''(x) = -\frac{1}{2}(x - 4)(x - 8)e^{-x/2}$.
(a) The function f is increasing on the interval(s) ________.
(b) The function f is concave up on the interval(s) ________.
(c) The function f is concave down on the interval(s) ________.

4. Consider the statement "The rise in the cost of living slowed during the first half of the year." If we graph the cost of living versus time for the first half of the year, how does the graph reflect this statement?

EXERCISE SET 4.1

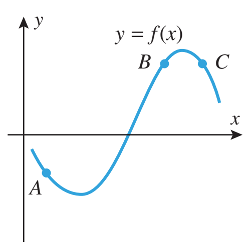
Graphing Utility C CAS

FOCUS ON CONCEPTS

1. In each part, sketch the graph of a function f with the stated properties, and discuss the signs of f' and f''.
(a) The function f is concave up and increasing on the interval $(-\infty, +\infty)$.
(b) The function f is concave down and increasing on the interval $(-\infty, +\infty)$.
(c) The function f is concave up and decreasing on the interval $(-\infty, +\infty)$.
(d) The function f is concave down and decreasing on the interval $(-\infty, +\infty)$.

2. In each part, sketch the graph of a function f with the stated properties.
(a) f is increasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave up on $(0, +\infty)$.
(b) f is increasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave down on $(0, +\infty)$.
(c) f is decreasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave up on $(0, +\infty)$.
(d) f is decreasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave down on $(0, +\infty)$.

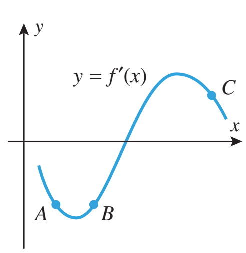
3. Use the graph of the equation $y = f(x)$ in the accompanying figure to find the signs of dy/dx and d^2y/dx^2 at the points A, B, and C.



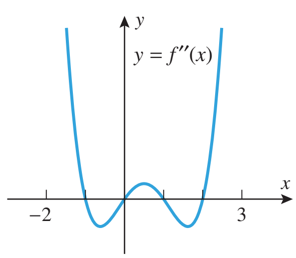
◀ **Figure Ex-3**

4. Use the graph of the equation $y = f'(x)$ in the accompanying figure to find the signs of dy/dx and d^2y/dx^2 at the points A, B, and C.

5. Use the graph of $y = f''(x)$ in the accompanying figure to determine the x-coordinates of all inflection points of f. Explain your reasoning.

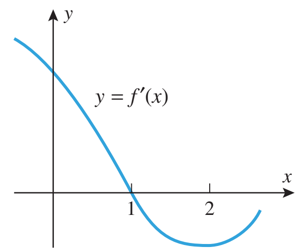


▲ **Figure Ex-4**



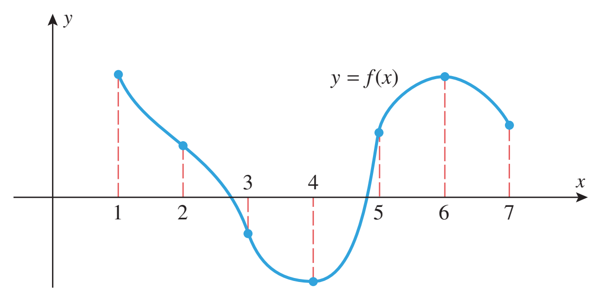
▲ **Figure Ex-5**

6. Use the graph of $y = f'(x)$ in the accompanying figure to replace the question mark with $<$, $=$, or $>$, as appropriate. Explain your reasoning.
(a) $f(0)$? $f(1)$ (b) $f(1)$? $f(2)$ (c) $f'(0)$? 0
(d) $f'(1)$? 0 (e) $f''(0)$? 0 (f) $f''(2)$? 0



◀ **Figure Ex-6**

7. In each part, use the graph of $y = f(x)$ in the accompanying figure to find the requested information.
(a) Find the intervals on which f is increasing.
(b) Find the intervals on which f is decreasing.
(c) Find the open intervals on which f is concave up.
(d) Find the open intervals on which f is concave down.
(e) Find all values of x at which f has an inflection point.



▲ **Figure Ex-7**

8. Use the graph in Exercise 7 to make a table that shows the signs of f' and f'' over the intervals (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), and (6, 7).

9–10 A sign chart is presented for the first and second derivatives of a function f. Assuming that f is continuous everywhere, find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points. ■

9.

INTERVAL	SIGN OF $f'(x)$	SIGN OF $f''(x)$
$x < 1$	−	+
$1 < x < 2$	+	+
$2 < x < 3$	+	−
$3 < x < 4$	−	−
$4 < x$	−	+

10.

INTERVAL	SIGN OF $f'(x)$	SIGN OF $f''(x)$
$x < 1$	+	+
$1 < x < 3$	+	−
$3 < x$	+	+

11–14 True–False Assume that f is differentiable everywhere. Determine whether the statement is true or false. Explain your answer.

11. If f is decreasing on $[0, 2]$, then $f(0) > f(1) > f(2)$.

12. If $f'(1) > 0$, then f is increasing on $[0, 2]$.

13. If f is increasing on $[0, 2]$, then $f'(1) > 0$.

14. If f' is increasing on $[0, 1]$ and f' is decreasing on $[1, 2]$, then f has an inflection point at $x = 1$.

15–32 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

15. $f(x) = x^2 - 3x + 8$

16. $f(x) = 5 - 4x - x^2$

17. $f(x) = (2x + 1)^3$

18. $f(x) = 5 + 12x - x^3$

19. $f(x) = 3x^4 - 4x^3$

20. $f(x) = x^4 - 5x^3 + 9x^2$

21. $f(x) = \dfrac{x - 2}{(x^2 - x + 1)^2}$

22. $f(x) = \dfrac{x}{x^2 + 2}$

23. $f(x) = \sqrt[3]{x^2 + x + 1}$

24. $f(x) = x^{4/3} - x^{1/3}$

25. $f(x) = (x^{2/3} - 1)^2$

26. $f(x) = x^{2/3} - x$

27. $f(x) = e^{-x^2/2}$

28. $f(x) = xe^{x^2}$

29. $f(x) = \ln\sqrt{x^2 + 4}$

30. $f(x) = x^3 \ln x$

31. $f(x) = \tan^{-1}(x^2 - 1)$

32. $f(x) = \sin^{-1} x^{2/3}$

33–38 Analyze the trigonometric function f over the specified interval, stating where f is increasing, decreasing, concave up, and concave down, and stating the x-coordinates of all inflection points. Confirm that your results are consistent with the graph of f generated with a graphing utility.

33. $f(x) = \sin x - \cos x$; $[-\pi, \pi]$

34. $f(x) = \sec x \tan x$; $(-\pi/2, \pi/2)$

35. $f(x) = 1 - \tan(x/2)$; $(-\pi, \pi)$

36. $f(x) = 2x + \cot x$; $(0, \pi)$

37. $f(x) = (\sin x + \cos x)^2$; $[-\pi, \pi]$

38. $f(x) = \sin^2 2x$; $[0, \pi]$

FOCUS ON CONCEPTS

39. In parts (a)–(c), sketch a continuous curve $y = f(x)$ with the stated properties.
(a) $f(2) = 4$, $f'(2) = 0$, $f''(x) > 0$ for all x
(b) $f(2) = 4$, $f'(2) = 0$, $f''(x) < 0$ for $x < 2$, $f''(x) > 0$ for $x > 2$
(c) $f(2) = 4$, $f''(x) < 0$ for $x \neq 2$ and $\lim_{x\to 2^+} f'(x) = +\infty$, $\lim_{x\to 2^-} f'(x) = -\infty$

40. In each part sketch a continuous curve $y = f(x)$ with the stated properties.
(a) $f(2) = 4$, $f'(2) = 0$, $f''(x) < 0$ for all x
(b) $f(2) = 4$, $f'(2) = 0$, $f''(x) > 0$ for $x < 2$, $f''(x) < 0$ for $x > 2$
(c) $f(2) = 4$, $f''(x) > 0$ for $x \neq 2$ and $\lim_{x\to 2^+} f'(x) = -\infty$, $\lim_{x\to 2^-} f'(x) = +\infty$

41–46 If f is increasing on an interval $[0, b)$, then it follows from Definition 4.1.1 that $f(0) < f(x)$ for each x in the interval $(0, b)$. Use this result in these exercises.

41. Show that $\sqrt[3]{1 + x} < 1 + \frac{1}{3}x$ if $x > 0$, and confirm the inequality with a graphing utility. [*Hint:* Show that the function $f(x) = 1 + \frac{1}{3}x - \sqrt[3]{1 + x}$ is increasing on $[0, +\infty)$.]

42. Show that $x < \tan x$ if $0 < x < \pi/2$, and confirm the inequality with a graphing utility. [*Hint:* Show that the function $f(x) = \tan x - x$ is increasing on $[0, \pi/2)$.]

43. Use a graphing utility to make a conjecture about the relative sizes of x and $\sin x$ for $x \geq 0$, and prove your conjecture.

44. Use a graphing utility to make a conjecture about the relative sizes of $1 - x^2/2$ and $\cos x$ for $x \geq 0$, and prove your conjecture. [*Hint:* Use the result of Exercise 43.]

45. (a) Show that $\ln(x + 1) \leq x$ if $x \geq 0$.
(b) Show that $\ln(x + 1) \geq x - \frac{1}{2}x^2$ if $x \geq 0$.
(c) Confirm the inequalities in parts (a) and (b) with a graphing utility.

46. (a) Show that $e^x \geq 1 + x$ if $x \geq 0$.
(b) Show that $e^x \geq 1 + x + \frac{1}{2}x^2$ if $x \geq 0$.
(c) Confirm the inequalities in parts (a) and (b) with a graphing utility.

47–48 Use a graphing utility to generate the graphs of f' and f'' over the stated interval; then use those graphs to estimate the x-coordinates of the inflection points of f, the intervals on which f is concave up or down, and the intervals on which f is increasing or decreasing. Check your estimates by graphing f.

47. $f(x) = x^4 - 24x^2 + 12x$, $\quad -5 \leq x \leq 5$

48. $f(x) = \dfrac{1}{1 + x^2}$, $\quad -5 \leq x \leq 5$

49–50 Use a CAS to find f'' and to approximate the x-coordinates of the inflection points to six decimal places. Confirm that your answer is consistent with the graph of f.

49. $f(x) = \dfrac{10x - 3}{3x^2 - 5x + 8}$

50. $f(x) = \dfrac{x^3 - 8x + 7}{\sqrt{x^2 + 1}}$

51. Use Definition 4.1.1 to prove that $f(x) = x^2$ is increasing on $[0, +\infty)$.

52. Use Definition 4.1.1 to prove that $f(x) = 1/x$ is decreasing on $(0, +\infty)$.

FOCUS ON CONCEPTS

53–54 Determine whether the statements are true or false. If a statement is false, find functions for which the statement fails to hold.

53. (a) If f and g are increasing on an interval, then so is $f + g$.
(b) If f and g are increasing on an interval, then so is $f \cdot g$.

54. (a) If f and g are concave up on an interval, then so is $f + g$.
(b) If f and g are concave up on an interval, then so is $f \cdot g$.

55. In each part, find functions f and g that are increasing on $(-\infty, +\infty)$ and for which $f - g$ has the stated property.
(a) $f - g$ is decreasing on $(-\infty, +\infty)$.
(b) $f - g$ is constant on $(-\infty, +\infty)$.
(c) $f - g$ is increasing on $(-\infty, +\infty)$.

56. In each part, find functions f and g that are positive and increasing on $(-\infty, +\infty)$ and for which f/g has the stated property.
(a) f/g is decreasing on $(-\infty, +\infty)$.
(b) f/g is constant on $(-\infty, +\infty)$.
(c) f/g is increasing on $(-\infty, +\infty)$.

57. (a) Prove that a general cubic polynomial

$$f(x) = ax^3 + bx^2 + cx + d \quad (a \neq 0)$$

has exactly one inflection point.
(b) Prove that if a cubic polynomial has three x-intercepts, then the inflection point occurs at the average value of the intercepts.
(c) Use the result in part (b) to find the inflection point of the cubic polynomial $f(x) = x^3 - 3x^2 + 2x$, and check your result by using f'' to determine where f is concave up and concave down.

58. From Exercise 57, the polynomial $f(x) = x^3 + bx^2 + 1$ has one inflection point. Use a graphing utility to reach a conclusion about the effect of the constant b on the location of the inflection point. Use f'' to explain what you have observed graphically.

59. Use Definition 4.1.1 to prove:
(a) If f is increasing on the intervals $(a, c]$ and $[c, b)$, then f is increasing on (a, b).
(b) If f is decreasing on the intervals $(a, c]$ and $[c, b)$, then f is decreasing on (a, b).

60. Use part (a) of Exercise 59 to show that $f(x) = x + \sin x$ is increasing on the interval $(-\infty, +\infty)$.

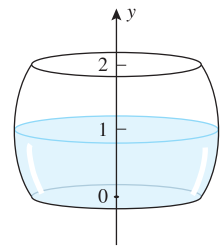
61. Use part (b) of Exercise 59 to show that $f(x) = \cos x - x$ is decreasing on the interval $(-\infty, +\infty)$.

62. Let $y = 1/(1 + x^2)$. Find the values of x for which y is increasing most rapidly or decreasing most rapidly.

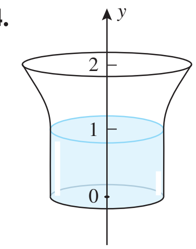
FOCUS ON CONCEPTS

63–66 Suppose that water is flowing at a constant rate into the container shown. Make a rough sketch of the graph of the water level y versus the time t. Make sure that your sketch conveys where the graph is concave up and concave down, and label the y-coordinates of the inflection points.

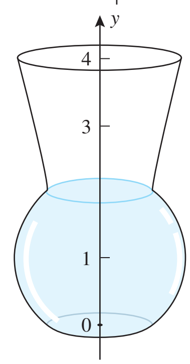
63.



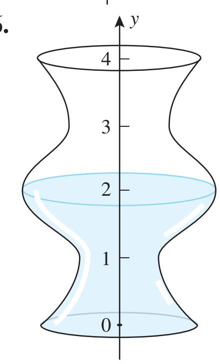
64.



65.



66.



67. Suppose that a population y grows according to the logistic model given by Formula (1).
(a) At what rate is y increasing at time $t = 0$?
(b) In words, describe how the rate of growth of y varies with time.
(c) At what time is the population growing most rapidly?

68. Suppose that the number of individuals at time t in a certain wildlife population is given by

$$N(t) = \frac{340}{1 + 9(0.77)^t}, \quad t \geq 0$$

where t is in years. Use a graphing utility to estimate the time at which the size of the population is increasing most rapidly.

69. Suppose that the spread of a flu virus on a college campus is modeled by the function

$$y(t) = \frac{1000}{1 + 999e^{-0.9t}}$$

where $y(t)$ is the number of infected students at time t (in days, starting with $t = 0$). Use a graphing utility to estimate the day on which the virus is spreading most rapidly.

70. The logistic growth model given in Formula (1) is equivalent to

$$ye^{kt} + Ay = Le^{kt}$$

where y is the population at time t $(t \geq 0)$ and A, k, and L

are positive constants. Use implicit differentiation to verify that

$$\frac{dy}{dt} = \frac{k}{L}y(L-y)$$

$$\frac{d^2y}{dt^2} = \frac{k^2}{L^2}y(L-y)(L-2y)$$

71. Assuming that A, k, and L are positive constants, verify that the graph of $y = L/(1 + Ae^{-kt})$ has an inflection point at $\left(\frac{1}{k}\ln A, \frac{1}{2}L\right)$.

72. Writing An approaching storm causes the air temperature to fall. Make a statement that indicates there is an inflection point in the graph of temperature versus time. Explain how the existence of an inflection point follows from your statement.

73. Writing Explain what the sign analyses of $f'(x)$ and $f''(x)$ tell us about the graph of $y = f(x)$.

✔ QUICK CHECK ANSWERS 4.1

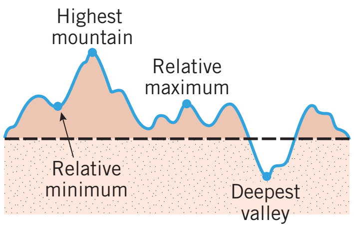
1. (a) $f(x_1) < f(x_2)$ (b) $f(x_1) > f(x_2)$ (c) increasing (d) $= 0$ **2.** (a) $-1, 3$ (b) $(-\infty, -1]$ and $[3, +\infty)$ (c) $(-\infty, 1)$ (d) $(1, -1.1)$ **3.** (a) $(-\infty, +\infty)$ (b) $(4, 8)$ (c) $(-\infty, 4)$, $(8, +\infty)$ **4.** The graph is increasing and concave down.

4.2 ANALYSIS OF FUNCTIONS II: RELATIVE EXTREMA; GRAPHING POLYNOMIALS

In this section we will develop methods for finding the high and low points on the graph of a function and we will discuss procedures for analyzing the graphs of polynomials.

RELATIVE MAXIMA AND MINIMA

If we imagine the graph of a function f to be a two-dimensional mountain range with hills and valleys, then the tops of the hills are called "relative maxima," and the bottoms of the valleys are called "relative minima" (Figure 4.2.1). The relative maxima are the high points in their *immediate vicinity*, and the relative minima are the low points. A relative maximum need not be the highest point in the entire mountain range, and a relative minimum need not be the lowest point—they are just high and low points *relative* to the nearby terrain. These ideas are captured in the following definition.



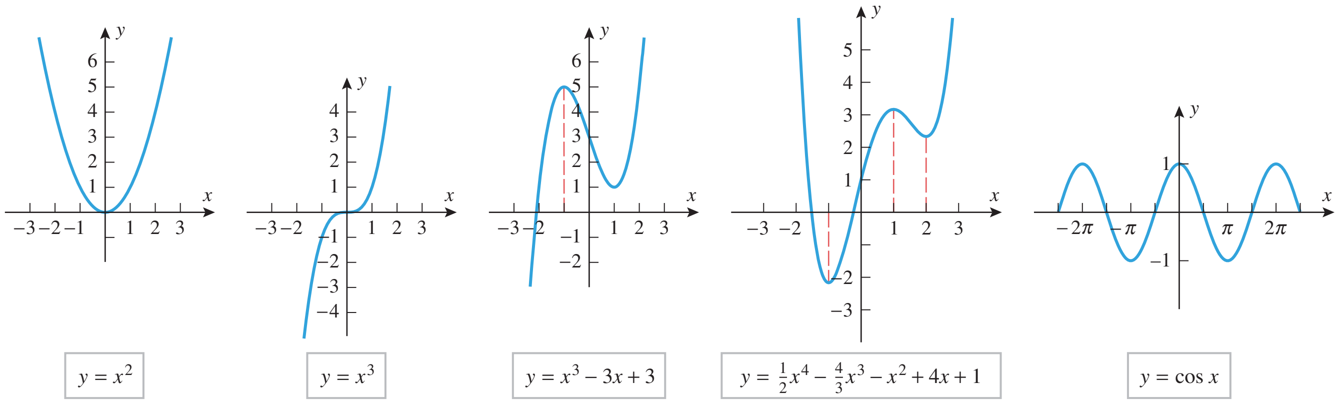
▲ **Figure 4.2.1**

4.2.1 DEFINITION A function f is said to have a ***relative maximum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval. Similarly, f is said to have a ***relative minimum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval. If f has either a relative maximum or a relative minimum at x_0, then f is said to have a ***relative extremum*** at x_0.

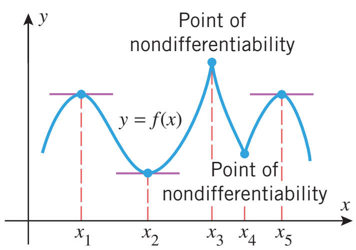
▶ **Example 1** We can see from Figure 4.2.2 that:

- $f(x) = x^2$ has a relative minimum at $x = 0$ but no relative maxima.
- $f(x) = x^3$ has no relative extrema.
- $f(x) = x^3 - 3x + 3$ has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.
- $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ has relative minima at $x = -1$ and $x = 2$ and a relative maximum at $x = 1$.

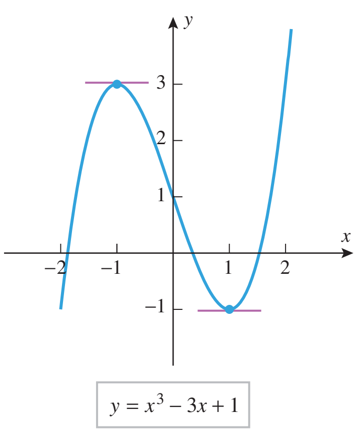
- $f(x) = \cos x$ has relative maxima at all even multiples of π and relative minima at all odd multiples of π. ◀



▲ **Figure 4.2.2**



▲ **Figure 4.2.3** The points x_1, x_2, x_3, x_4, and x_5 are critical points. Of these, x_1, x_2, and x_5 are stationary points.



▲ **Figure 4.2.4**

What is the maximum number of critical points that a polynomial of degree n can have? Why?

The relative extrema for the five functions in Example 1 occur at points where the graphs of the functions have horizontal tangent lines. Figure 4.2.3 illustrates that a relative extremum can also occur at a point where a function is not differentiable. In general, we define a ***critical point*** for a function f to be a point in the domain of f at which either the graph of f has a horizontal tangent line or f is not differentiable. To distinguish between the two types of critical points we call x a ***stationary point*** of f if $f'(x) = 0$. The following theorem, which is proved in Appendix D, states that the critical points for a function form a complete set of candidates for relative extrema on the interior of the domain of the function.

4.2.2 THEOREM *Suppose that f is a function defined on an open interval containing the point x_0. If f has a relative extremum at $x = x_0$, then $x = x_0$ is a critical point of f; that is, either $f'(x_0) = 0$ or f is not differentiable at x_0.*

▶ **Example 2** Find all critical points of $f(x) = x^3 - 3x + 1$.

Solution. The function f, being a polynomial, is differentiable everywhere, so its critical points are all stationary points. To find these points we must solve the equation $f'(x) = 0$. Since

$$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$$

we conclude that the critical points occur at $x = -1$ and $x = 1$. This is consistent with the graph of f in Figure 4.2.4. ◀

▶ **Example 3** Find all critical points of $f(x) = 3x^{5/3} - 15x^{2/3}$.

Solution. The function f is continuous everywhere and its derivative is

$$f'(x) = 5x^{2/3} - 10x^{-1/3} = 5x^{-1/3}(x-2) = \frac{5(x-2)}{x^{1/3}}$$

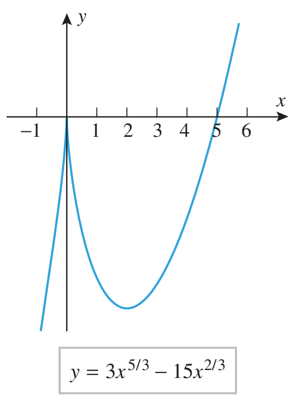
We see from this that $f'(x) = 0$ if $x = 2$ and $f'(x)$ is undefined if $x = 0$. Thus $x = 0$ and $x = 2$ are critical points and $x = 2$ is a stationary point. This is consistent with the graph of f shown in Figure 4.2.5. ◂

TECHNOLOGY MASTERY

Your graphing utility may have trouble producing portions of the graph in Figure 4.2.5 because of the fractional exponents. If this is the case for you, graph the function

$$y = 3(|x|/x)|x|^{5/3} - 15|x|^{2/3}$$

which is equivalent to $f(x)$ for $x \neq 0$. Appendix A explores the method suggested here in more detail.

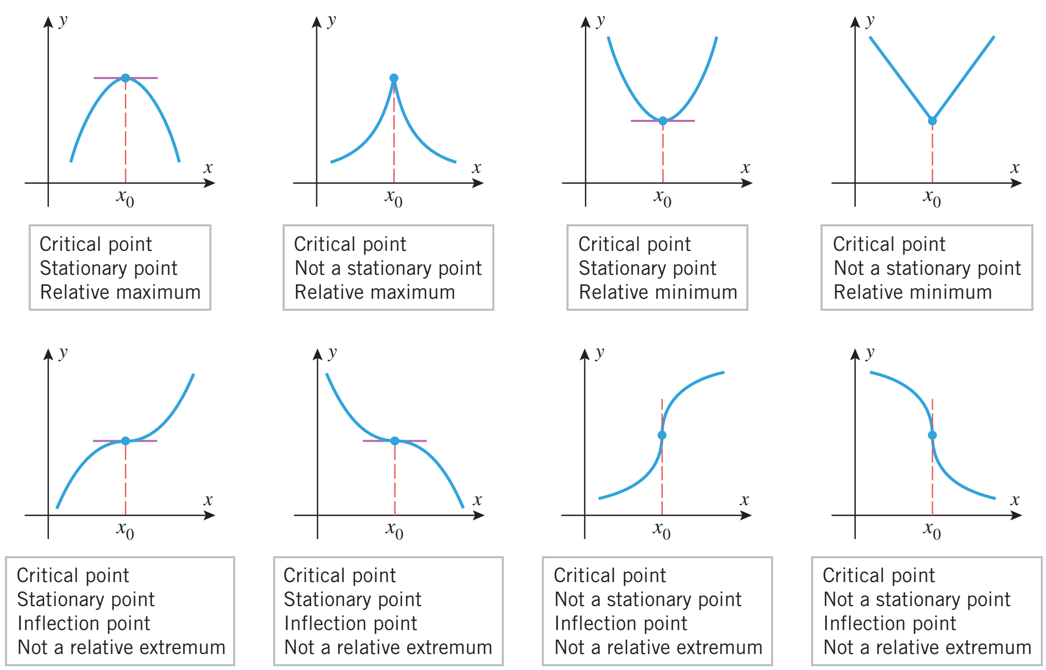


▲ **Figure 4.2.5**

FIRST DERIVATIVE TEST

Theorem 4.2.2 asserts that the relative extrema must occur at critical points, but it does *not* say that a relative extremum occurs at *every* critical point. For example, for the eight critical points in Figure 4.2.6, relative extrema occur at each x_0 in the top row but not at any x_0 in the bottom row. Moreover, at the critical points in the first row the derivatives have opposite signs on the two sides of x_0, whereas at the critical points in the second row the signs of the derivatives are the same on both sides. This suggests:

A function f has a relative extremum at those critical points where f' changes sign.



▲ **Figure 4.2.6**

We can actually take this a step further. At the two relative maxima in Figure 4.2.6 the derivative is positive on the left side and negative on the right side, and at the two relative minima the derivative is negative on the left side and positive on the right side. All of this is summarized more precisely in the following theorem.

4.2.3 THEOREM (*First Derivative Test*) *Suppose that f is continuous at a critical point x_0.*

(*a*) *If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.*

(*b*) *If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.*

(*c*) *If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0, then f does not have a relative extremum at x_0.*

Informally stated, parts (*a*) and (*b*) of Theorem 4.2.3 tell us that for a continuous function, relative maxima occur at critical points where the derivative changes from + to − and relative minima where it changes from − to +.

Use the first derivative test to confirm the behavior at x_0 of each graph in Figure 4.2.6.

PROOF We will prove part (*a*) and leave parts (*b*) and (*c*) as exercises. We are assuming that $f'(x) > 0$ on the interval (a, x_0) and that $f'(x) < 0$ on the interval (x_0, b), and we want to show that

$$f(x_0) \geq f(x)$$

for all x in the interval (a, b). However, the two hypotheses, together with Theorem 4.1.2 and its associated marginal note imply that f is increasing on the interval $(a, x_0]$ and decreasing on the interval $[x_0, b)$. Thus, $f(x_0) \geq f(x)$ for all x in (a, b) with equality only at x_0. ■

▶ **Example 4** We showed in Example 3 that the function $f(x) = 3x^{5/3} - 15x^{2/3}$ has critical points at $x = 0$ and $x = 2$. Figure 4.2.5 suggests that f has a relative maximum at $x = 0$ and a relative minimum at $x = 2$. Confirm this using the first derivative test.

Solution. We showed in Example 3 that

$$f'(x) = \frac{5(x-2)}{x^{1/3}}$$

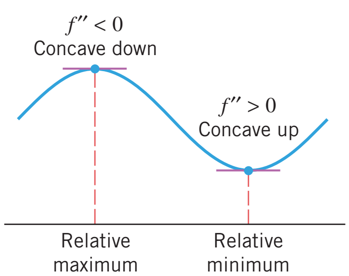
A sign analysis of this derivative is shown in Table 4.2.1. The sign of f' changes from + to − at $x = 0$, so there is a relative maximum at that point. The sign changes from − to + at $x = 2$, so there is a relative minimum at that point. ◀

Table 4.2.1

INTERVAL	$5(x-2)/x^{1/3}$	$f'(x)$
$x < 0$	$(-)/(-)$	+
$0 < x < 2$	$(-)/(+)$	−
$x > 2$	$(+)/(+)$	+

SECOND DERIVATIVE TEST

There is another test for relative extrema that is based on the following geometric observation: A function f has a relative maximum at a stationary point if the graph of f is concave down on an open interval containing that point, and it has a relative minimum if it is concave up (Figure 4.2.7).



▲ **Figure 4.2.7**

4.2.4 THEOREM (*Second Derivative Test*) *Suppose that f is twice differentiable at the point x_0.*

(*a*) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.*

(*b*) *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.*

(*c*) *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.*

The second derivative test is often easier to apply than the first derivative test. However, the first derivative test can be used at any critical point of a continuous function, while the second derivative test applies only at stationary points where the second derivative exists.

We will prove parts (a) and (c) and leave part (b) as an exercise.

PROOF (a) We are given that $f'(x_0) = 0$ and $f''(x_0) > 0$, and we want to show that f has a relative minimum at x_0. Expressing $f''(x_0)$ as a limit and using the two given conditions we obtain

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x)}{x - x_0} > 0$$

This implies that for x sufficiently close to but different from x_0 we have

$$\frac{f'(x)}{x - x_0} > 0 \tag{1}$$

Thus, there is an open interval extending left from x_0 and an open interval extending right from x_0 on which (1) holds. On the open interval extending left the denominator in (1) is negative, so $f'(x) < 0$, and on the open interval extending right the denominator is positive, so $f'(x) > 0$. It now follows from part (b) of the first derivative test (Theorem 4.2.3) that f has a relative minimum at x_0.

PROOF (c) To prove this part of the theorem we need only provide functions for which $f'(x_0) = 0$ and $f''(x_0) = 0$ at some point x_0, but with one having a relative minimum at x_0, one having a relative maximum at x_0, and one having neither at x_0. We leave it as an exercise for you to show that three such functions are $f(x) = x^4$ (relative minimum at $x = 0$), $f(x) = -x^4$ (relative maximum at $x = 0$), and $f(x) = x^3$ (neither a relative maximum nor a relative minimum at x_0). ■

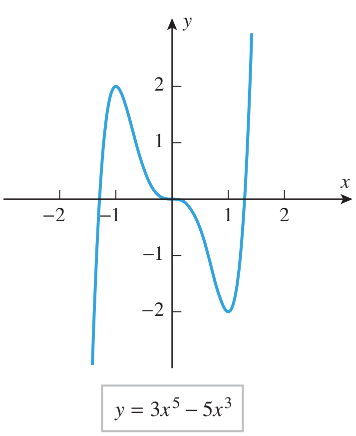
▶ Example 5 Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

Solution. We have

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x + 1)(x - 1)$$
$$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$$

Solving $f'(x) = 0$ yields the stationary points $x = 0$, $x = -1$, and $x = 1$. As shown in the following table, we can conclude from the second derivative test that f has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

STATIONARY POINT	$30x(2x^2 - 1)$	$f''(x)$	SECOND DERIVATIVE TEST
$x = -1$	-30	$-$	f has a relative maximum
$x = 0$	0	0	Inconclusive
$x = 1$	30	$+$	f has a relative minimum



▲ Figure 4.2.8

The test is inconclusive at $x = 0$, so we will try the first derivative test at that point. A sign analysis of f' is given in the following table:

INTERVAL	$15x^2(x + 1)(x - 1)$	$f'(x)$
$-1 < x < 0$	$(+)(+)(-)$	$-$
$0 < x < 1$	$(+)(+)(-)$	$-$

Since there is no sign change in f' at $x = 0$, there is neither a relative maximum nor a relative minimum at that point. All of this is consistent with the graph of f shown in Figure 4.2.8. ◀

GEOMETRIC IMPLICATIONS OF MULTIPLICITY

Our final goal in this section is to outline a general procedure that can be used to analyze and graph polynomials. To do so, it will be helpful to understand how the graph of a polynomial behaves in the vicinity of its roots. For example, it would be nice to know what property of the polynomial in Example 5 produced the inflection point and horizontal tangent at the root $x = 0$.

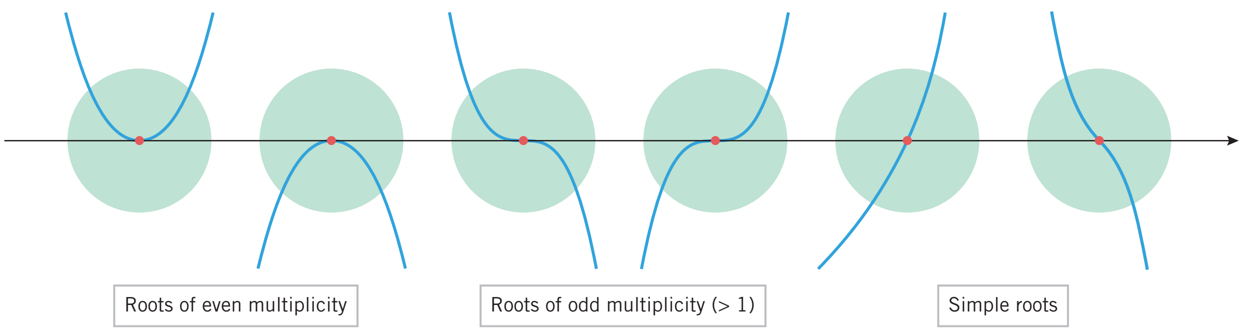
Recall that a root $x = r$ of a polynomial $p(x)$ has ***multiplicity m*** if $(x - r)^m$ divides $p(x)$ but $(x - r)^{m+1}$ does not. A root of multiplicity 1 is called a ***simple root***. Figure 4.2.9 and the following theorem show that the behavior of a polynomial in the vicinity of a real root is determined by the multiplicity of that root (we omit the proof).

4.2.5 THE GEOMETRIC IMPLICATIONS OF MULTIPLICITY *Suppose that $p(x)$ is a polynomial with a root of multiplicity m at $x = r$.*

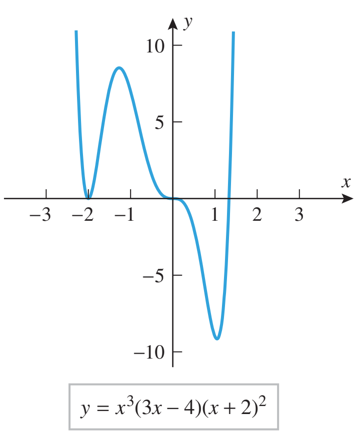
(a) *If m is even, then the graph of $y = p(x)$ is tangent to the x-axis at $x = r$, does not cross the x-axis there, and does not have an inflection point there.*

(b) *If m is odd and greater than 1, then the graph is tangent to the x-axis at $x = r$, crosses the x-axis there, and also has an inflection point there.*

(c) *If $m = 1$ (so that the root is simple), then the graph is not tangent to the x-axis at $x = r$, crosses the x-axis there, and may or may not have an inflection point there.*



▲ **Figure 4.2.9**



▲ **Figure 4.2.10**

▶ **Example 6** Make a conjecture about the behavior of the graph of

$$y = x^3(3x - 4)(x + 2)^2$$

in the vicinity of its x-intercepts, and test your conjecture by generating the graph.

Solution. The x-intercepts occur at $x = 0$, $x = \frac{4}{3}$, and $x = -2$. The root $x = 0$ has multiplicity 3, which is odd, so at that point the graph should be tangent to the x-axis, cross the x-axis, and have an inflection point there. The root $x = -2$ has multiplicity 2, which is even, so the graph should be tangent to but not cross the x-axis there. The root $x = \frac{4}{3}$ is simple, so at that point the curve should cross the x-axis without being tangent to it. All of this is consistent with the graph in Figure 4.2.10. ◀

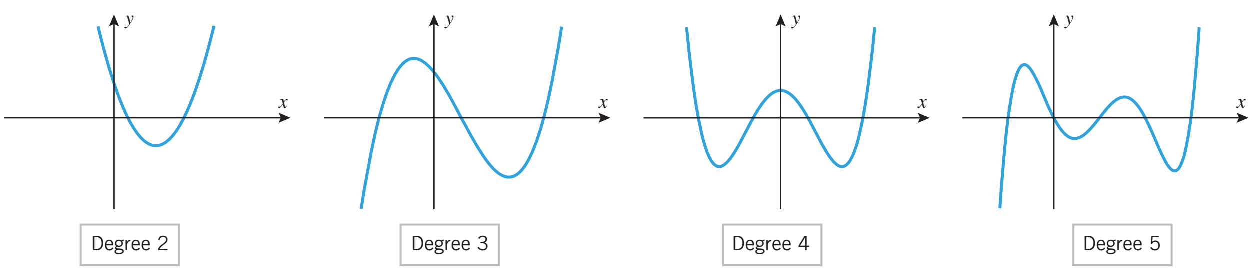
ANALYSIS OF POLYNOMIALS

Historically, the term "curve sketching" meant using calculus to help draw the graph of a function by hand—the graph was the goal. Since graphs can now be produced with great precision using calculators and computers, the purpose of curve sketching has changed. Today, we typically start with a graph produced by a calculator or computer, then use curve sketching to identify important features of the graph that the calculator or computer might have missed. Thus, the goal of curve sketching is no longer the graph itself, but rather the information it reveals about the function.

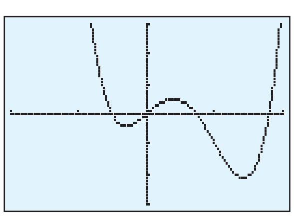
Polynomials are among the simplest functions to graph and analyze. Their significant features are symmetry, intercepts, relative extrema, inflection points, and the behavior as $x \to +\infty$ and as $x \to -\infty$. Figure 4.2.11 shows the graphs of four polynomials in x. The graphs in Figure 4.2.11 have properties that are common to all polynomials:

- The natural domain of a polynomial is $(-\infty, +\infty)$.
- Polynomials are continuous everywhere.
- Polynomials are differentiable everywhere, so their graphs have no corners or vertical tangent lines.
- The graph of a nonconstant polynomial eventually increases or decreases without bound as $x \to +\infty$ and as $x \to -\infty$. This is because the limit of a nonconstant polynomial as $x \to +\infty$ or as $x \to -\infty$ is $\pm\infty$, depending on the sign of the term of highest degree and whether the polynomial has even or odd degree [see Formulas (17) and (18) of Section 1.3 and the related discussion].
- The graph of a polynomial of degree $n\,(> 2)$ has at most n x-intercepts, at most $n-1$ relative extrema, and at most $n-2$ inflection points. This is because the x-intercepts, relative extrema, and inflection points of a polynomial $p(x)$ are among the real solutions of the equations $p(x) = 0$, $p'(x) = 0$, and $p''(x) = 0$, and the polynomials in these equations have degree n, $n-1$, and $n-2$, respectively. Thus, for example, the graph of a quadratic polynomial has at most two x-intercepts, one relative extremum, and no inflection points; and the graph of a cubic polynomial has at most three x-intercepts, two relative extrema, and one inflection point.

For each of the graphs in Figure 4.2.11, count the number of x-intercepts, relative extrema, and inflection points, and confirm that your count is consistent with the degree of the polynomial.



▲ **Figure 4.2.11**



$[-2, 2] \times [-3, 3]$

$y = 3x^4 - 6x^3 + 2x$

▲ **Figure 4.2.12**

► **Example 7** Figure 4.2.12 shows the graph of

$$y = 3x^4 - 6x^3 + 2x$$

produced on a graphing calculator. Confirm that the graph is not missing any significant features.

Solution. We can be confident that the graph shows all significant features of the polynomial because the polynomial has degree 4 and we can account for four roots, three relative extrema, and two inflection points. Moreover, the graph suggests the correct behavior as

$x \to +\infty$ and as $x \to -\infty$, since

$$\lim_{x\to+\infty}(3x^4 - 6x^3 + 2x) = \lim_{x\to+\infty} 3x^4 = +\infty$$
$$\lim_{x\to-\infty}(3x^4 - 6x^3 + 2x) = \lim_{x\to-\infty} 3x^4 = +\infty \blacktriangleleft$$

Example 8 Sketch the graph of the equation

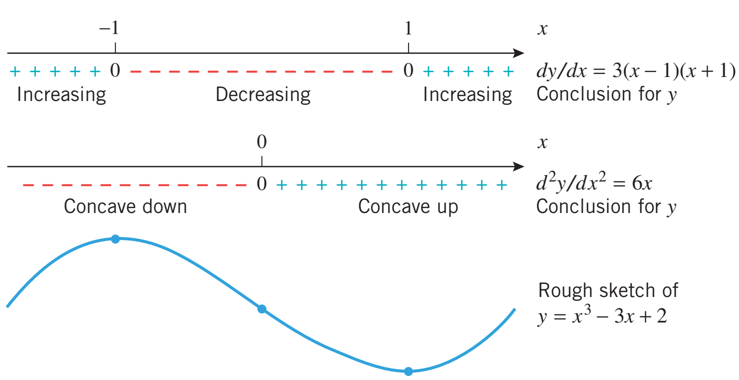
$$y = x^3 - 3x + 2$$

and identify the locations of the intercepts, relative extrema, and inflection points.

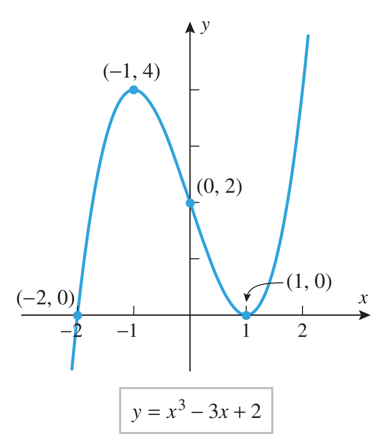
Solution. The following analysis will produce the information needed to sketch the graph:

A review of polynomial factoring is given in Appendix C.

- *x-intercepts:* Factoring the polynomial yields
$$x^3 - 3x + 2 = (x+2)(x-1)^2$$
which tells us that the x-intercepts are $x = -2$ and $x = 1$.
- *y-intercept:* Setting $x = 0$ yields $y = 2$.
- *End behavior:* We have
$$\lim_{x\to+\infty}(x^3 - 3x + 2) = \lim_{x\to+\infty} x^3 = +\infty$$
$$\lim_{x\to-\infty}(x^3 - 3x + 2) = \lim_{x\to-\infty} x^3 = -\infty$$
so the graph increases without bound as $x \to +\infty$ and decreases without bound as $x \to -\infty$.
- *Derivatives:*
$$\frac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1)$$
$$\frac{d^2y}{dx^2} = 6x$$
- *Increase, decrease, relative extrema, inflection points:* Figure 4.2.13 gives a sign analysis of the first and second derivatives and indicates its geometric significance. There are stationary points at $x = -1$ and $x = 1$. Since the sign of dy/dx changes from $+$ to $-$ at $x = -1$, there is a relative maximum there, and since it changes from $-$ to $+$ at $x = 1$, there is a relative minimum there. The sign of d^2y/dx^2 changes from $-$ to $+$ at $x = 0$, so there is an inflection point there.
- *Final sketch:* Figure 4.2.14 shows the final sketch with the coordinates of the intercepts, relative extrema, and inflection point labeled. $\blacktriangleleft$



▲ **Figure 4.2.13**



▲ **Figure 4.2.14**

✔QUICK CHECK EXERCISES 4.2

(See page 254 for answers.)

1. A function f has a relative maximum at x_0 if there is an open interval containing x_0 on which $f(x)$ is ________ $f(x_0)$ for every x in the interval.
2. Suppose that f is defined everywhere and $x = 2, 3, 5, 7$ are critical points for f. If $f'(x)$ is positive on the intervals $(-\infty, 2)$ and $(5, 7)$, and if $f'(x)$ is negative on the intervals $(2, 3)$, $(3, 5)$, and $(7, +\infty)$, then f has relative maxima at $x =$ ________ and f has relative minima at $x =$ ________.
3. Suppose that f is defined everywhere and $x = -2$ and $x = 1$ are critical points for f. If $f''(x) = 2x + 1$, then f has a relative ________ at $x = -2$ and f has a relative ________ at $x = 1$.
4. Let $f(x) = (x^2 - 4)^2$. Then $f'(x) = 4x(x^2 - 4)$ and $f''(x) = 4(3x^2 - 4)$. Identify the locations of the (a) relative maxima, (b) relative minima, and (c) inflection points on the graph of f.

EXERCISE SET 4.2

Graphing Utility   C CAS

FOCUS ON CONCEPTS

1. In each part, sketch the graph of a continuous function f with the stated properties.
   (a) f is concave up on the interval $(-\infty, +\infty)$ and has exactly one relative extremum.
   (b) f is concave up on the interval $(-\infty, +\infty)$ and has no relative extrema.
   (c) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to +\infty$ as $x \to +\infty$.
   (d) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to -\infty$ as $x \to +\infty$.
2. In each part, sketch the graph of a continuous function f with the stated properties.
   (a) f has exactly one relative extremum on $(-\infty, +\infty)$, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.
   (b) f has exactly two relative extrema on $(-\infty, +\infty)$, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.
   (c) f has exactly one inflection point and one relative extremum on $(-\infty, +\infty)$.
   (d) f has infinitely many relative extrema, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.

3. (a) Use both the first and second derivative tests to show that $f(x) = 3x^2 - 6x + 1$ has a relative minimum at $x = 1$.
   (b) Use both the first and second derivative tests to show that $f(x) = x^3 - 3x + 3$ has a relative minimum at $x = 1$ and a relative maximum at $x = -1$.
4. (a) Use both the first and second derivative tests to show that $f(x) = \sin^2 x$ has a relative minimum at $x = 0$.
   (b) Use both the first and second derivative tests to show that $g(x) = \tan^2 x$ has a relative minimum at $x = 0$.
   (c) Give an informal verbal argument to explain without calculus why the functions in parts (a) and (b) have relative minima at $x = 0$.
5. (a) Show that both of the functions $f(x) = (x - 1)^4$ and $g(x) = x^3 - 3x^2 + 3x - 2$ have stationary points at $x = 1$.
   (b) What does the second derivative test tell you about the nature of these stationary points?
   (c) What does the first derivative test tell you about the nature of these stationary points?
6. (a) Show that $f(x) = 1 - x^5$ and $g(x) = 3x^4 - 8x^3$ both have stationary points at $x = 0$.
   (b) What does the second derivative test tell you about the nature of these stationary points?
   (c) What does the first derivative test tell you about the nature of these stationary points?

7–14 Locate the critical points and identify which critical points are stationary points. ■

7. $f(x) = 4x^4 - 16x^2 + 17$
8. $f(x) = 3x^4 + 12x$
9. $f(x) = \dfrac{x+1}{x^2+3}$
10. $f(x) = \dfrac{x^2}{x^3+8}$
11. $f(x) = \sqrt[3]{x^2 - 25}$
12. $f(x) = x^2(x-1)^{2/3}$
13. $f(x) = |\sin x|$
14. $f(x) = \sin|x|$

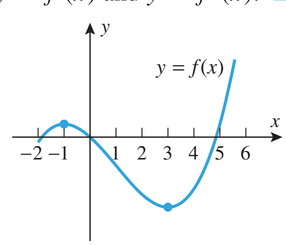
15–18 True–False Assume that f is continuous everywhere. Determine whether the statement is true or false. Explain your answer. ■

15. If f has a relative maximum at $x = 1$, then $f(1) \geq f(2)$.
16. If f has a relative maximum at $x = 1$, then $x = 1$ is a critical point for f.
17. If $f''(x) > 0$, then f has a relative minimum at $x = 1$.
18. If $p(x)$ is a polynomial such that $p'(x)$ has a simple root at $x = 1$, then p has a relative extremum at $x = 1$.

FOCUS ON CONCEPTS

19–20 The graph of a function $f(x)$ is given. Sketch graphs of $y = f'(x)$ and $y = f''(x)$. ■

19.



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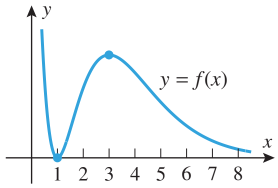
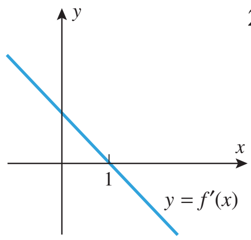


21–24 Use the graph of f' shown in the figure to estimate all values of x at which f has (a) relative minima, (b) relative maxima, and (c) inflection points. (d) Draw a rough sketch of the graph of a function f with the given derivative. ■

21.



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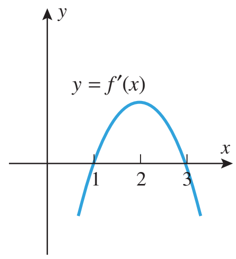

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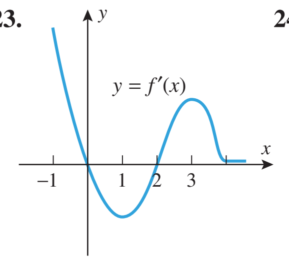

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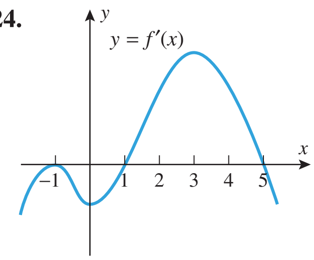


25–32 Use the given derivative to find all critical points of f, and at each critical point determine whether a relative maximum, relative minimum, or neither occurs. Assume in each case that f is continuous everywhere. ■

25. $f'(x) = x^2(x^3 - 5)$ **26.** $f'(x) = 4x^3 - 9x$

27. $f'(x) = \dfrac{2 - 3x}{\sqrt[3]{x + 2}}$ **28.** $f'(x) = \dfrac{x^2 - 7}{\sqrt[3]{x^2 + 4}}$

29. $f'(x) = xe^{1-x^2}$ **30.** $f'(x) = x^4(e^x - 3)$

31. $f'(x) = \ln\left(\dfrac{2}{1 + x^2}\right)$ **32.** $f'(x) = e^{2x} - 5e^x + 6$

33–36 Find the relative extrema using both first and second derivative tests. ■

33. $f(x) = 1 + 8x - 3x^2$ **34.** $f(x) = x^4 - 12x^3$

35. $f(x) = \sin 2x, \quad 0 < x < \pi$ **36.** $f(x) = (x - 3)e^x$

37–50 Use any method to find the relative extrema of the function f. ■

37. $f(x) = x^4 - 4x^3 + 4x^2$ **38.** $f(x) = x(x - 4)^3$

39. $f(x) = x^3(x + 1)^2$ **40.** $f(x) = x^2(x + 1)^3$

41. $f(x) = 2x + 3x^{2/3}$ **42.** $f(x) = 2x + 3x^{1/3}$

43. $f(x) = \dfrac{x + 3}{x - 2}$ **44.** $f(x) = \dfrac{x^2}{x^4 + 16}$

45. $f(x) = \ln(2 + x^2)$ **46.** $f(x) = \ln|2 + x^3|$

47. $f(x) = e^{2x} - e^x$ **48.** $f(x) = (xe^x)^2$

49. $f(x) = |3x - x^2|$ **50.** $f(x) = |1 + \sqrt[3]{x}|$

51–60 Give a graph of the polynomial and label the coordinates of the intercepts, stationary points, and inflection points. Check your work with a graphing utility. ■

51. $p(x) = x^2 - 3x - 4$ **52.** $p(x) = 1 + 8x - x^2$

53. $p(x) = 2x^3 - 3x^2 - 36x + 5$

54. $p(x) = 2 - x + 2x^2 - x^3$

55. $p(x) = (x + 1)^2(2x - x^2)$

56. $p(x) = x^4 - 6x^2 + 5$

57. $p(x) = x^4 - 2x^3 + 2x - 1$ **58.** $p(x) = 4x^3 - 9x^4$

59. $p(x) = x(x^2 - 1)^2$ **60.** $p(x) = x(x^2 - 1)^3$

61. In each part: (i) Make a conjecture about the behavior of the graph in the vicinity of its x-intercepts. (ii) Make a rough sketch of the graph based on your conjecture and the limits of the polynomial as $x \to +\infty$ and as $x \to -\infty$. (iii) Compare your sketch to the graph generated with a graphing utility.
(a) $y = x(x - 1)(x + 1)$ (b) $y = x^2(x - 1)^2(x + 1)^2$
(c) $y = x^2(x - 1)^2(x + 1)^3$ (d) $y = x(x - 1)^5(x + 1)^4$

62. Sketch the graph of $y = (x - a)^m(x - b)^n$ for the stated values of m and n, assuming that $a < b$ (six graphs in total).
(a) $m = 1, \; n = 1, 2, 3$ (b) $m = 2, \; n = 2, 3$
(c) $m = 3, \; n = 3$

63–66 Find the relative extrema in the interval $0 < x < 2\pi$, and confirm that your results are consistent with the graph of f generated with a graphing utility. ■

63. $f(x) = |\sin 2x|$ **64.** $f(x) = \sqrt{3}x + 2\sin x$

65. $f(x) = \cos^2 x$ **66.** $f(x) = \dfrac{\sin x}{2 - \cos x}$

67–70 Use a graphing utility to make a conjecture about the relative extrema of f, and then check your conjecture using either the first or second derivative test. ■

67. $f(x) = x \ln x$ **68.** $f(x) = \dfrac{2}{e^x + e^{-x}}$

69. $f(x) = x^2 e^{-2x}$ **70.** $f(x) = 10 \ln x - x$

71–72 Use a graphing utility to generate the graphs of f' and f'' over the stated interval, and then use those graphs to estimate the x-coordinates of the relative extrema of f. Check that your estimates are consistent with the graph of f. ■

71. $f(x) = x^4 - 24x^2 + 12x, \quad -5 \le x \le 5$

72. $f(x) = \sin \frac{1}{2}x \cos x, \quad -\pi/2 \le x \le \pi/2$

C **73–76** Use a CAS to graph f' and f'', and then use those graphs to estimate the x-coordinates of the relative extrema of f. Check that your estimates are consistent with the graph of f. ■

73. $f(x) = \dfrac{10x^3 - 3}{3x^2 - 5x + 8}$ **74.** $f(x) = \dfrac{\tan^{-1}(x^2 - x)}{x^2 + 4}$

75. $f(x)=\sqrt{x^4+\cos^2 x}$ **76.** $f(x)=x^2(e^{2x}-e^x)$

77. In each part, find k so that f has a relative extremum at the point where $x=3$.

(a) $f(x)=x^2+\dfrac{k}{x}$ (b) $f(x)=\dfrac{x}{x^2+k}$

78. (a) Use a CAS to graph the function
$$f(x)=\frac{x^4+1}{x^2+1}$$
and use the graph to estimate the x-coordinates of the relative extrema.

(b) Find the exact x-coordinates by using the CAS to solve the equation $f'(x)=0$.

79. Functions similar to
$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$
arise in a wide variety of statistical problems.

(a) Use the first derivative test to show that f has a relative maximum at $x=0$, and confirm this by using a graphing utility to graph f.

(b) Sketch the graph of
$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-(x-\mu)^2/2}$$
where μ is a constant, and label the coordinates of the relative extrema.

80. Functions of the form
$$f(x)=\frac{x^n e^{-x}}{n!}, \quad x>0$$
where n is a positive integer, arise in the statistical study of traffic flow.

(a) Use a graphing utility to generate the graph of f for $n=2, 3, 4,$ and 5, and make a conjecture about the number and locations of the relative extrema of f.

(b) Confirm your conjecture using the first derivative test.

81. Let h and g have relative maxima at x_0. Prove or disprove:

(a) $h+g$ has a relative maximum at x_0

(b) $h-g$ has a relative maximum at x_0.

82. Sketch some curves that show that the three parts of the first derivative test (Theorem 4.2.3) can be false without the assumption that f is continuous at x_0.

83. **Writing** Discuss the relative advantages or disadvantages of using the first derivative test versus using the second derivative test to classify candidates for relative extrema on the interior of the domain of a function. Include specific examples to illustrate your points.

84. **Writing** If $p(x)$ is a polynomial, discuss the usefulness of knowing zeros for p, p', and p'' when determining information about the graph of p.

✔ QUICK CHECK ANSWERS 4.2

1. less than or equal to **2.** 2, 7; 5 **3.** maximum; minimum **4.** (a) (0, 16) (b) (−2, 0) and (2, 0) (c) $(-2/\sqrt{3}, 64/9)$ and $(2/\sqrt{3}, 64/9)$

4.3 ANALYSIS OF FUNCTIONS III: RATIONAL FUNCTIONS, CUSPS, AND VERTICAL TANGENTS

In this section we will discuss procedures for graphing rational functions and other kinds of curves. We will also discuss the interplay between calculus and technology in curve sketching.

PROPERTIES OF GRAPHS

In many problems, the properties of interest in the graph of a function are:

- symmetries
- periodicity
- x-intercepts
- y-intercepts
- relative extrema
- concavity
- intervals of increase and decrease
- inflection points
- asymptotes
- behavior as $x\to+\infty$ or as $x\to-\infty$

Some of these properties may not be relevant in certain cases; for example, asymptotes are characteristic of rational functions but not of polynomials, and periodicity is characteristic of trigonometric functions but not of polynomial or rational functions. Thus, when analyzing the graph of a function f, it helps to know something about the general properties of the family to which it belongs.

In a given problem you will usually have a definite objective for your analysis of a graph. For example, you may be interested in showing all of the important characteristics of the function, you may only be interested in the behavior of the graph as $x \to +\infty$ or as $x \to -\infty$, or you may be interested in some specific feature such as a particular inflection point. Thus, your objectives in the problem will dictate those characteristics on which you want to focus.

GRAPHING RATIONAL FUNCTIONS

Recall that a rational function is a function of the form $f(x) = P(x)/Q(x)$ in which $P(x)$ and $Q(x)$ are polynomials. Graphs of rational functions are more complicated than those of polynomials because of the possibility of asymptotes and discontinuities (see Figure 0.3.11, for example). If $P(x)$ and $Q(x)$ have no common factors, then the information obtained in the following steps will usually be sufficient to obtain an accurate sketch of the graph of a rational function.

Graphing a Rational Function $f(x) = P(x)/Q(x)$ if $P(x)$ and $Q(x)$ have no Common Factors

Step 1. (symmetries). Determine whether there is symmetry about the y-axis or the origin.

Step 2. (x- and y-intercepts). Find the x- and y-intercepts.

Step 3. (vertical asymptotes). Find the values of x for which $Q(x) = 0$. The graph has a vertical asymptote at each such value.

Step 4. (sign of $f(x)$). The only places where $f(x)$ can change sign are at the x-intercepts or vertical asymptotes. Mark the points on the x-axis at which these occur and calculate a sample value of $f(x)$ in each of the open intervals determined by these points. This will tell you whether $f(x)$ is positive or negative over that interval.

Step 5. (end behavior). Determine the end behavior of the graph by computing the limits of $f(x)$ as $x \to +\infty$ and as $x \to -\infty$. If either limit has a finite value L, then the line $y = L$ is a horizontal asymptote.

Step 6. (derivatives). Find $f'(x)$ and $f''(x)$.

Step 7. (conclusions and graph). Analyze the sign changes of $f'(x)$ and $f''(x)$ to determine the intervals where $f(x)$ is increasing, decreasing, concave up, and concave down. Determine the locations of all stationary points, relative extrema, and inflection points. Use the sign analysis of $f(x)$ to determine the behavior of the graph in the vicinity of the vertical asymptotes. Sketch a graph of f that exhibits these conclusions.

▶ **Example 1** Sketch a graph of the equation

$$y = \frac{2x^2 - 8}{x^2 - 16}$$

and identify the locations of the intercepts, relative extrema, inflection points, and asymptotes.

Solution. The numerator and denominator have no common factors, so we will use the procedure just outlined.

- *Symmetries:* Replacing x by $-x$ does not change the equation, so the graph is symmetric about the y-axis.
- x- and y-*intercepts:* Setting $y = 0$ yields the x-intercepts $x = -2$ and $x = 2$. Setting $x = 0$ yields the y-intercept $y = \frac{1}{2}$.
- *Vertical asymptotes:* We observed above that the numerator and denominator of y have no common factors, so the graph has vertical asymptotes at the points where the denominator of y is zero, namely, at $x = -4$ and $x = 4$.
- *Sign of y:* The set of points where x-intercepts or vertical asymptotes occur is $\{-4, -2, 2, 4\}$. These points divide the x-axis into the open intervals

$$(-\infty, -4), \quad (-4, -2), \quad (-2, 2), \quad (2, 4), \quad (4, +\infty)$$

We can find the sign of y on each interval by choosing an arbitrary test point in the interval and evaluating $y = f(x)$ at the test point (Table 4.3.1). This analysis is summarized on the first line of Figure 4.3.1*a*.

Table 4.3.1

SIGN ANALYSIS OF $y = \dfrac{2x^2 - 8}{x^2 - 16}$

INTERVAL	TEST POINT	VALUE OF y	SIGN OF y
$(-\infty, -4)$	-5	$14/3$	+
$(-4, -2)$	-3	$-10/7$	−
$(-2, 2)$	0	$1/2$	+
$(2, 4)$	3	$-10/7$	−
$(4, +\infty)$	5	$14/3$	+

- *End behavior:* The limits

$$\lim_{x\to+\infty} \frac{2x^2 - 8}{x^2 - 16} = \lim_{x\to+\infty} \frac{2 - (8/x^2)}{1 - (16/x^2)} = 2$$

$$\lim_{x\to-\infty} \frac{2x^2 - 8}{x^2 - 16} = \lim_{x\to-\infty} \frac{2 - (8/x^2)}{1 - (16/x^2)} = 2$$

yield the horizontal asymptote $y = 2$.

- *Derivatives:*

$$\frac{dy}{dx} = \frac{(x^2 - 16)(4x) - (2x^2 - 8)(2x)}{(x^2 - 16)^2} = -\frac{48x}{(x^2 - 16)^2}$$

$$\frac{d^2y}{dx^2} = \frac{48(16 + 3x^2)}{(x^2 - 16)^3} \quad \text{(verify)}$$

Conclusions and graph:

- The sign analysis of y in Figure 4.3.1*a* reveals the behavior of the graph in the vicinity of the vertical asymptotes: The graph increases without bound as $x \to -4^-$ and decreases without bound as $x \to -4^+$; and the graph decreases without bound as $x \to 4^-$ and increases without bound as $x \to 4^+$ (Figure 4.3.1*b*).
- The sign analysis of dy/dx in Figure 4.3.1*a* shows that the graph is increasing to the left of $x = 0$ and is decreasing to the right of $x = 0$. Thus, there is a relative maximum at the stationary point $x = 0$. There are no relative minima.
- The sign analysis of d^2y/dx^2 in Figure 4.3.1*a* shows that the graph is concave up to the left of $x = -4$, is concave down between $x = -4$ and $x = 4$, and is concave up to the right of $x = 4$. There are no inflection points.

The procedure we stated for graphing a rational function $P(x)/Q(x)$ applies only if the polynomials $P(x)$ and $Q(x)$ have no common factors. How would you find the graph if those polynomials have common factors?

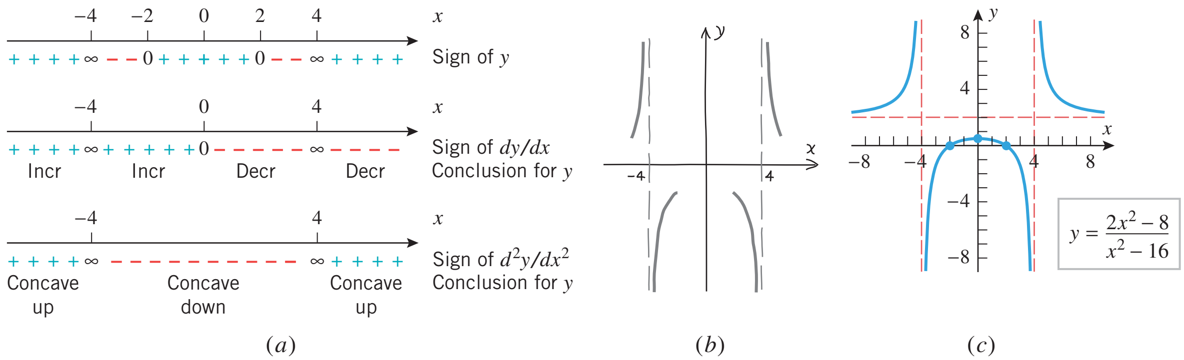
The graph is shown in Figure 4.3.1*c*. ◀

▶ **Example 2** Sketch a graph of

$$y = \frac{x^2 - 1}{x^3}$$

and identify the locations of all asymptotes, intercepts, relative extrema, and inflection points.

Solution. The numerator and denominator have no common factors, so we will use the procedure outlined previously.



▲ **Figure 4.3.1**

Table 4.3.2

SIGN ANALYSIS OF $y = \dfrac{x^2-1}{x^3}$

INTERVAL	TEST POINT	VALUE OF y	SIGN OF y
$(-\infty, -1)$	-2	$-\frac{3}{8}$	$-$
$(-1, 0)$	$-\frac{1}{2}$	6	$+$
$(0, 1)$	$\frac{1}{2}$	-6	$-$
$(1, +\infty)$	2	$\frac{3}{8}$	$+$

- *Symmetries:* Replacing x by $-x$ and y by $-y$ yields an equation that simplifies to the original equation, so the graph is symmetric about the origin.
- *x- and y-intercepts:* Setting $y = 0$ yields the x-intercepts $x = -1$ and $x = 1$. Setting $x = 0$ leads to a division by zero, so there is no y-intercept.
- *Vertical asymptotes:* Setting $x^3 = 0$ yields the solution $x = 0$. This is not a root of $x^2 - 1$, so $x = 0$ is a vertical asymptote.
- *Sign of y:* The set of points where x-intercepts or vertical asymptotes occur is $\{-1, 0, 1\}$. These points divide the x-axis into the open intervals

$$(-\infty, -1), \quad (-1, 0), \quad (0, 1), \quad (1, +\infty)$$

Table 4.3.2 uses the method of test points to produce the sign of y on each of these intervals.

- *End behavior:* The limits

$$\lim_{x\to+\infty} \frac{x^2-1}{x^3} = \lim_{x\to+\infty}\left(\frac{1}{x} - \frac{1}{x^3}\right) = 0$$
$$\lim_{x\to-\infty} \frac{x^2-1}{x^3} = \lim_{x\to-\infty}\left(\frac{1}{x} - \frac{1}{x^3}\right) = 0$$

yield the horizontal asymptote $y = 0$.

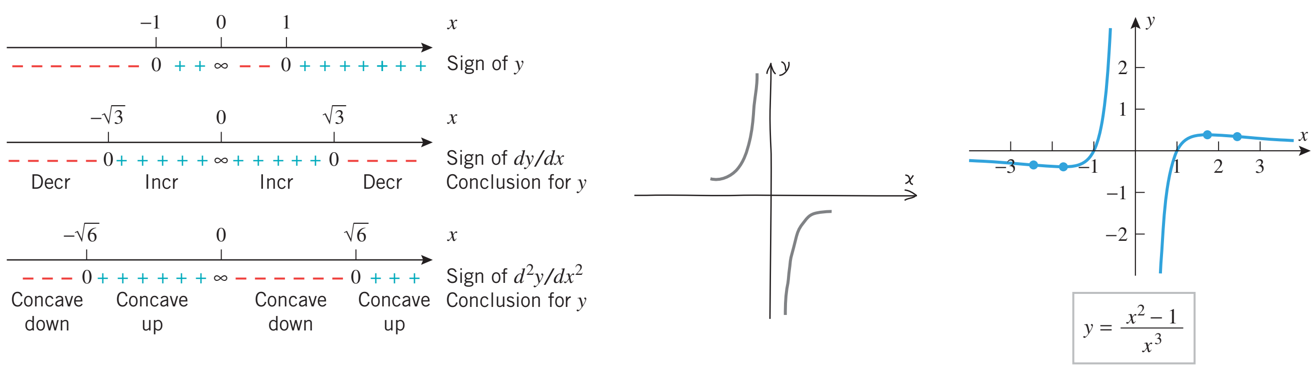
- *Derivatives:*

$$\frac{dy}{dx} = \frac{x^3(2x) - (x^2-1)(3x^2)}{(x^3)^2} = \frac{3-x^2}{x^4} = \frac{(\sqrt{3}+x)(\sqrt{3}-x)}{x^4}$$
$$\frac{d^2y}{dx^2} = \frac{x^4(-2x) - (3-x^2)(4x^3)}{(x^4)^2} = \frac{2(x^2-6)}{x^5} = \frac{2(x-\sqrt{6})(x+\sqrt{6})}{x^5}$$

Conclusions and graph:

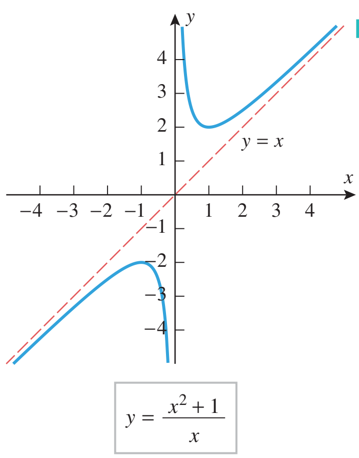
- The sign analysis of y in Figure 4.3.2*a* reveals the behavior of the graph in the vicinity of the vertical asymptote $x = 0$: The graph increases without bound as $x \to 0^-$ and decreases without bound as $x \to 0^+$ (Figure 4.3.2*b*).
- The sign analysis of dy/dx in Figure 4.3.2*a* shows that there is a relative minimum at $x = -\sqrt{3}$ and a relative maximum at $x = \sqrt{3}$.
- The sign analysis of d^2y/dx^2 in Figure 4.3.2*a* shows that the graph changes concavity at the vertical asymptote $x = 0$ and that there are inflection points at $x = -\sqrt{6}$ and $x = \sqrt{6}$.

The graph is shown in Figure 4.3.2c. To produce a slightly more accurate sketch, we used a graphing utility to help plot the relative extrema and inflection points. You should confirm that the approximate coordinates of the inflection points are $(-2.45, -0.34)$ and $(2.45, 0.34)$ and that the approximate coordinates of the relative minimum and relative maximum are $(-1.73, -0.38)$ and $(1.73, 0.38)$, respectively. ◀

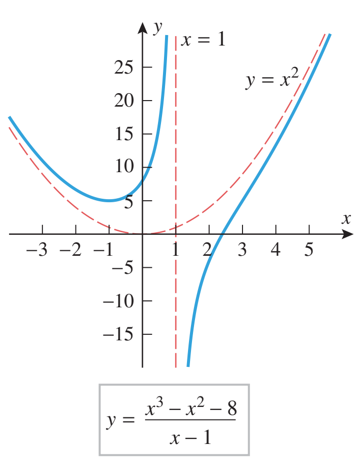


(a) (b) (c)

▲ **Figure 4.3.2**



▲ **Figure 4.3.3**



▲ **Figure 4.3.4**

RATIONAL FUNCTIONS WITH OBLIQUE OR CURVILINEAR ASYMPTOTES

In the rational functions of Examples 1 and 2, the degree of the numerator did not exceed the degree of the denominator, and the asymptotes were either vertical or horizontal. If the numerator of a rational function has greater degree than the denominator, then other kinds of "asymptotes" are possible. For example, consider the rational functions

$$f(x) = \frac{x^2+1}{x} \quad \text{and} \quad g(x) = \frac{x^3-x^2-8}{x-1} \tag{1}$$

By division we can rewrite these as

$$f(x) = x + \frac{1}{x} \quad \text{and} \quad g(x) = x^2 - \frac{8}{x-1}$$

Since the second terms both approach 0 as $x \to +\infty$ or as $x \to -\infty$, it follows that

$$(f(x) - x) \to 0 \quad \text{as } x \to +\infty \text{ or as } x \to -\infty$$
$$(g(x) - x^2) \to 0 \quad \text{as } x \to +\infty \text{ or as } x \to -\infty$$

Geometrically, this means that the graph of $y = f(x)$ eventually gets closer and closer to the line $y = x$ as $x \to +\infty$ or as $x \to -\infty$. The line $y = x$ is called an ***oblique*** or ***slant asymptote*** of f. Similarly, the graph of $y = g(x)$ eventually gets closer and closer to the parabola $y = x^2$ as $x \to +\infty$ or as $x \to -\infty$. The parabola is called a ***curvilinear asymptote*** of g. The graphs of the functions in (1) are shown in Figures 4.3.3 and 4.3.4.

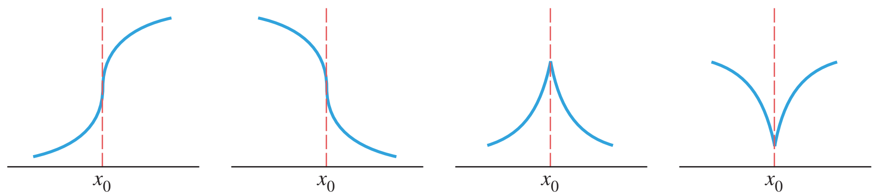
In general, if $f(x) = P(x)/Q(x)$ is a rational function, then we can find quotient and remainder polynomials $q(x)$ and $r(x)$ such that

$$f(x) = q(x) + \frac{r(x)}{Q(x)}$$

and the degree of $r(x)$ is less than the degree of $Q(x)$. Then $r(x)/Q(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$, so $y = q(x)$ is an asymptote of f. This asymptote will be an oblique line if the degree of $P(x)$ is one greater than the degree of $Q(x)$, and it will be curvilinear if the degree of $P(x)$ exceeds that of $Q(x)$ by two or more. Problems involving these kinds of asymptotes are given in the exercises (Exercises 17 and 18).

GRAPHS WITH VERTICAL TANGENTS AND CUSPS

Figure 4.3.5 shows four curve elements that are commonly found in graphs of functions that involve radicals or fractional exponents. In all four cases, the function is not differentiable at x_0 because the secant line through $(x_0, f(x_0))$ and $(x, f(x))$ approaches a vertical position as x approaches x_0 from either side. Thus, in each case, the curve has a vertical tangent line at $(x_0, f(x_0))$. In parts (a) and (b) of the figure, there is an inflection point at x_0 because there is a change in concavity at that point. In parts (c) and (d), where $f'(x)$ approaches $+\infty$ from one side of x_0 and $-\infty$ from the other side, we say that the graph has a ***cusp*** at x_0.



(a) $\lim_{x\to x_0^+} f'(x) = +\infty$, $\lim_{x\to x_0^-} f'(x) = +\infty$

(b) $\lim_{x\to x_0^+} f'(x) = -\infty$, $\lim_{x\to x_0^-} f'(x) = -\infty$

(c) $\lim_{x\to x_0^+} f'(x) = -\infty$, $\lim_{x\to x_0^-} f'(x) = +\infty$

(d) $\lim_{x\to x_0^+} f'(x) = +\infty$, $\lim_{x\to x_0^-} f'(x) = -\infty$

Figure 4.3.5

The steps that are used to sketch the graph of a rational function can serve as guidelines for sketching graphs of other types of functions. This is illustrated in Examples 3, 4, and 5.

Example 3 Sketch the graph of $y = (x-4)^{2/3}$.

- *Symmetries:* There are no symmetries about the coordinate axes or the origin (verify). However, the graph of $y = (x-4)^{2/3}$ is symmetric about the line $x = 4$ since it is a translation (4 units to the right) of the graph of $y = x^{2/3}$, which is symmetric about the y-axis.
- *x- and y-intercepts:* Setting $y = 0$ yields the x-intercept $x = 4$. Setting $x = 0$ yields the y-intercept $y = \sqrt[3]{16} \approx 2.5$.
- *Vertical asymptotes:* None, since $f(x) = (x-4)^{2/3}$ is continuous everywhere.
- *End behavior:* The graph has no horizontal asymptotes since

$$\lim_{x\to+\infty} (x-4)^{2/3} = +\infty \quad \text{and} \quad \lim_{x\to-\infty} (x-4)^{2/3} = +\infty$$

- *Derivatives:*

$$\frac{dy}{dx} = f'(x) = \frac{2}{3}(x-4)^{-1/3} = \frac{2}{3(x-4)^{1/3}}$$
$$\frac{d^2y}{dx^2} = f''(x) = -\frac{2}{9}(x-4)^{-4/3} = -\frac{2}{9(x-4)^{4/3}}$$

- *Vertical tangent lines:* There is a vertical tangent line and cusp at $x = 4$ of the type in Figure 4.3.5d since $f(x) = (x-4)^{2/3}$ is continuous at $x = 4$ and

$$\lim_{x\to4^+} f'(x) = \lim_{x\to4^+} \frac{2}{3(x-4)^{1/3}} = +\infty$$
$$\lim_{x\to4^-} f'(x) = \lim_{x\to4^-} \frac{2}{3(x-4)^{1/3}} = -\infty$$

Conclusions and graph:

- The function $f(x) = (x-4)^{2/3} = ((x-4)^{1/3})^2$ is nonnegative for all x. There is a zero for f at $x = 4$.

- There is a critical point at $x = 4$ since f is not differentiable there. We saw above that a cusp occurs at this point. The sign analysis of dy/dx in Figure 4.3.6*a* and the first derivative test show that there is a relative minimum at this cusp since $f'(x) < 0$ if $x < 4$ and $f'(x) > 0$ if $x > 4$.
- The sign analysis of d^2y/dx^2 in Figure 4.3.6*a* shows that the graph is concave down on both sides of the cusp.

The graph is shown in Figure 4.3.6*b*. ◀

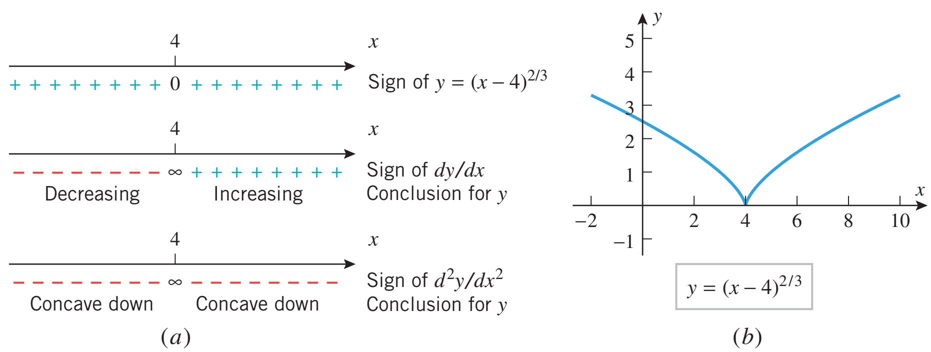


Figure 4.3.6

▶ **Example 4** Sketch the graph of $y = 6x^{1/3} + 3x^{4/3}$.

Solution. It will help in our analysis to write

$$f(x) = 6x^{1/3} + 3x^{4/3} = 3x^{1/3}(2 + x)$$

- *Symmetries:* There are no symmetries about the coordinate axes or the origin (verify).
- *x- and y-intercepts:* Setting $y = 3x^{1/3}(2 + x) = 0$ yields the x-intercepts $x = 0$ and $x = -2$. Setting $x = 0$ yields the y-intercept $y = 0$.
- *Vertical asymptotes:* None, since $f(x) = 6x^{1/3} + 3x^{4/3}$ is continuous everywhere.
- *End behavior:* The graph has no horizontal asymptotes since

$$\lim_{x\to+\infty}(6x^{1/3} + 3x^{4/3}) = \lim_{x\to+\infty} 3x^{1/3}(2 + x) = +\infty$$

$$\lim_{x\to-\infty}(6x^{1/3} + 3x^{4/3}) = \lim_{x\to-\infty} 3x^{1/3}(2 + x) = +\infty$$

- *Derivatives:*

$$\frac{dy}{dx} = f'(x) = 2x^{-2/3} + 4x^{1/3} = 2x^{-2/3}(1 + 2x) = \frac{2(2x + 1)}{x^{2/3}}$$

$$\frac{d^2y}{dx^2} = f''(x) = -\frac{4}{3}x^{-5/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-5/3}(-1 + x) = \frac{4(x - 1)}{3x^{5/3}}$$

- *Vertical tangent lines:* There is a vertical tangent line at $x = 0$ since f is continuous there and

$$\lim_{x\to0^+} f'(x) = \lim_{x\to0^+} \frac{2(2x + 1)}{x^{2/3}} = +\infty$$

$$\lim_{x\to0^-} f'(x) = \lim_{x\to0^-} \frac{2(2x + 1)}{x^{2/3}} = +\infty$$

This and the change in concavity at $x = 0$ mean that $(0, 0)$ is an inflection point of the type in Figure 4.3.5*a*.

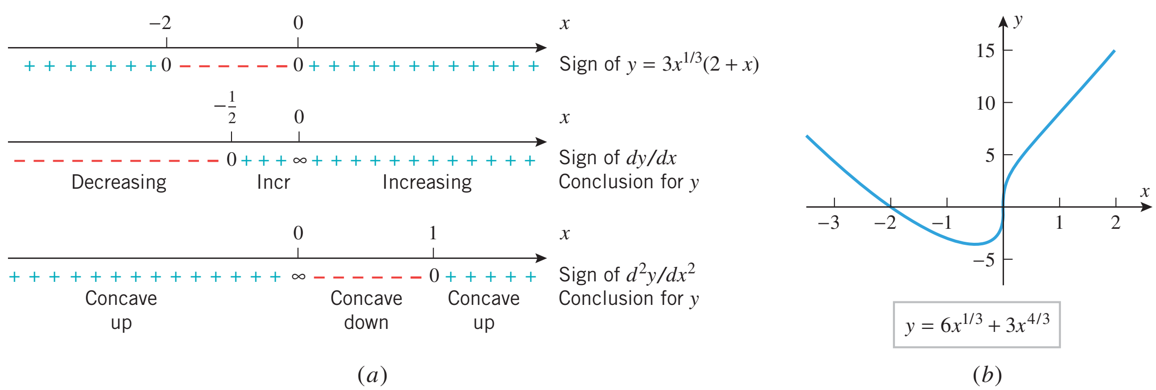
Conclusions and graph:

- From the sign analysis of y in Figure 4.3.7*a*, the graph is below the x-axis between the x-intercepts $x = -2$ and $x = 0$ and is above the x-axis if $x < -2$ or $x > 0$.
- From the formula for dy/dx we see that there is a stationary point at $x = -\frac{1}{2}$ and a critical point at $x = 0$ at which f is not differentiable. We saw above that a vertical tangent line and inflection point are at that critical point.
- The sign analysis of dy/dx in Figure 4.3.7*a* and the first derivative test show that there is a relative minimum at the stationary point at $x = -\frac{1}{2}$ (verify).
- The sign analysis of d^2y/dx^2 in Figure 4.3.7*a* shows that in addition to the inflection point at the vertical tangent there is an inflection point at $x = 1$ at which the graph changes from concave down to concave up.

TECHNOLOGY MASTERY

The graph in Figure 4.3.7*b* was generated with a graphing utility. However, the inflection point at $x = 1$ is so subtle that it is not evident from this graph. See if you can produce a version of this graph with your graphing utility that makes the inflection point evident.

The graph is shown in Figure 4.3.7*b*. ◀



▲ **Figure 4.3.7**

GRAPHING OTHER KINDS OF FUNCTIONS

We have discussed methods for graphing polynomials, rational functions, and functions with cusps and vertical tangent lines. The same calculus tools that we used to analyze these functions can also be used to analyze and graph trigonometric functions, logarithmic and exponential functions, and an endless variety of other kinds of functions.

▶ **Example 5** Sketch the graph of $y = e^{-x^2/2}$ and identify the locations of all relative extrema and inflection points.

Solution.

- *Symmetries:* Replacing x by $-x$ does not change the equation, so the graph is symmetric about the y-axis.
- *x- and y-intercepts:* Setting $y = 0$ leads to the equation $e^{-x^2/2} = 0$, which has no solutions since all powers of e have positive values. Thus, there are no x-intercepts. Setting $x = 0$ yields the y-intercept $y = 1$.
- *Vertical asymptotes:* There are no vertical asymptotes since $e^{-x^2/2}$ is continuous on $(-\infty, +\infty)$.
- *End behavior:* The x-axis ($y = 0$) is a horizontal asymptote since

$$\lim_{x\to-\infty} e^{-x^2/2} = \lim_{x\to+\infty} e^{-x^2/2} = 0$$

- *Derivatives:*

$$\frac{dy}{dx} = e^{-x^2/2}\frac{d}{dx}\left[-\frac{x^2}{2}\right] = -xe^{-x^2/2}$$

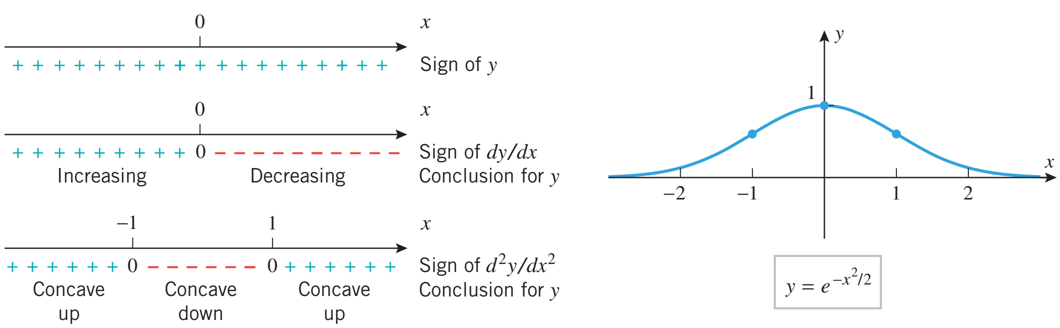
$$\frac{d^2y}{dx^2} = -x\frac{d}{dx}[e^{-x^2/2}] + e^{-x^2/2}\frac{d}{dx}[-x]$$

$$= x^2e^{-x^2/2} - e^{-x^2/2} = (x^2-1)e^{-x^2/2}$$

Conclusions and graph:

- The sign analysis of y in Figure 4.3.8a is based on the fact that $e^{-x^2/2} > 0$ for all x. This shows that the graph is always above the x-axis.
- The sign analysis of dy/dx in Figure 4.3.8a is based on the fact that $dy/dx = -xe^{-x^2/2}$ has the same sign as $-x$. This analysis and the first derivative test show that there is a stationary point at $x = 0$ at which there is a relative maximum. The value of y at the relative maximum is $y = e^0 = 1$.
- The sign analysis of d^2y/dx^2 in Figure 4.3.8a is based on the fact that $d^2y/dx^2 = (x^2-1)e^{-x^2/2}$ has the same sign as x^2-1. This analysis shows that there are inflection points at $x = -1$ and $x = 1$. The graph changes from concave up to concave down at $x = -1$ and from concave down to concave up at $x = 1$. The coordinates of the inflection points are $(-1, e^{-1/2}) \approx (-1, 0.61)$ and $(1, e^{-1/2}) \approx (1, 0.61)$.

The graph is shown in Figure 4.3.8b. ◀

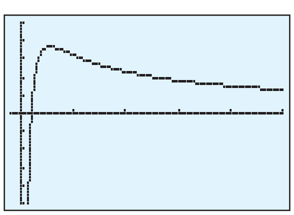


(a) (b)

▲ **Figure 4.3.8**

GRAPHING USING CALCULUS AND TECHNOLOGY TOGETHER

Thus far in this chapter we have used calculus to produce graphs of functions; the graph was the end result. Now we will work in the reverse direction by *starting* with a graph produced by a graphing utility. Our goal will be to use the tools of calculus to determine the exact locations of relative extrema, inflection points, and other features suggested by that graph and to determine whether the graph may be missing some important features that we would like to see.



$[-1, 25] \times [-0.5, 0.5]$
$x\text{Scl} = 5,\ y\text{Scl} = 0.2$

$y = \dfrac{\ln x}{x}$

▲ **Figure 4.3.9**

▶ **Example 6** Use a graphing utility to generate the graph of $f(x) = (\ln x)/x$, and discuss what it tells you about relative extrema, inflection points, asymptotes, and end behavior. Use calculus to find the locations of all key features of the graph.

Solution. Figure 4.3.9 shows a graph of f produced by a graphing utility. The graph suggests that there is an x-intercept near $x = 1$, a relative maximum somewhere between

$x = 0$ and $x = 5$, an inflection point near $x = 5$, a vertical asymptote at $x = 0$, and possibly a horizontal asymptote $y = 0$. For a more precise analysis of this information we need to consider the derivatives

$$f'(x) = \frac{x\left(\dfrac{1}{x}\right) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f''(x) = \frac{x^2\left(-\dfrac{1}{x}\right) - (1 - \ln x)(2x)}{x^4} = \frac{2x \ln x - 3x}{x^4} = \frac{2 \ln x - 3}{x^3}$$

- *Relative extrema:* Solving $f'(x) = 0$ yields the stationary point $x = e$ (verify). Since
$$f''(e) = \frac{2-3}{e^3} = -\frac{1}{e^3} < 0$$
there is a relative maximum at $x = e \approx 2.7$ by the second derivative test.
- *Inflection points:* Since $f(x) = (\ln x)/x$ is only defined for positive values of x, the second derivative $f''(x)$ has the same sign as $2 \ln x - 3$. We leave it for you to use the inequalities $(2 \ln x - 3) < 0$ and $(2 \ln x - 3) > 0$ to show that $f''(x) < 0$ if $x < e^{3/2}$ and $f''(x) > 0$ if $x > e^{3/2}$. Thus, there is an inflection point at $x = e^{3/2} \approx 4.5$.
- *Asymptotes:* Applying L'Hôpital's rule we have
$$\lim_{x \to +\infty} \frac{\ln x}{x} = \lim_{x \to +\infty} \frac{(1/x)}{1} = \lim_{x \to +\infty} \frac{1}{x} = 0$$
so that $y = 0$ is a horizontal asymptote. Also, there is a vertical asymptote at $x = 0$ since
$$\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$$
(why?).
- *Intercepts:* Setting $f(x) = 0$ yields $(\ln x)/x = 0$. The only real solution of this equation is $x = 1$, so there is an x-intercept at this point. ◀

✔ QUICK CHECK EXERCISES 4.3 *(See page 266 for answers.)*

1. Let $f(x) = \dfrac{3(x+1)(x-3)}{(x+2)(x-4)}$. Given that
$$f'(x) = \frac{-30(x-1)}{(x+2)^2(x-4)^2}, \qquad f''(x) = \frac{90(x^2 - 2x + 4)}{(x+2)^3(x-4)^3}$$
determine the following properties of the graph of f.
(a) The x- and y-intercepts are ______.
(b) The vertical asymptotes are ______.
(c) The horizontal asymptote is ______.
(d) The graph is above the x-axis on the intervals ______.
(e) The graph is increasing on the intervals ______.
(f) The graph is concave up on the intervals ______.
(g) The relative maximum point on the graph is ______.

2. Let $f(x) = \dfrac{x^2 - 4}{x^{8/3}}$. Given that
$$f'(x) = \frac{-2(x^2 - 16)}{3x^{11/3}}, \qquad f''(x) = \frac{2(5x^2 - 176)}{9x^{14/3}}$$
determine the following properties of the graph of f.
(a) The x-intercepts are ______.
(b) The vertical asymptote is ______.
(c) The horizontal asymptote is ______.
(d) The graph is above the x-axis on the intervals ______.
(e) The graph is increasing on the intervals ______.
(f) The graph is concave up on the intervals ______.
(g) Inflection points occur at $x =$ ______.

3. Let $f(x) = (x-2)^2 e^{x/2}$. Given that
$$f'(x) = \tfrac{1}{2}(x^2 - 4)e^{x/2}, \qquad f''(x) = \tfrac{1}{4}(x^2 + 4x - 4)e^{x/2}$$
determine the following properties of the graph of f.
(a) The horizontal asymptote is ______.
(b) The graph is above the x-axis on the intervals ______.
(c) The graph is increasing on the intervals ______.
(d) The graph is concave up on the intervals ______.
(e) The relative minimum point on the graph is ______.
(f) The relative maximum point on the graph is ______.
(g) Inflection points occur at $x =$ ______.

EXERCISE SET 4.3 Graphing Utility

1–14 Give a graph of the rational function and label the coordinates of the stationary points and inflection points. Show the horizontal and vertical asymptotes and label them with their equations. Label point(s), if any, where the graph crosses a horizontal asymptote. Check your work with a graphing utility.

1. $\dfrac{2x-6}{4-x}$
2. $\dfrac{8}{x^2-4}$
3. $\dfrac{x}{x^2-4}$
4. $\dfrac{x^2}{x^2-4}$
5. $\dfrac{x^2}{x^2+4}$
6. $\dfrac{(x^2-1)^2}{x^4+1}$
7. $\dfrac{x^3+1}{x^3-1}$
8. $2-\dfrac{1}{3x^2+x^3}$
9. $\dfrac{4}{x^2}-\dfrac{2}{x}+3$
10. $\dfrac{3(x+1)^2}{(x-1)^2}$
11. $\dfrac{(3x+1)^2}{(x-1)^2}$
12. $3+\dfrac{x+1}{(x-1)^4}$
13. $\dfrac{x^2+x}{1-x^2}$
14. $\dfrac{x^2}{1-x^3}$

15–16 In each part, make a rough sketch of the graph using asymptotes and appropriate limits but no derivatives. Compare your graph to that generated with a graphing utility.

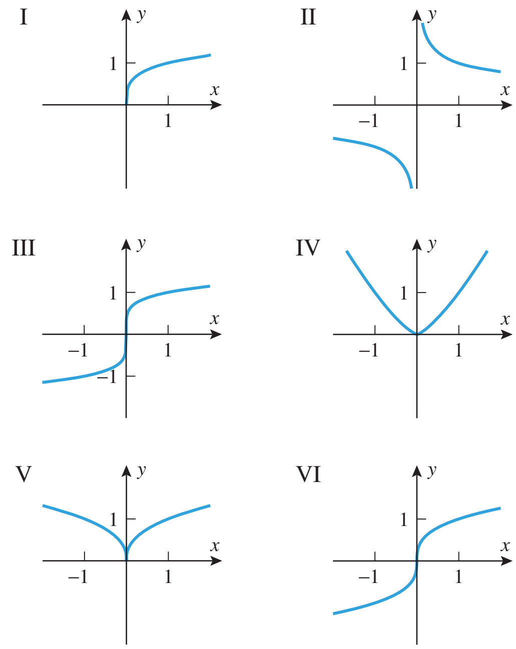
15. (a) $y=\dfrac{3x^2-8}{x^2-4}$ (b) $y=\dfrac{x^2+2x}{x^2-1}$
16. (a) $y=\dfrac{2x-x^2}{x^2+x-2}$ (b) $y=\dfrac{x^2}{x^2-x-2}$
17. Show that $y=x+3$ is an oblique asymptote of the graph of $f(x)=x^2/(x-3)$. Sketch the graph of $y=f(x)$ showing this asymptotic behavior.
18. Show that $y=3-x^2$ is a curvilinear asymptote of the graph of $f(x)=(2+3x-x^3)/x$. Sketch the graph of $y=f(x)$ showing this asymptotic behavior.

19–24 Sketch a graph of the rational function and label the coordinates of the stationary points and inflection points. Show the horizontal, vertical, oblique, and curvilinear asymptotes and label them with their equations. Label point(s), if any, where the graph crosses an asymptote. Check your work with a graphing utility.

19. $x^2-\dfrac{1}{x}$
20. $\dfrac{x^2-2}{x}$
21. $\dfrac{(x-2)^3}{x^2}$
22. $x-\dfrac{1}{x}-\dfrac{1}{x^2}$
23. $\dfrac{x^3-4x-8}{x+2}$
24. $\dfrac{x^5}{x^2+1}$

FOCUS ON CONCEPTS

25. In each part, match the function with graphs I–VI.
(a) $x^{1/3}$ (b) $x^{1/4}$ (c) $x^{1/5}$
(d) $x^{2/5}$ (e) $x^{4/3}$ (f) $x^{-1/3}$



▲ Figure Ex-25

26. Sketch the general shape of the graph of $y=x^{1/n}$, and then explain in words what happens to the shape of the graph as n increases if
(a) n is a positive even integer
(b) n is a positive odd integer.

27–30 True–False Determine whether the statement is true or false. Explain your answer.

27. Suppose that $f(x)=P(x)/Q(x)$, where P and Q are polynomials with no common factors. If $y=5$ is a horizontal asymptote for the graph of f, then P and Q have the same degree.
28. If the graph of f has a vertical asymptote at $x=1$, then f cannot be continuous at $x=1$.
29. If the graph of f' has a vertical asymptote at $x=1$, then f cannot be continuous at $x=1$.
30. If the graph of f has a cusp at $x=1$, then f cannot have an inflection point at $x=1$.

31–38 Give a graph of the function and identify the locations of all critical points and inflection points. Check your work with a graphing utility.

31. $\sqrt{4x^2-1}$
32. $\sqrt[3]{x^2-4}$
33. $2x+3x^{2/3}$
34. $2x^2-3x^{4/3}$
35. $4x^{1/3}-x^{4/3}$
36. $5x^{2/3}+x^{5/3}$
37. $\dfrac{8+x}{2+\sqrt[3]{x}}$
38. $\dfrac{8(\sqrt{x}-1)}{x}$

39–44 Give a graph of the function and identify the locations of all relative extrema and inflection points. Check your work with a graphing utility.

39. $x + \sin x$ **40.** $x - \tan x$

41. $\sqrt{3}\cos x + \sin x$ **42.** $\sin x + \cos x$

43. $\sin^2 x - \cos x, \quad -\pi \le x \le 3\pi$

44. $\sqrt{\tan x}, \quad 0 \le x < \pi/2$

45–54 Using L'Hôpital's rule (Section 3.6) one can verify that

$$\lim_{x\to+\infty} \frac{e^x}{x} = +\infty, \quad \lim_{x\to+\infty} \frac{x}{e^x} = 0, \quad \lim_{x\to-\infty} xe^x = 0$$

In these exercises: (a) Use these results, as necessary, to find the limits of $f(x)$ as $x \to +\infty$ and as $x \to -\infty$. (b) Sketch a graph of $f(x)$ and identify all relative extrema, inflection points, and asymptotes (as appropriate). Check your work with a graphing utility.

45. $f(x) = xe^x$ **46.** $f(x) = xe^{-x}$

47. $f(x) = x^2e^{-2x}$ **48.** $f(x) = x^2e^{2x}$

49. $f(x) = x^2e^{-x^2}$ **50.** $f(x) = e^{-1/x^2}$

51. $f(x) = \dfrac{e^x}{1-x}$ **52.** $f(x) = x^{2/3}e^x$

53. $f(x) = x^2e^{1-x}$ **54.** $f(x) = x^3e^{x-1}$

55–60 Using L'Hôpital's rule (Section 3.6) one can verify that

$$\lim_{x\to+\infty} \frac{\ln x}{x^r} = 0, \quad \lim_{x\to+\infty} \frac{x^r}{\ln x} = +\infty, \quad \lim_{x\to 0^+} x^r \ln x = 0$$

for any positive real number r. In these exercises: (a) Use these results, as necessary, to find the limits of $f(x)$ as $x \to +\infty$ and as $x \to 0^+$. (b) Sketch a graph of $f(x)$ and identify all relative extrema, inflection points, and asymptotes (as appropriate). Check your work with a graphing utility.

55. $f(x) = x \ln x$ **56.** $f(x) = x^2 \ln x$

57. $f(x) = x^2 \ln(2x)$ **58.** $f(x) = \ln(x^2 + 1)$

59. $f(x) = x^{2/3} \ln x$ **60.** $f(x) = x^{-1/3} \ln x$

FOCUS ON CONCEPTS

61. Consider the family of curves $y = xe^{-bx}$ $(b > 0)$.
(a) Use a graphing utility to generate some members of this family.
(b) Discuss the effect of varying b on the shape of the graph, and discuss the locations of the relative extrema and inflection points.

62. Consider the family of curves $y = e^{-bx^2}$ $(b > 0)$.
(a) Use a graphing utility to generate some members of this family.
(b) Discuss the effect of varying b on the shape of the graph, and discuss the locations of the relative extrema and inflection points.

63. (a) Determine whether the following limits exist, and if so, find them:

$$\lim_{x\to+\infty} e^x \cos x, \quad \lim_{x\to-\infty} e^x \cos x$$

(b) Sketch the graphs of the equations $y = e^x$, $y = -e^x$, and $y = e^x \cos x$ in the same coordinate system, and label any points of intersection.
(c) Use a graphing utility to generate some members of the family $y = e^{ax} \cos bx$ $(a > 0$ and $b > 0)$, and discuss the effect of varying a and b on the shape of the curve.

64. Consider the family of curves $y = x^n e^{-x^2/n}$, where n is a positive integer.
(a) Use a graphing utility to generate some members of this family.
(b) Discuss the effect of varying n on the shape of the graph, and discuss the locations of the relative extrema and inflection points.

65. The accompanying figure shows the graph of the *derivative* of a function h that is defined and continuous on the interval $(-\infty, +\infty)$. Assume that the graph of h' has a vertical asymptote at $x = 3$ and that

$$h'(x) \to 0^+ \text{ as } x \to -\infty$$
$$h'(x) \to -\infty \text{ as } x \to +\infty$$

(a) What are the critical points for $h(x)$?
(b) Identify the intervals on which $h(x)$ is increasing.
(c) Identify the x-coordinates of relative extrema for $h(x)$ and classify each as a relative maximum or relative minimum.
(d) Estimate the x-coordinates of inflection points for $h(x)$.

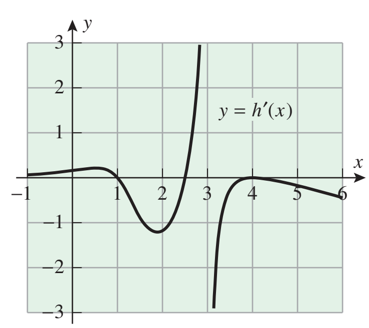


Figure Ex-65

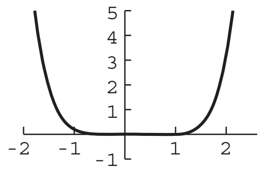
66. Let $f(x) = (1 - 2x)h(x)$, where $h(x)$ is as given in Exercise 65. Suppose that $x = 5$ is a critical point for $f(x)$.
(a) Estimate $h(5)$.
(b) Use the second derivative test to determine whether $f(x)$ has a relative maximum or a relative minimum at $x = 5$.

67. A rectangular plot of land is to be fenced off so that the area enclosed will be 400 ft^2. Let L be the length of fencing needed and x the length of one side of the rectangle. Show that $L = 2x + 800/x$ for $x > 0$, and sketch the graph of L versus x for $x > 0$.

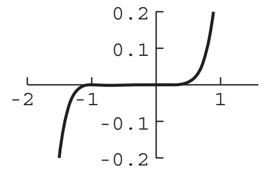
68. A box with a square base and open top is to be made from sheet metal so that its volume is 500 in^3. Let S be the area of the surface of the box and x the length of a side of the square base. Show that $S = x^2 + 2000/x$ for $x > 0$, and sketch the graph of S versus x for $x > 0$.

69. The accompanying figure shows a computer-generated graph of the polynomial $y = 0.1x^5(x - 1)$ using a viewing window of $[-2, 2.5] \times [-1, 5]$. Show that the choice of the vertical scale caused the computer to miss important features of the graph. Find the features that were missed and make your own sketch of the graph that shows the missing features.

70. The accompanying figure shows a computer-generated graph of the polynomial $y = 0.1x^5(x + 1)^2$ using a viewing window of $[-2, 1.5] \times [-0.2, 0.2]$. Show that the choice of the vertical scale caused the computer to miss important features of the graph. Find the features that were missed and make your own sketch of the graph that shows the missing features.



▲ **Figure Ex-69**



▲ **Figure Ex-70**

71. Writing Suppose that $x = x_0$ is a point at which a function f is continuous but not differentiable and that $f'(x)$ approaches different finite limits as x approaches x_0 from either side. Invent your own term to describe the graph of f at such a point and discuss the appropriateness of your term.

72. Writing Suppose that the graph of a function f is obtained using a graphing utility. Discuss the information that calculus techniques can provide about f to add to what can already be inferred about f from the graph as shown on your utility's display.

✔ QUICK CHECK ANSWERS 4.3

1. (a) $(-1, 0), (3, 0), \left(0, \frac{9}{8}\right)$ (b) $x = -2$ and $x = 4$ (c) $y = 3$ (d) $(-\infty, -2), (-1, 3),$ and $(4, +\infty)$ (e) $(-\infty, -2)$ and $(-2, 1]$ (f) $(-\infty, -2)$ and $(4, +\infty)$ (g) $\left(1, \frac{4}{3}\right)$ **2.** (a) $(-2, 0), (2, 0)$ (b) $x = 0$ (c) $y = 0$ (d) $(-\infty, -2)$ and $(2, +\infty)$ (e) $(-\infty, -4]$ and $(0, 4]$ (f) $(-\infty, -4\sqrt{11/5})$ and $(4\sqrt{11/5}, +\infty)$ (g) $\pm 4\sqrt{11/5} \approx \pm 5.93$ **3.** (a) $y = 0$ (as $x \to -\infty$) (b) $(-\infty, 2)$ and $(2, +\infty)$ (c) $(-\infty, -2]$ and $[2, +\infty)$ (d) $(-\infty, -2 - 2\sqrt{2})$ and $(-2 + 2\sqrt{2}, +\infty)$ (e) $(2, 0)$ (f) $(-2, 16e^{-1}) \approx (-2, 5.89)$ (g) $-2 \pm 2\sqrt{2}$

4.4 ABSOLUTE MAXIMA AND MINIMA

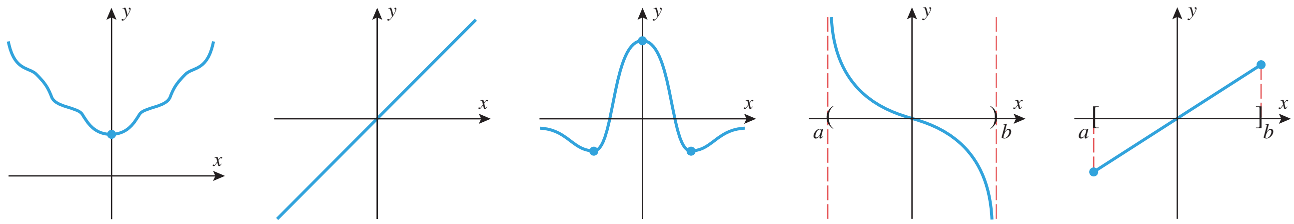
At the beginning of Section 4.2 we observed that if the graph of a function f is viewed as a two-dimensional mountain range (Figure 4.2.1), then the relative maxima and minima correspond to the tops of the hills and the bottoms of the valleys; that is, they are the high and low points in their immediate vicinity. In this section we will be concerned with the more encompassing problem of finding the highest and lowest points over the entire mountain range, that is, we will be looking for the top of the highest hill and the bottom of the deepest valley. In mathematical terms, we will be looking for the largest and smallest values of a function over an interval.

■ ABSOLUTE EXTREMA

We will begin with some terminology for describing the largest and smallest values of a function on an interval.

4.4.1 DEFINITION Consider an interval in the domain of a function f and a point x_0 in that interval. We say that f has an ***absolute maximum*** at x_0 if $f(x) \le f(x_0)$ for all x in the interval, and we say that f has an ***absolute minimum*** at x_0 if $f(x_0) \le f(x)$ for all x in the interval. We say that f has an ***absolute extremum*** at x_0 if it has either an absolute maximum or an absolute minimum at that point.

If f has an absolute maximum at the point x_0 on an interval, then $f(x_0)$ is the largest value of f on the interval, and if f has an absolute minimum at x_0, then $f(x_0)$ is the smallest value of f on the interval. In general, there is no guarantee that a function will actually have an absolute maximum or minimum on a given interval (Figure 4.4.1).



f has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$.
(*a*)

f has no absolute extrema on $(-\infty, +\infty)$.
(*b*)

f has an absolute maximum and minimum on $(-\infty, +\infty)$.
(*c*)

f has no absolute extrema on (a, b).
(*d*)

f has an absolute maximum and minimum on $[a, b]$.
(*e*)

▲ **Figure 4.4.1**

THE EXTREME VALUE THEOREM

Parts (*a*)–(*d*) of Figure 4.4.1 show that a continuous function may or may not have absolute maxima or minima on an infinite interval or on a finite open interval. However, the following theorem shows that a continuous function must have both an absolute maximum and an absolute minimum on every *finite closed* interval [see part (*e*) of Figure 4.4.1].

The hypotheses in the Extreme-Value Theorem are essential. That is, if either the interval is not closed or f is not continuous on the interval, then f need not have absolute extrema on the interval (Exercises 4–6).

4.4.2 THEOREM (*Extreme-Value Theorem*) *If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.*

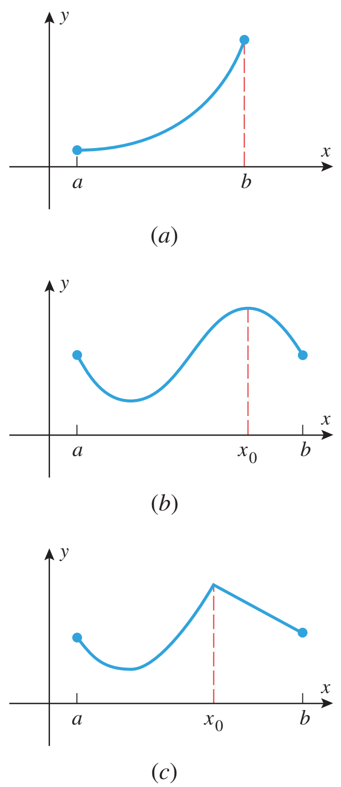
REMARK Although the proof of this theorem is too difficult to include here, you should be able to convince yourself of its validity with a little experimentation—try graphing various continuous functions over the interval $[0, 1]$, and convince yourself that there is no way to avoid having a highest and lowest point on a graph. As a physical analogy, if you imagine the graph to be a roller-coaster track starting at $x = 0$ and ending at $x = 1$, the roller coaster will have to pass through a highest point and a lowest point during the trip.

The Extreme-Value Theorem is an example of what mathematicians call an ***existence theorem***. Such theorems state conditions under which certain objects exist, in this case absolute extrema. However, knowing that an object exists and finding it are two separate things. We will now address methods for determining the locations of absolute extrema under the conditions of the Extreme-Value Theorem.

If f is continuous on the finite closed interval $[a, b]$, then the absolute extrema of f occur either at the endpoints of the interval or inside on the open interval (a, b). If the absolute extrema happen to fall inside, then the following theorem tells us that they must occur at critical points of f.

Theorem 4.4.3 is also valid on infinite open intervals, that is, intervals of the form $(-\infty, +\infty)$, $(a, +\infty)$, and $(-\infty, b)$.

4.4.3 THEOREM *If f has an absolute extremum on an open interval (a, b), then it must occur at a critical point of f.*



▲ **Figure 4.4.2** In part (*a*) the absolute maximum occurs at an endpoint of $[a, b]$, in part (*b*) it occurs at a stationary point in (a, b), and in part (*c*) it occurs at a critical point in (a, b) where f is not differentiable.

PROOF If f has an absolute maximum on (a, b) at x_0, then $f(x_0)$ is also a relative maximum for f; for if $f(x_0)$ is the largest value of f on all (a, b), then $f(x_0)$ is certainly the largest value for f in the immediate vicinity of x_0. Thus, x_0 is a critical point of f by Theorem 4.2.2. The proof for absolute minima is similar. ■

It follows from this theorem that if f is continuous on the finite closed interval $[a, b]$, then the absolute extrema occur either at the endpoints of the interval or at critical points inside the interval (Figure 4.4.2). Thus, we can use the following procedure to find the absolute extrema of a continuous function on a finite closed interval $[a, b]$.

A Procedure for Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval [a, b]

Step 1. Find the critical points of f in (a, b).

Step 2. Evaluate f at all the critical points and at the endpoints a and b.

Step 3. The largest of the values in Step 2 is the absolute maximum value of f on $[a, b]$ and the smallest value is the absolute minimum.

▶ **Example 1** Find the absolute maximum and minimum values of the function $f(x) = 2x^3 - 15x^2 + 36x$ on the interval $[1, 5]$, and determine where these values occur.

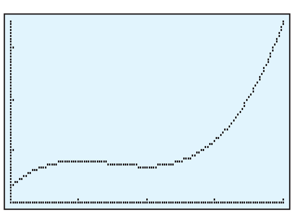
Solution. Since f is continuous and differentiable everywhere, the absolute extrema must occur either at endpoints of the interval or at solutions to the equation $f'(x) = 0$ in the open interval $(1, 5)$. The equation $f'(x) = 0$ can be written as

$$6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3) = 0$$

Thus, there are stationary points at $x = 2$ and at $x = 3$. Evaluating f at the endpoints, at $x = 2$, and at $x = 3$ yields

$$\begin{aligned} f(1) &= 2(1)^3 - 15(1)^2 + 36(1) = 23 \\ f(2) &= 2(2)^3 - 15(2)^2 + 36(2) = 28 \\ f(3) &= 2(3)^3 - 15(3)^2 + 36(3) = 27 \\ f(5) &= 2(5)^3 - 15(5)^2 + 36(5) = 55 \end{aligned}$$

from which we conclude that the absolute minimum of f on $[1, 5]$ is 23, occurring at $x = 1$, and the absolute maximum of f on $[1, 5]$ is 55, occurring at $x = 5$. This is consistent with the graph of f in Figure 4.4.3. ◀



$[1, 5] \times [20, 55]$
$x\text{Scl} = 1$, $y\text{Scl} = 10$

$y = 2x^3 - 15x^2 + 36x$

▲ **Figure 4.4.3**

▶ **Example 2** Find the absolute extrema of $f(x) = 6x^{4/3} - 3x^{1/3}$ on the interval $[-1, 1]$, and determine where these values occur.

Solution. Note that f is continuous everywhere and therefore the Extreme-Value Theorem guarantees that f has a maximum and a minimum value in the interval $[-1, 1]$. Differentiating, we obtain

$$f'(x) = 8x^{1/3} - x^{-2/3} = x^{-2/3}(8x - 1) = \frac{8x - 1}{x^{2/3}}$$

Thus, $f'(x) = 0$ at $x = \frac{1}{8}$, and $f'(x)$ is undefined at $x = 0$. Evaluating f at these critical points and endpoints yields Table 4.4.1, from which we conclude that an absolute minimum value of $-\frac{9}{8}$ occurs at $x = \frac{1}{8}$, and an absolute maximum value of 9 occurs at $x = -1$. ◀

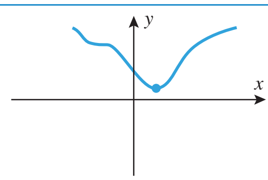
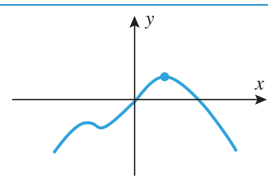
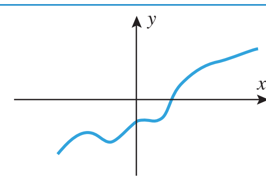
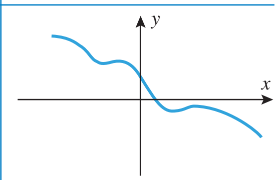
Table 4.4.1

x	-1	0	$\frac{1}{8}$	1
$f(x)$	9	0	$-\frac{9}{8}$	3

ABSOLUTE EXTREMA ON INFINITE INTERVALS

We observed earlier that a continuous function may or may not have absolute extrema on an infinite interval (see Figure 4.4.1). However, certain conclusions about the existence of absolute extrema of a continuous function f on $(-\infty, +\infty)$ can be drawn from the behavior of $f(x)$ as $x \to -\infty$ and as $x \to +\infty$ (Table 4.4.2).

Table 4.4.2
ABSOLUTE EXTREMA ON INFINITE INTERVALS

LIMITS	$\lim\limits_{x\to-\infty} f(x) = +\infty$ $\lim\limits_{x\to+\infty} f(x) = +\infty$	$\lim\limits_{x\to-\infty} f(x) = -\infty$ $\lim\limits_{x\to+\infty} f(x) = -\infty$	$\lim\limits_{x\to-\infty} f(x) = -\infty$ $\lim\limits_{x\to+\infty} f(x) = +\infty$	$\lim\limits_{x\to-\infty} f(x) = +\infty$ $\lim\limits_{x\to+\infty} f(x) = -\infty$
CONCLUSION IF f IS CONTINUOUS EVERYWHERE	f has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$.	f has an absolute maximum but no absolute minimum on $(-\infty, +\infty)$.	f has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$.	f has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$.
GRAPH				

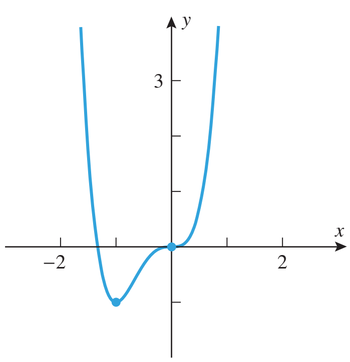
Example 3 What can you say about the existence of absolute extrema on $(-\infty, +\infty)$ for polynomials?

Solution. If $p(x)$ is a polynomial of odd degree, then

$$\lim_{x\to+\infty} p(x) \quad \text{and} \quad \lim_{x\to-\infty} p(x) \tag{1}$$

have opposite signs (one is $+\infty$ and the other is $-\infty$), so there are no absolute extrema. On the other hand, if $p(x)$ has even degree, then the limits in (1) have the same sign (both $+\infty$ or both $-\infty$). If the leading coefficient is positive, then both limits are $+\infty$, and there is an absolute minimum but no absolute maximum; if the leading coefficient is negative, then both limits are $-\infty$, and there is an absolute maximum but no absolute minimum. ◀

Example 4 Determine by inspection whether $p(x) = 3x^4 + 4x^3$ has any absolute extrema. If so, find them and state where they occur.



$p(x) = 3x^4 + 4x^3$

▲ **Figure 4.4.4**

Solution. Since $p(x)$ has even degree and the leading coefficient is positive, $p(x) \to +\infty$ as $x \to \pm\infty$. Thus, there is an absolute minimum but no absolute maximum. From Theorem 4.4.3 [applied to the interval $(-\infty, +\infty)$], the absolute minimum must occur at a critical point of p. Since p is differentiable everywhere, we can find all critical points by solving the equation $p'(x) = 0$. This equation is

$$12x^3 + 12x^2 = 12x^2(x+1) = 0$$

from which we conclude that the critical points are $x = 0$ and $x = -1$. Evaluating p at these critical points yields

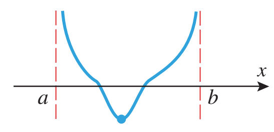
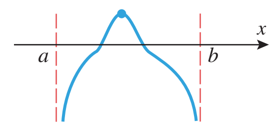
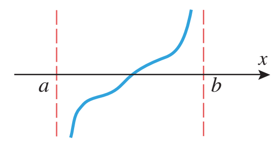
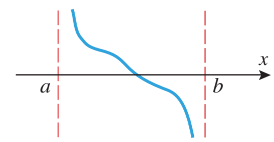
$$p(0) = 0 \quad \text{and} \quad p(-1) = -1$$

Therefore, p has an absolute minimum of -1 at $x = -1$ (Figure 4.4.4). ◀

ABSOLUTE EXTREMA ON OPEN INTERVALS

We know that a continuous function may or may not have absolute extrema on an open interval. However, certain conclusions about the existence of absolute extrema of a continuous function f on a finite open interval (a, b) can be drawn from the behavior of $f(x)$ as $x \to a^+$ and as $x \to b^-$ (Table 4.4.3). Similar conclusions can be drawn for intervals of the form $(-\infty, b)$ or $(a, +\infty)$.

Table 4.4.3
ABSOLUTE EXTREMA ON OPEN INTERVALS

LIMITS	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = -\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = -\infty$
CONCLUSION IF f IS CONTINUOUS ON (a, b)	f has an absolute minimum but no absolute maximum on (a, b).	f has an absolute maximum but no absolute minimum on (a, b).	f has neither an absolute maximum nor an absolute minimum on (a, b).	f has neither an absolute maximum nor an absolute minimum on (a, b).
GRAPH				

Example 5 Determine whether the function

$$f(x) = \frac{1}{x^2 - x}$$

has any absolute extrema on the interval $(0, 1)$. If so, find them and state where they occur.

Solution. Since f is continuous on the interval $(0, 1)$ and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x^2 - x} = \lim_{x\to 0^+} \frac{1}{x(x-1)} = -\infty$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x^2 - x} = \lim_{x\to 1^-} \frac{1}{x(x-1)} = -\infty$$

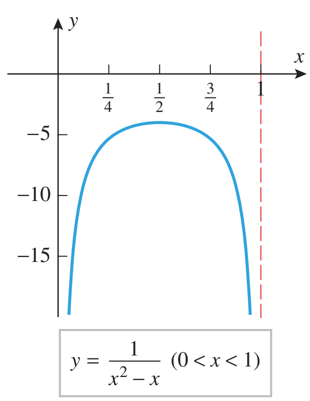
the function f has an absolute maximum but no absolute minimum on the interval $(0, 1)$. By Theorem 4.4.3 the absolute maximum must occur at a critical point of f in the interval $(0, 1)$. We have

$$f'(x) = -\frac{2x - 1}{(x^2 - x)^2}$$

so the only solution of the equation $f'(x) = 0$ is $x = \frac{1}{2}$. Although f is not differentiable at $x = 0$ or at $x = 1$, these values are doubly disqualified since they are neither in the domain of f nor in the interval $(0, 1)$. Thus, the absolute maximum occurs at $x = \frac{1}{2}$, and this absolute maximum is

$$f\left(\tfrac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^2 - \frac{1}{2}} = -4$$

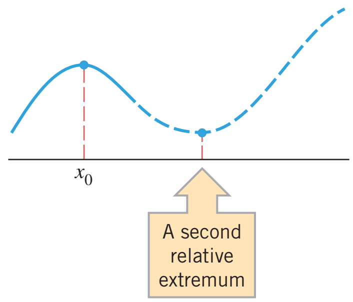
(Figure 4.4.5). ◄



▲ **Figure 4.4.5**

ABSOLUTE EXTREMA OF FUNCTIONS WITH ONE RELATIVE EXTREMUM

If a continuous function has only one relative extremum on a finite or infinite interval, then that relative extremum must of necessity also be an absolute extremum. To understand why



▲ **Figure 4.4.6**

this is so, suppose that f has a relative maximum at x_0 in an interval, and there are no other relative extrema of f on the interval. If $f(x_0)$ is *not* the absolute maximum of f on the interval, then the graph of f has to make an upward turn somewhere on the interval to rise above $f(x_0)$. However, this cannot happen because in the process of making an upward turn it would produce a second relative extremum (Figure 4.4.6). Thus, $f(x_0)$ must be the absolute maximum as well as a relative maximum. This idea is captured in the following theorem, which we state without proof.

4.4.4 THEOREM *Suppose that f is continuous and has exactly one relative extremum on an interval, say at x_0.*

(*a*) *If f has a relative minimum at x_0, then $f(x_0)$ is the absolute minimum of f on the interval.*

(*b*) *If f has a relative maximum at x_0, then $f(x_0)$ is the absolute maximum of f the interval.*

This theorem is often helpful in situations where other methods are difficult or tedious to apply.

▶ **Example 6** Find the absolute extrema, if any, of the function $f(x) = e^{(x^3-3x^2)}$ on the interval $(0, +\infty)$.

Solution. We have

$$\lim_{x\to+\infty} f(x) = +\infty$$

(verify), so f does not have an absolute maximum on the interval $(0, +\infty)$. However, the continuity of f together with the fact that

$$\lim_{x\to 0^+} f(x) = e^0 = 1$$

is finite allow for the possibility that f has an absolute minimum on $(0, +\infty)$. If so, it would have to occur at a critical point of f, so we consider

$$f'(x) = e^{(x^3-3x^2)}(3x^2 - 6x) = 3x(x-2)e^{(x^3-3x^2)}$$

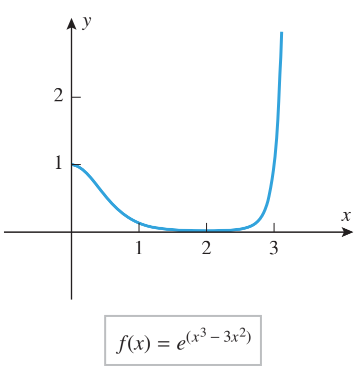
Since $e^{(x^3-3x^2)} > 0$ for all values of x, we see that $x = 0$ and $x = 2$ are the only critical points of f. Of these, only $x = 2$ is in the interval $(0, +\infty)$, so this is the point at which an absolute minimum could occur. To see whether an absolute minimum actually does occur at this point, we can apply part (a) of Theorem 4.4.4. Since

$$\begin{aligned} f''(x) &= e^{(x^3-3x^2)}(3x^2-6x)^2 + e^{(x^3-3x^2)}(6x-6) \\ &= [(3x^2-6x)^2 + (6x-6)]e^{(x^3-3x^2)} \end{aligned}$$

we have

$$f''(2) = (0+6)e^{-4} = 6e^{-4} > 0$$

so a relative minimum occurs at $x = 2$ by the second derivative test. Thus, $f(x)$ has an absolute minimum at $x = 2$, and this absolute minimum is $f(2) = e^{-4} \approx 0.0183$ (Figure 4.4.7). ◀



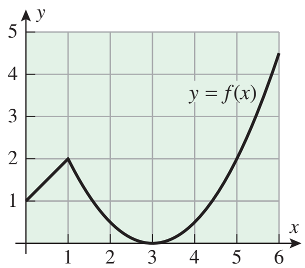
▲ **Figure 4.4.7**

Does the function in Example 6 have an absolute minimum on the interval $(-\infty, +\infty)$?

✔ QUICK CHECK EXERCISES 4.4

(See page 274 for answers.)

1. Use the accompanying graph to find the x-coordinates of the relative extrema and absolute extrema of f on $[0, 6]$.



◀ **Figure Ex-1**

2. Suppose that a function f is continuous on $[-4, 4]$ and has critical points at $x = -3, 0, 2$. Use the accompanying table to determine the absolute maximum and absolute minimum values, if any, for f on the indicated intervals.
(a) $[1, 4]$ (b) $[-2, 2]$ (c) $[-4, 4]$ (d) $(-4, 4)$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	2224	-1333	0	1603	2096	2293	2400	2717	6064

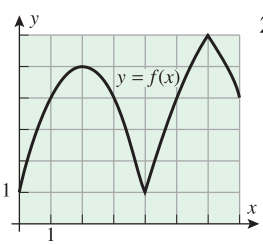
3. Let $f(x) = x^3 - 3x^2 - 9x + 25$. Use the derivative $f'(x) = 3(x + 1)(x - 3)$ to determine the absolute maximum and absolute minimum values, if any, for f on each of the given intervals.
(a) $[0, 4]$ (b) $[-2, 4]$ (c) $[-4, 2]$
(d) $[-5, 10]$ (e) $(-5, 4)$

EXERCISE SET 4.4 Graphing Utility C CAS

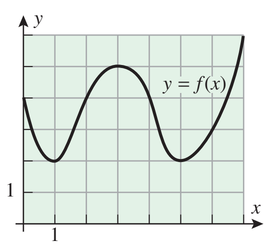
FOCUS ON CONCEPTS

1–2 Use the graph to find x-coordinates of the relative extrema and absolute extrema of f on $[0, 7]$. ■

1.



2.



3. In each part, sketch the graph of a continuous function f with the stated properties on the interval $[0, 10]$.
(a) f has an absolute minimum at $x = 0$ and an absolute maximum at $x = 10$.
(b) f has an absolute minimum at $x = 2$ and an absolute maximum at $x = 7$.
(c) f has relative minima at $x = 1$ and $x = 8$, has relative maxima at $x = 3$ and $x = 7$, has an absolute minimum at $x = 5$, and has an absolute maximum at $x = 10$.

4. In each part, sketch the graph of a continuous function f with the stated properties on the interval $(-\infty, +\infty)$.
(a) f has no relative extrema or absolute extrema.
(b) f has an absolute minimum at $x = 0$ but no absolute maximum.
(c) f has an absolute maximum at $x = -5$ and an absolute minimum at $x = 5$.

5. Let
$$f(x) = \begin{cases} \dfrac{1}{1-x}, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$
Explain why f has a minimum value but no maximum value on the closed interval $[0, 1]$.

6. Let
$$f(x) = \begin{cases} x, & 0 < x < 1 \\ \frac{1}{2}, & x = 0, 1 \end{cases}$$
Explain why f has neither a minimum value nor a maximum value on the closed interval $[0, 1]$.

7–16 Find the absolute maximum and minimum values of f on the given closed interval, and state where those values occur. ■

7. $f(x) = 4x^2 - 12x + 10$; $[1, 2]$

8. $f(x) = 8x - x^2$; $[0, 6]$

9. $f(x) = (x - 2)^3$; $[1, 4]$

10. $f(x) = 2x^3 + 3x^2 - 12x$; $[-3, 2]$

11. $f(x) = \dfrac{3x}{\sqrt{4x^2 + 1}}$; $[-1, 1]$

12. $f(x) = (x^2 + x)^{2/3}$; $[-2, 3]$

13. $f(x) = x - 2\sin x$; $[-\pi/4, \pi/2]$

14. $f(x) = \sin x - \cos x$; $[0, \pi]$

15. $f(x) = 1 + |9 - x^2|$; $[-5, 1]$

16. $f(x) = |6 - 4x|$; $[-3, 3]$

17–20 True–False Determine whether the statement is true or false. Explain your answer. ■

17. If a function f is continuous on $[a, b]$, then f has an absolute maximum on $[a, b]$.

18. If a function f is continuous on (a, b), then f has an absolute minimum on (a, b).

19. If a function f has an absolute minimum on (a, b), then there is a critical point of f in (a, b).

20. If a function f is continuous on $[a, b]$ and f has no relative extreme values in (a, b), then the absolute maximum value of f exists and occurs either at $x = a$ or at $x = b$.

21–28 Find the absolute maximum and minimum values of f, if any, on the given interval, and state where those values occur.

21. $f(x) = x^2 - x - 2;\ (-\infty, +\infty)$

22. $f(x) = 3 - 4x - 2x^2;\ (-\infty, +\infty)$

23. $f(x) = 4x^3 - 3x^4;\ (-\infty, +\infty)$

24. $f(x) = x^4 + 4x;\ (-\infty, +\infty)$

25. $f(x) = 2x^3 - 6x + 2;\ (-\infty, +\infty)$

26. $f(x) = x^3 - 9x + 1;\ (-\infty, +\infty)$

27. $f(x) = \dfrac{x^2+1}{x+1};\ (-5, -1)$

28. $f(x) = \dfrac{x-2}{x+1};\ (-1, 5]$

29–42 Use a graphing utility to estimate the absolute maximum and minimum values of f, if any, on the stated interval, and then use calculus methods to find the exact values.

29. $f(x) = (x^2 - 2x)^2;\ (-\infty, +\infty)$

30. $f(x) = (x-1)^2(x+2)^2;\ (-\infty, +\infty)$

31. $f(x) = x^{2/3}(20 - x);\ [-1, 20]$

32. $f(x) = \dfrac{x}{x^2+2};\ [-1, 4]$

33. $f(x) = 1 + \dfrac{1}{x};\ (0, +\infty)$

34. $f(x) = \dfrac{2x^2 - 3x + 3}{x^2 - 2x + 2};\ [1, +\infty)$

35. $f(x) = \dfrac{2 - \cos x}{\sin x};\ [\pi/4, 3\pi/4]$

36. $f(x) = \sin^2 x + \cos x;\ [-\pi, \pi]$

37. $f(x) = x^3 e^{-2x};\ [1, 4]$

38. $f(x) = \dfrac{\ln(2x)}{x};\ [1, e]$

39. $f(x) = 5\ln(x^2+1) - 3x;\ [0, 4]$

40. $f(x) = (x^2 - 1)e^x;\ [-2, 2]$

41. $f(x) = \sin(\cos x);\ [0, 2\pi]$

42. $f(x) = \cos(\sin x);\ [0, \pi]$

43. Find the absolute maximum and minimum values of

$$f(x) = \begin{cases} 4x - 2, & x < 1 \\ (x-2)(x-3), & x \geq 1 \end{cases}$$

on $\left[\frac{1}{2}, \frac{7}{2}\right]$.

44. Let $f(x) = x^2 + px + q$. Find the values of p and q such that $f(1) = 3$ is an extreme value of f on $[0, 2]$. Is this value a maximum or minimum?

45–46 If f is a periodic function, then the locations of all absolute extrema on the interval $(-\infty, +\infty)$ can be obtained by finding the locations of the absolute extrema for one period and using the periodicity to locate the rest. Use this idea in these exercises to find the absolute maximum and minimum values of the function, and state the x-values at which they occur.

45. $f(x) = 2\cos x + \cos 2x$ **46.** $f(x) = 3\cos\dfrac{x}{3} + 2\cos\dfrac{x}{2}$

47–48 One way of proving that $f(x) \leq g(x)$ for all x in a given interval is to show that $0 \leq g(x) - f(x)$ for all x in the interval; and one way of proving the latter inequality is to show that the absolute minimum value of $g(x) - f(x)$ on the interval is nonnegative. Use this idea to prove the inequalities in these exercises.

47. Prove that $\sin x \leq x$ for all x in the interval $[0, 2\pi]$.

48. Prove that $\cos x \geq 1 - (x^2/2)$ for all x in the interval $[0, 2\pi]$.

49. What is the smallest possible slope for a tangent to the graph of the equation $y = x^3 - 3x^2 + 5x$?

50. (a) Show that $f(x) = \sec x + \csc x$ has a minimum value but no maximum value on the interval $(0, \pi/2)$.
(b) Find the minimum value in part (a).

C **51.** Show that the absolute minimum value of

$$f(x) = x^2 + \frac{x^2}{(8-x)^2}, \quad x > 8$$

occurs at $x = 10$ by using a CAS to find $f'(x)$ and to solve the equation $f'(x) = 0$.

52. The concentration $C(t)$ of a drug in the bloodstream t hours after it has been injected is commonly modeled by an equation of the form

$$C(t) = \frac{K(e^{-bt} - e^{-at})}{a - b}$$

where $K > 0$ and $a > b > 0$.
(a) At what time does the maximum concentration occur?
(b) Let $K = 1$ for simplicity, and use a graphing utility to check your result in part (a) by graphing $C(t)$ for various values of a and b.

53. Suppose that the equations of motion of a paper airplane during the first 12 seconds of flight are

$$x = t - 2\sin t, \quad y = 2 - 2\cos t \qquad (0 \leq t \leq 12)$$

What are the highest and lowest points in the trajectory, and when is the airplane at those points?

54. The accompanying figure shows the path of a fly whose equations of motion are

$$x = \frac{\cos t}{2 + \sin t}, \quad y = 3 + \sin(2t) - 2\sin^2 t \qquad (0 \leq t \leq 2\pi)$$

(a) How high and low does it fly?
(b) How far left and right of the origin does it fly?

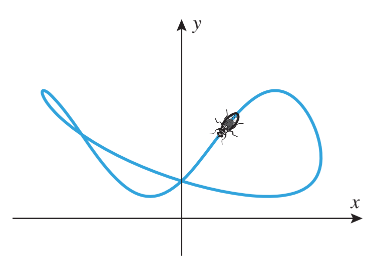


Figure Ex-54

55. Let $f(x) = ax^2 + bx + c$, where $a > 0$. Prove that $f(x) \geq 0$ for all x if and only if $b^2 - 4ac \leq 0$. [*Hint:* Find the minimum of $f(x)$.]

56. Prove Theorem 4.4.3 in the case where the extreme value is a minimum.

57. **Writing** Suppose that f is continuous and positive-valued everywhere and that the x-axis is an asymptote for the graph of f, both as $x \to -\infty$ and as $x \to +\infty$. Explain why f cannot have an absolute minimum but may have a relative minimum.

58. **Writing** Explain the difference between a relative maximum and an absolute maximum. Sketch a graph that illustrates a function with a relative maximum that is not an absolute maximum, and sketch another graph illustrating an absolute maximum that is not a relative maximum. Explain how these graphs satisfy the given conditions.

✔ QUICK CHECK ANSWERS 4.4

1. There is a relative minimum at $x = 3$, a relative maximum at $x = 1$, an absolute minimum at $x = 3$, and an absolute maximum at $x = 6$. **2.** (a) max, 6064; min, 2293 (b) max, 2400; min, 0 (c) max, 6064; min, -1333 (d) no max; min, -1333 **3.** (a) max, $f(0) = 25$; min, $f(3) = -2$ (b) max, $f(-1) = 30$; min, $f(3) = -2$ (c) max, $f(-1) = 30$; min, $f(-4) = -51$ (d) max, $f(10) = 635$; min, $f(-5) = -130$ (e) max, $f(-1) = 30$; no min

4.5 APPLIED MAXIMUM AND MIMIMUM PROBLEMS

In this section we will show how the methods discussed in the last section can be used to solve various applied optimization problems.

CLASSIFICATION OF OPTIMIZATION PROBLEMS

The applied optimization problems that we will consider in this section fall into the following two categories:

- Problems that reduce to maximizing or minimizing a continuous function over a finite closed interval.
- Problems that reduce to maximizing or minimizing a continuous function over an infinite interval or a finite interval that is not closed.

For problems of the first type the Extreme-Value Theorem (4.4.2) guarantees that the problem has a solution, and we know that the solution can be obtained by examining the values of the function at the critical points and at the endpoints. However, for problems of the second type there may or may not be a solution. If the function is continuous and has exactly one relative extremum of the appropriate type on the interval, then Theorem 4.4.4 guarantees the existence of a solution and provides a method for finding it. In cases where this theorem is not applicable some ingenuity may be required to solve the problem.

PROBLEMS INVOLVING FINITE CLOSED INTERVALS

In his *On a Method for the Evaluation of Maxima and Minima*, the seventeenth century French mathematician Pierre de Fermat solved an optimization problem very similar to the one posed in our first example. Fermat's work on such optimization problems prompted the French mathematician Laplace to proclaim Fermat the "true inventor of the differential calculus." Although this honor must still reside with Newton and Leibniz, it is the case that Fermat developed procedures that anticipated parts of differential calculus.

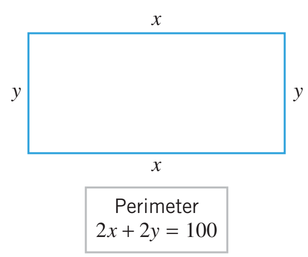
▶ **Example 1** A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running feet of chicken wire is available for the fence?

Solution. Let

$$x = \text{length of the rectangle (ft)}$$
$$y = \text{width of the rectangle (ft)}$$
$$A = \text{area of the rectangle (ft}^2)$$

Then

$$A = xy \tag{1}$$



▲ **Figure 4.5.1**

Since the perimeter of the rectangle is 100 ft, the variables x and y are related by the equation

$$2x + 2y = 100 \quad \text{or} \quad y = 50 - x \tag{2}$$

(See Figure 4.5.1.) Substituting (2) in (1) yields

$$A = x(50 - x) = 50x - x^2 \tag{3}$$

Because x represents a length, it cannot be negative, and because the two sides of length x cannot have a combined length exceeding the total perimeter of 100 ft, the variable x must satisfy

$$0 \leq x \leq 50 \tag{4}$$

Thus, we have reduced the problem to that of finding the value (or values) of x in [0, 50], for which A is maximum. Since A is a polynomial in x, it is continuous on [0, 50], and so the maximum must occur at an endpoint of this interval or at a critical point.

From (3) we obtain

$$\frac{dA}{dx} = 50 - 2x$$

Setting $dA/dx = 0$ we obtain

$$50 - 2x = 0$$

Pierre de Fermat (1601–1665) Fermat, the son of a successful French leather merchant, was a lawyer who practiced mathematics as a hobby. He received a Bachelor of Civil Laws degree from the University of Orleans in 1631 and subsequently held various government positions, including a post as councillor to the Toulouse parliament. Although he was apparently financially successful, confidential documents of that time suggest that his performance in office and as a lawyer was poor, perhaps because he devoted so much time to mathematics. Throughout his life, Fermat fought all efforts to have his mathematical results published. He had the unfortunate habit of scribbling his work in the margins of books and often sent his results to friends without keeping copies for himself. As a result, he never received credit for many major achievements until his name was raised from obscurity in the mid-nineteenth century. It is now known that Fermat, simultaneously and independently of Descartes, developed analytic geometry. Unfortunately, Descartes and Fermat argued bitterly over various problems so that there was never any real cooperation between these two great geniuses.

Fermat solved many fundamental calculus problems. He obtained the first procedure for differentiating polynomials, and solved many important maximization, minimization, area, and tangent problems. His work served to inspire Isaac Newton. Fermat is best known for his work in number theory, the study of properties of and relationships between whole numbers. He was the first mathematician to make substantial contributions to this field after the ancient Greek mathematician Diophantus. Unfortunately, none of Fermat's contemporaries appreciated his work in this area, a fact that eventually pushed Fermat into isolation and obscurity in later life. In addition to his work in calculus and number theory, Fermat was one of the founders of probability theory and made major contributions to the theory of optics. Outside mathematics, Fermat was a classical scholar of some note, was fluent in French, Italian, Spanish, Latin, and Greek, and he composed a considerable amount of Latin poetry.

One of the great mysteries of mathematics is shrouded in Fermat's work in number theory. In the margin of a book by Diophantus, Fermat scribbled that for integer values of n greater than 2, the equation $x^n + y^n = z^n$ has no nonzero integer solutions for x, y, and z. He stated, "I have discovered a truly marvelous proof of this, which however the margin is not large enough to contain." This result, which became known as "Fermat's last theorem," appeared to be true, but its proof evaded the greatest mathematical geniuses for 300 years until Professor Andrew Wiles of Princeton University presented a proof in June 1993 in a dramatic series of three lectures that drew international media attention (see *New York Times*, June 27, 1993). As it turned out, that proof had a serious gap that Wiles and Richard Taylor fixed and published in 1995. A prize of 100,000 German marks was offered in 1908 for the solution, but it is worthless today because of inflation.

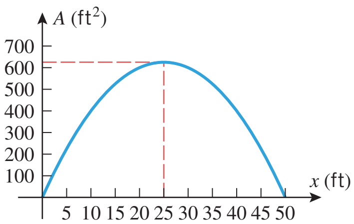
or $x = 25$. Thus, the maximum occurs at one of the values

$$x = 0, \quad x = 25, \quad x = 50$$

Substituting these values in (3) yields Table 4.5.1, which tells us that the maximum area of 625 ft^2 occurs at $x = 25$, which is consistent with the graph of (3) in Figure 4.5.2. From (2) the corresponding value of y is 25, so the rectangle of perimeter 100 ft with greatest area is a square with sides of length 25 ft. ◀

Table 4.5.1

x	0	25	50
A	0	625	0



▲ **Figure 4.5.2**

In Example 1 we included $x = 0$ and $x = 50$ as possible values of x, even though these correspond to rectangles with two sides of length zero. If we view this as a purely mathematical problem, then there is nothing wrong with this. However, if we view this as an applied problem in which the rectangle will be formed from physical material, then it would make sense to exclude these values.

Example 1 illustrates the following five-step procedure that can be used for solving many applied maximum and minimum problems.

A Procedure for Solving Applied Maximum and Minimum Problems

Step 1. Draw an appropriate figure and label the quantities relevant to the problem.

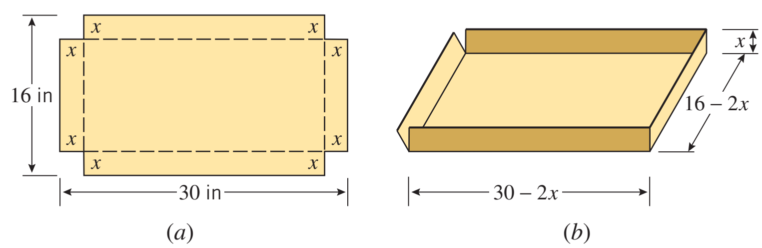
Step 2. Find a formula for the quantity to be maximized or minimized.

Step 3. Using the conditions stated in the problem to eliminate variables, express the quantity to be maximized or minimized as a function of one variable.

Step 4. Find the interval of possible values for this variable from the physical restrictions in the problem.

Step 5. If applicable, use the techniques of the preceding section to obtain the maximum or minimum.

▶ **Example 2** An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides (Figure 4.5.3). What size should the squares be to obtain a box with the largest volume?



▶ **Figure 4.5.3**

Solution. For emphasis, we explicitly list the steps of the five-step problem-solving procedure given above as an outline for the solution of this problem. (In later examples we will follow these guidelines without listing the steps.)

- *Step 1:* Figure 4.5.3a illustrates the cardboard piece with squares removed from its corners. Let

  x = length (in inches) of the sides of the squares to be cut out

  V = volume (in cubic inches) of the resulting box

- *Step 2:* Because we are removing a square of side x from each corner, the resulting box will have dimensions $16 - 2x$ by $30 - 2x$ by x (Figure 4.5.3b). Since the volume of a box is the product of its dimensions, we have

$$V = (16 - 2x)(30 - 2x)x = 480x - 92x^2 + 4x^3 \tag{5}$$

- *Step 3:* Note that our volume expression is already in terms of the single variable x.
- *Step 4:* The variable x in (5) is subject to certain restrictions. Because x represents a length, it cannot be negative, and because the width of the cardboard is 16 inches, we cannot cut out squares whose sides are more than 8 inches long. Thus, the variable x in (5) must satisfy

$$0 \le x \le 8$$

and hence we have reduced our problem to finding the value (or values) of x in the interval [0, 8] for which (5) is a maximum.

- *Step 5:* From (5) we obtain

$$\begin{aligned}\frac{dV}{dx} &= 480 - 184x + 12x^2 = 4(120 - 46x + 3x^2)\\ &= 4(x-12)(3x-10)\end{aligned}$$

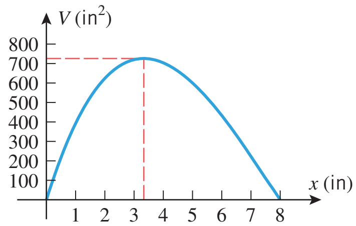
Setting $dV/dx = 0$ yields

$$x = \tfrac{10}{3} \quad \text{and} \quad x = 12$$

Since $x = 12$ falls outside the interval [0, 8], the maximum value of V occurs either at the critical point $x = \frac{10}{3}$ or at the endpoints $x = 0$, $x = 8$. Substituting these values into (5) yields Table 4.5.2, which tells us that the greatest possible volume $V = \frac{19{,}600}{27}\ \text{in}^3 \approx 726\ \text{in}^3$ occurs when we cut out squares whose sides have length $\frac{10}{3}$ inches. This is consistent with the graph of (5) shown in Figure 4.5.4. ◀

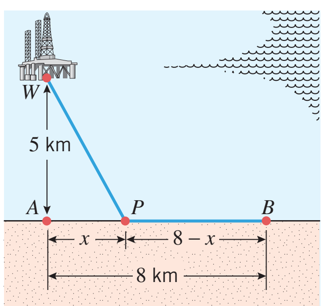
Table 4.5.2

x	0	$\frac{10}{3}$	8
V	0	$\frac{19{,}600}{27} \approx 726$	0



▲ **Figure 4.5.4**

▶ **Example 3** Figure 4.5.5 shows an offshore oil well located at a point W that is 5 km from the closest point A on a straight shoreline. Oil is to be piped from W to a shore point B that is 8 km from A by piping it on a straight line under water from W to some shore point P between A and B and then on to B via pipe along the shoreline. If the cost of laying pipe is \$1,000,000/km under water and \$500,000/km over land, where should the point P be located to minimize the cost of laying the pipe?



▲ **Figure 4.5.5**

Solution. Let

$$\begin{aligned}x &= \text{distance (in kilometers) between } A \text{ and } P\\ c &= \text{cost (in millions of dollars) for the entire pipeline}\end{aligned}$$

From Figure 4.5.5 the length of pipe under water is the distance between W and P. By the Theorem of Pythagoras that length is

$$\sqrt{x^2 + 25} \tag{6}$$

Also from Figure 4.5.5, the length of pipe over land is the distance between P and B, which is

$$8 - x \tag{7}$$

From (6) and (7) it follows that the total cost c (in millions of dollars) for the pipeline is

$$c = 1(\sqrt{x^2+25}) + \tfrac{1}{2}(8-x) = \sqrt{x^2+25} + \tfrac{1}{2}(8-x) \tag{8}$$

Because the distance between A and B is 8 km, the distance x between A and P must satisfy

$$0 \le x \le 8$$

We have thus reduced our problem to finding the value (or values) of x in the interval [0, 8] for which c is a minimum. Since c is a continuous function of x on the closed interval [0, 8], we can use the methods developed in the preceding section to find the minimum.

From (8) we obtain
$$\frac{dc}{dx} = \frac{x}{\sqrt{x^2+25}} - \frac{1}{2}$$

Setting $dc/dx = 0$ and solving for x yields
$$\frac{x}{\sqrt{x^2+25}} = \frac{1}{2} \tag{9}$$
$$x^2 = \frac{1}{4}(x^2+25)$$
$$x = \pm\frac{5}{\sqrt{3}}$$

TECHNOLOGY MASTERY

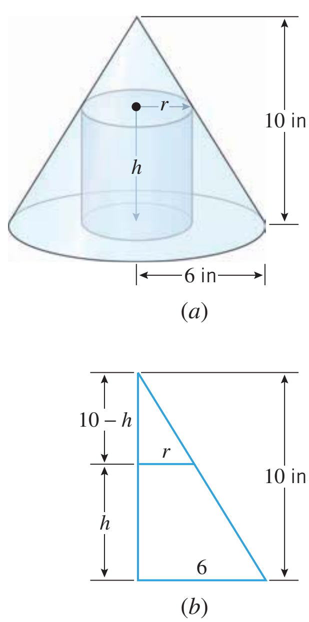
If you have a CAS, use it to check all of the computations in Example 3. Specifically, differentiate c with respect to x, solve the equation $dc/dx = 0$, and perform all of the numerical calculations.

The number $-5/\sqrt{3}$ is not a solution of (9) and must be discarded, leaving $x = 5/\sqrt{3}$ as the only critical point. Since this point lies in the interval $[0, 8]$, the minimum must occur at one of the values
$$x = 0, \quad x = 5/\sqrt{3}, \quad x = 8$$

Substituting these values into (8) yields Table 4.5.3, which tells us that the least possible cost of the pipeline (to the nearest dollar) is $c = \$8{,}330{,}127$, and this occurs when the point P is located at a distance of $5/\sqrt{3} \approx 2.89$ km from A. ◀

Table 4.5.3

x	0	$\frac{5}{\sqrt{3}}$	8
c	9	$\frac{10}{\sqrt{3}} + \left(4 - \frac{5}{2\sqrt{3}}\right) \approx 8.330127$	$\sqrt{89} \approx 9.433981$



(a) (b)

▲ **Figure 4.5.6**

▶ **Example 4** Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches (Figure 4.5.6*a*).

Solution. Let
$$\begin{aligned} r &= \text{radius (in inches) of the cylinder} \\ h &= \text{height (in inches) of the cylinder} \\ V &= \text{volume (in cubic inches) of the cylinder} \end{aligned}$$

The formula for the volume of the inscribed cylinder is
$$V = \pi r^2 h \tag{10}$$

To eliminate one of the variables in (10) we need a relationship between r and h. Using similar triangles (Figure 4.5.6*b*) we obtain
$$\frac{10-h}{r} = \frac{10}{6} \quad \text{or} \quad h = 10 - \frac{5}{3}r \tag{11}$$

Substituting (11) into (10) we obtain
$$V = \pi r^2\left(10 - \frac{5}{3}r\right) = 10\pi r^2 - \frac{5}{3}\pi r^3 \tag{12}$$

which expresses V in terms of r alone. Because r represents a radius, it cannot be negative, and because the radius of the inscribed cylinder cannot exceed the radius of the cone, the variable r must satisfy
$$0 \le r \le 6$$

Thus, we have reduced the problem to that of finding the value (or values) of r in [0, 6] for which (12) is a maximum. Since V is a continuous function of r on [0, 6], the methods developed in the preceding section apply.

From (12) we obtain

$$\frac{dV}{dr} = 20\pi r - 5\pi r^2 = 5\pi r(4 - r)$$

Setting $dV/dr = 0$ gives

$$5\pi r(4 - r) = 0$$

so $r = 0$ and $r = 4$ are critical points. Since these lie in the interval [0, 6], the maximum must occur at one of the values

$$r = 0, \quad r = 4, \quad r = 6$$

Table 4.5.4

r	0	4	6
V	0	$\frac{160}{3}\pi$	0

Substituting these values into (12) yields Table 4.5.4, which tells us that the maximum volume $V = \frac{160}{3}\pi \approx 168$ in^3 occurs when the inscribed cylinder has radius 4 in. When $r = 4$ it follows from (11) that $h = \frac{10}{3}$. Thus, the inscribed cylinder of largest volume has radius $r = 4$ in and height $h = \frac{10}{3}$ in. ◀

PROBLEMS INVOLVING INTERVALS THAT ARE NOT BOTH FINITE AND CLOSED

▶ **Example 5** A closed cylindrical can is to hold 1 liter (1000 cm^3) of liquid. How should we choose the height and radius to minimize the amount of material needed to manufacture the can?

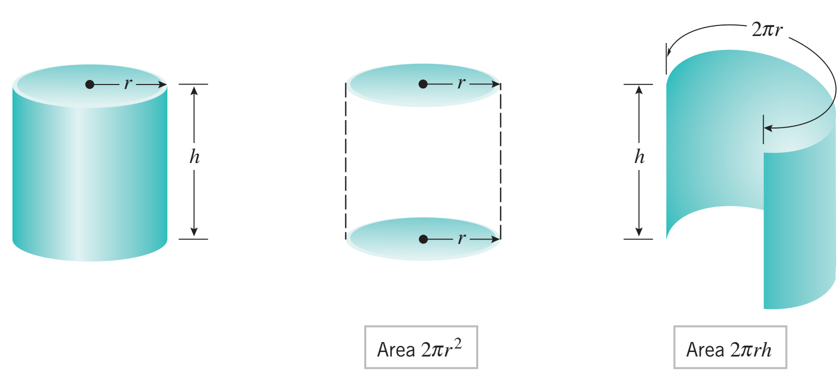
Solution. Let

h = height (in cm) of the can
r = radius (in cm) of the can
S = surface area (in cm^2) of the can

Assuming there is no waste or overlap, the amount of material needed for manufacture will be the same as the surface area of the can. Since the can consists of two circular disks of radius r and a rectangular sheet with dimensions h by $2\pi r$ (Figure 4.5.7), the surface area will be

$$S = 2\pi r^2 + 2\pi rh \tag{13}$$

Since S depends on two variables, r and h, we will look for some condition in the problem that will allow us to express one of these variables in terms of the other. For this purpose,



▲ **Figure 4.5.7**

observe that the volume of the can is 1000 cm^3, so it follows from the formula $V = \pi r^2 h$ for the volume of a cylinder that

$$1000 = \pi r^2 h \quad \text{or} \quad h = \frac{1000}{\pi r^2} \tag{14–15}$$

Substituting (15) in (13) yields

$$S = 2\pi r^2 + \frac{2000}{r} \tag{16}$$

Thus, we have reduced the problem to finding a value of r in the interval $(0, +\infty)$ for which S is minimum. Since S is a continuous function of r on the interval $(0, +\infty)$ and

$$\lim_{r\to 0^+} \left(2\pi r^2 + \frac{2000}{r}\right) = +\infty \quad \text{and} \quad \lim_{r\to +\infty} \left(2\pi r^2 + \frac{2000}{r}\right) = +\infty$$

the analysis in Table 4.4.3 implies that S does have a minimum on the interval $(0, +\infty)$. Since this minimum must occur at a critical point, we calculate

$$\frac{dS}{dr} = 4\pi r - \frac{2000}{r^2} \tag{17}$$

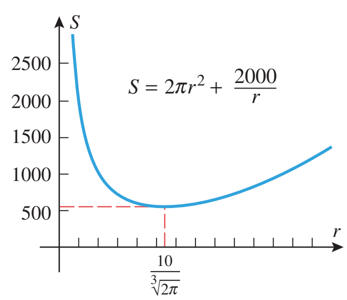
Setting $dS/dr = 0$ gives

$$r = \frac{10}{\sqrt[3]{2\pi}} \approx 5.4 \tag{18}$$

Since (18) is the only critical point in the interval $(0, +\infty)$, this value of r yields the minimum value of S. From (15) the value of h corresponding to this r is

$$h = \frac{1000}{\pi(10/\sqrt[3]{2\pi})^2} = \frac{20}{\sqrt[3]{2\pi}} = 2r$$

It is not an accident here that the minimum occurs when the height of the can is equal to the diameter of its base (Exercise 29).



▲ **Figure 4.5.8**

In Example 5, the surface area S has no absolute maximum, since S increases without bound as the radius r approaches 0 (Figure 4.5.8). Thus, had we asked for the dimensions of the can requiring the *maximum* amount of material for its manufacture, there would have been no solution to the problem. Optimization problems with no solution are sometimes called *ill posed.*

Second Solution. The conclusion that a minimum occurs at the value of r in (18) can be deduced from Theorem 4.4.4 and the second derivative test by noting that

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive if $r > 0$ and hence is positive if $r = 10/\sqrt[3]{2\pi}$. This implies that a relative minimum, and therefore a minimum, occurs at the critical point $r = 10/\sqrt[3]{2\pi}$.

Third Solution. An alternative justification that the critical point $r = 10/\sqrt[3]{2\pi}$ corresponds to a minimum for S is to view the graph of S versus r (Figure 4.5.8). ◄

▶ **Example 6** Find a point on the curve $y = x^2$ that is closest to the point $(18, 0)$.

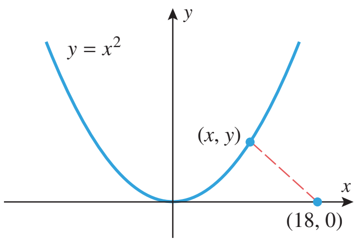
Solution. The distance L between $(18, 0)$ and an arbitrary point (x, y) on the curve $y = x^2$ (Figure 4.5.9) is given by

$$L = \sqrt{(x-18)^2 + (y-0)^2}$$

Since (x, y) lies on the curve, x and y satisfy $y = x^2$; thus,

$$L = \sqrt{(x-18)^2 + x^4} \tag{19}$$

Because there are no restrictions on x, the problem reduces to finding a value of x in $(-\infty, +\infty)$ for which (19) is a minimum. The distance L and the square of the distance L^2



▲ Figure 4.5.9

are minimized at the same value (see Exercise 66). Thus, the minimum value of L in (19) and the minimum value of

$$S = L^2 = (x - 18)^2 + x^4 \tag{20}$$

occur at the same x-value.

From (20),

$$\frac{dS}{dx} = 2(x - 18) + 4x^3 = 4x^3 + 2x - 36 \tag{21}$$

so the critical points satisfy $4x^3 + 2x - 36 = 0$ or, equivalently,

$$2x^3 + x - 18 = 0 \tag{22}$$

To solve for x we will begin by checking the divisors of -18 to see whether the polynomial on the left side has any integer roots (see Appendix C). These divisors are ± 1, ± 2, ± 3, ± 6, ± 9, and ± 18. A check of these values shows that $x = 2$ is a root, so $x - 2$ is a factor of the polynomial. After dividing the polynomial by this factor we can rewrite (22) as

$$(x - 2)(2x^2 + 4x + 9) = 0$$

Thus, the remaining solutions of (22) satisfy the quadratic equation

$$2x^2 + 4x + 9 = 0$$

But this equation has no real solutions (using the quadratic formula), so $x = 2$ is the only critical point of S. To determine the nature of this critical point we will use the second derivative test. From (21),

$$\frac{d^2S}{dx^2} = 12x^2 + 2, \quad \text{so} \quad \left.\frac{d^2S}{dx^2}\right|_{x=2} = 50 > 0$$

which shows that a relative minimum occurs at $x = 2$. Since $x = 2$ yields the only relative extremum for L, it follows from Theorem 4.4.4 that an absolute minimum value of L also occurs at $x = 2$. Thus, the point on the curve $y = x^2$ closest to (18, 0) is

$$(x, y) = (x, x^2) = (2, 4) \blacktriangleleft$$

AN APPLICATION TO ECONOMICS

Three functions of importance to an economist or a manufacturer are

$C(x)$ = total cost of producing x units of a product during some time period

$R(x)$ = total revenue from selling x units of the product during the time period

$P(x)$ = total profit obtained by selling x units of the product during the time period

These are called, respectively, the ***cost function***, ***revenue function***, and ***profit function***. If all units produced are sold, then these are related by

$$P(x) = R(x) - C(x) \tag{23}$$
$$[\text{profit}] = [\text{revenue}] - [\text{cost}]$$

The total cost $C(x)$ of producing x units can be expressed as a sum

$$C(x) = a + M(x) \tag{24}$$

where a is a constant, called ***overhead***, and $M(x)$ is a function representing ***manufacturing cost***. The overhead, which includes such fixed costs as rent and insurance, does not depend on x; it must be paid even if nothing is produced. On the other hand, the manufacturing cost $M(x)$, which includes such items as cost of materials and labor, depends on the number of items manufactured. It is shown in economics that with suitable simplifying assumptions, $M(x)$ can be expressed in the form

$$M(x) = bx + cx^2$$

where b and c are constants. Substituting this in (24) yields

$$C(x) = a + bx + cx^2 \tag{25}$$

If a manufacturing firm can sell all the items it produces for p dollars apiece, then its total revenue $R(x)$ (in dollars) will be

$$R(x) = px \tag{26}$$

and its total profit $P(x)$ (in dollars) will be

$$P(x) = [\text{total revenue}] - [\text{total cost}] = R(x) - C(x) = px - C(x)$$

Thus, if the cost function is given by (25),

$$P(x) = px - (a + bx + cx^2) \tag{27}$$

Depending on such factors as number of employees, amount of machinery available, economic conditions, and competition, there will be some upper limit l on the number of items a manufacturer is capable of producing and selling. Thus, during a fixed time period the variable x in (27) will satisfy

$$0 \le x \le l$$

By determining the value or values of x in $[0, l]$ that maximize (27), the firm can determine how many units of its product must be manufactured and sold to yield the greatest profit. This is illustrated in the following numerical example.

Jim Karageorge/Getty Images

A pharmaceutical firm's profit is a function of the number of units produced.

▶ **Example 7** A liquid form of antibiotic manufactured by a pharmaceutical firm is sold in bulk at a price of \$200 per unit. If the total production cost (in dollars) for x units is

$$C(x) = 500{,}000 + 80x + 0.003x^2$$

and if the production capacity of the firm is at most 30,000 units in a specified time, how many units of antibiotic must be manufactured and sold in that time to maximize the profit?

Solution. Since the total revenue for selling x units is $R(x) = 200x$, the profit $P(x)$ on x units will be

$$P(x) = R(x) - C(x) = 200x - (500{,}000 + 80x + 0.003x^2) \tag{28}$$

Since the production capacity is at most 30,000 units, x must lie in the interval [0, 30,000]. From (28)

$$\frac{dP}{dx} = 200 - (80 + 0.006x) = 120 - 0.006x$$

Setting $dP/dx = 0$ gives

$$120 - 0.006x = 0 \quad \text{or} \quad x = 20{,}000$$

Since this critical point lies in the interval [0, 30,000], the maximum profit must occur at one of the values

$$x = 0, \quad x = 20{,}000, \quad \text{or} \quad x = 30{,}000$$

Substituting these values in (28) yields Table 4.5.5, which tells us that the maximum profit $P = \$700{,}000$ occurs when $x = 20{,}000$ units are manufactured and sold in the specified time. ◀

Table 4.5.5

x	0	20,000	30,000
$P(x)$	−500,000	700,000	400,000

MARGINAL ANALYSIS

Economists call $P'(x)$, $R'(x)$, and $C'(x)$ the ***marginal profit***, ***marginal revenue***, and ***marginal cost***, respectively; and they interpret these quantities as the *additional* profit, revenue, and cost that result from producing and selling one additional unit of the product when the production and sales levels are at x units. These interpretations follow from the local linear approximations of the profit, revenue, and cost functions. For example, it follows from Formula (2) of Section 3.5 that when the production and sales levels are at x units the local linear approximation of the profit function is

$$P(x + \Delta x) \approx P(x) + P'(x)\Delta x$$

Thus, if $\Delta x = 1$ (one additional unit produced and sold), this formula implies

$$P(x + 1) \approx P(x) + P'(x)$$

and hence the *additional* profit that results from producing and selling one additional unit can be approximated as

$$P(x + 1) - P(x) \approx P'(x)$$

Similarly, $R(x + 1) - R(x) \approx R'(x)$ and $C(x + 1) - C(x) \approx C'(x)$.

A BASIC PRINCIPLE OF ECONOMICS

It follows from (23) that $P'(x) = 0$ has the same solution as $C'(x) = R'(x)$, and this implies that the maximum profit must occur at a point where the marginal revenue is equal to the marginal cost; that is:

If profit is maximum, then the cost of manufacturing and selling an additional unit of a product is approximately equal to the revenue generated by the additional unit.

In Example 7, the maximum profit occurs when $x = 20{,}000$ units. Note that

$$C(20{,}001) - C(20{,}000) = \$200.003 \quad \text{and} \quad R(20{,}001) - R(20{,}000) = \$200$$

which is consistent with this basic economic principle.

✔QUICK CHECK EXERCISES 4.5 (See page 288 for answers.)

1. A positive number x and its reciprocal are added together. The smallest possible value of this sum is obtained by minimizing $f(x) =$ ________ for x in the interval ________.

2. Two nonnegative numbers, x and y, have a sum equal to 10. The largest possible product of the two numbers is obtained by maximizing $f(x) =$ ________ for x in the interval ________.

3. A rectangle in the xy-plane has one corner at the origin, an adjacent corner at the point $(x, 0)$, and a third corner at a point on the line segment from $(0, 4)$ to $(3, 0)$. The largest possible area of the rectangle is obtained by maximizing $A(x) =$ ________ for x in the interval ________.

4. An open box is to be made from a 20-inch by 32-inch piece of cardboard by cutting out x-inch by x-inch squares from the four corners and bending up the sides. The largest possible volume of the box is obtained by maximizing $V(x) =$ ________ for x in the interval ________.

EXERCISE SET 4.5

1. Find a number in the closed interval $\left[\frac{1}{2}, \frac{3}{2}\right]$ such that the sum of the number and its reciprocal is
(a) as small as possible
(b) as large as possible.

2. How should two nonnegative numbers be chosen so that their sum is 1 and the sum of their squares is
(a) as large as possible
(b) as small as possible?

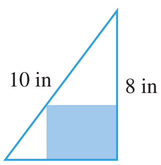
3. A rectangular field is to be bounded by a fence on three sides and by a straight stream on the fourth side. Find the dimensions of the field with maximum area that can be enclosed using 1000 ft of fence.

4. The boundary of a field is a right triangle with a straight stream along its hypotenuse and with fences along its other two sides. Find the dimensions of the field with maximum area that can be enclosed using 1000 ft of fence.

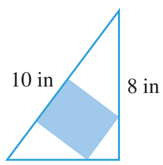
5. A rectangular plot of land is to be fenced in using two kinds of fencing. Two opposite sides will use heavy-duty fencing selling for \$3 a foot, while the remaining two sides will use standard fencing selling for \$2 a foot. What are the dimensions of the rectangular plot of greatest area that can be fenced in at a cost of \$6000?

6. A rectangle is to be inscribed in a right triangle having sides of length 6 in, 8 in, and 10 in. Find the dimensions of the rectangle with greatest area assuming the rectangle is positioned as in Figure Ex-6.

7. Solve the problem in Exercise 6 assuming the rectangle is positioned as in Figure Ex-7.



▲ **Figure Ex-6**



▲ **Figure Ex-7**

8. A rectangle has its two lower corners on the x-axis and its two upper corners on the curve $y = 16 - x^2$. For all such rectangles, what are the dimensions of the one with largest area?

9. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10.

10. Find the point P in the first quadrant on the curve $y = x^{-2}$ such that a rectangle with sides on the coordinate axes and a vertex at P has the smallest possible perimeter.

11. A rectangular area of 3200 ft^2 is to be fenced off. Two opposite sides will use fencing costing \$1 per foot and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle of least cost.

12. Show that among all rectangles with perimeter p, the square has the maximum area.

13. Show that among all rectangles with area A, the square has the minimum perimeter.

14. A wire of length 12 in can be bent into a circle, bent into a square, or cut into two pieces to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be
(a) a maximum (b) a minimum?

15. A rectangle R in the plane has corners at $(\pm 8, \pm 12)$, and a 100 by 100 square S is positioned in the plane so that its sides are parallel to the coordinate axes and the lower left corner of S is on the line $y = -3x$. What is the largest possible area of a region in the plane that is contained in both R and S?

16. Solve the problem in Exercise 15 if S is a 16 by 16 square.

17. Solve the problem in Exercise 15 if S is positioned with its lower left corner on the line $y = -6x$.

18. A rectangular page is to contain 42 square inches of printable area. The margins at the top and bottom of the page are each 1 inch, one side margin is 1 inch, and the other side margin is 2 inches. What should the dimensions of the page be so that the least amount of paper is used?

19. A box with a square base is taller than it is wide. In order to send the box through the U.S. mail, the height of the box and the perimeter of the base can sum to no more than 108 in. What is the maximum volume for such a box?

20. A box with a square base is wider than it is tall. In order to send the box through the U.S. mail, the width of the box and the perimeter of one of the (nonsquare) sides of the box can sum to no more than 108 in. What is the maximum volume for such a box?

21. An open box is to be made from a 3 ft by 8 ft rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sides. Find the maximum volume that the box can have.

22. A closed rectangular container with a square base is to have a volume of 2250 in^3. The material for the top and bottom of the container will cost \$2 per in^2, and the material for the sides will cost \$3 per in^2. Find the dimensions of the container of least cost.

23. A closed rectangular container with a square base is to have a volume of 2000 cm^3. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimensions of the container of least cost.

24. A container with square base, vertical sides, and open top is to be made from 1000 ft^2 of material. Find the dimensions of the container with greatest volume.

25. A rectangular container with two square sides and an open top is to have a volume of V cubic units. Find the dimensions of the container with minimum surface area.

26. A church window consisting of a rectangle topped by a semicircle is to have a perimeter p. Find the radius of the semicircle if the area of the window is to be maximum.

27. Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius R.

28. Find the dimensions of the right circular cylinder of greatest surface area that can be inscribed in a sphere of radius R.

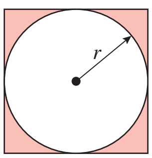
29. A closed, cylindrical can is to have a volume of V cubic units. Show that the can of minimum surface area is achieved when the height is equal to the diameter of the base.

30. A closed cylindrical can is to have a surface area of S square units. Show that the can of maximum volume is achieved when the height is equal to the diameter of the base.

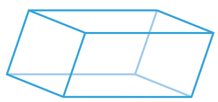
31. A cylindrical can, open at the top, is to hold 500 cm^3 of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can.

32. A soup can in the shape of a right circular cylinder of radius r and height h is to have a prescribed volume V. The top and bottom are cut from squares as shown in Figure Ex-32. If the shaded corners are wasted, but there is no other waste, find the ratio r/h for the can requiring the least material (including waste).

33. A box-shaped wire frame consists of two identical wire squares whose vertices are connected by four straight wires of equal length (Figure Ex-33). If the frame is to be made from a wire of length L, what should the dimensions be to obtain a box of greatest volume?



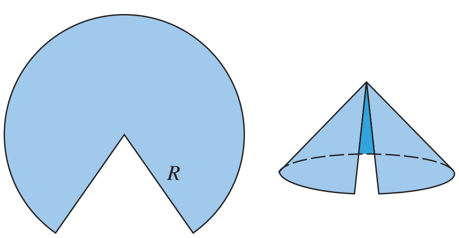
▲ Figure Ex-32


▲ Figure Ex-33

34. Suppose that the sum of the surface areas of a sphere and a cube is a constant.
(a) Show that the sum of their volumes is smallest when the diameter of the sphere is equal to the length of an edge of the cube.
(b) When will the sum of their volumes be greatest?

35. Find the height and radius of the cone of slant height L whose volume is as large as possible.

36. A cone is made from a circular sheet of radius R by cutting out a sector and gluing the cut edges of the remaining piece together (Figure Ex-36). What is the maximum volume attainable for the cone?



▲ Figure Ex-36

37. A cone-shaped paper drinking cup is to hold 100 cm^3 of water. Find the height and radius of the cup that will require the least amount of paper.

38. Find the dimensions of the isosceles triangle of least area that can be circumscribed about a circle of radius R.

39. Find the height and radius of the right circular cone with least volume that can be circumscribed about a sphere of radius R.

40. A commercial cattle ranch currently allows 20 steers per acre of grazing land; on the average its steers weigh 2000 lb at market. Estimates by the Agriculture Department indicate that the average market weight per steer will be reduced by 50 lb for each additional steer added per acre of grazing land. How many steers per acre should be allowed in order for the ranch to get the largest possible total market weight for its cattle?

41. A company mines low-grade nickel ore. If the company mines x tons of ore, it can sell the ore for $p = 225 - 0.25x$ dollars per ton. Find the revenue and marginal revenue functions. At what level of production would the company obtain the maximum revenue?

42. A fertilizer producer finds that it can sell its product at a price of $p = 300 - 0.1x$ dollars per unit when it produces x units of fertilizer. The total production cost (in dollars) for x units is

$$C(x) = 15{,}000 + 125x + 0.025x^2$$

If the production capacity of the firm is at most 1000 units of fertilizer in a specified time, how many units must be manufactured and sold in that time to maximize the profit?

43. (a) A chemical manufacturer sells sulfuric acid in bulk at a price of \$100 per unit. If the daily total production cost in dollars for x units is

$$C(x) = 100{,}000 + 50x + 0.0025x^2$$

and if the daily production capacity is at most 7000 units, how many units of sulfuric acid must be manufactured and sold daily to maximize the profit?
(b) Would it benefit the manufacturer to expand the daily production capacity?
(c) Use marginal analysis to approximate the effect on profit if daily production could be increased from 7000 to 7001 units.

44. A firm determines that x units of its product can be sold daily at p dollars per unit, where

$$x = 1000 - p$$

The cost of producing x units per day is

$$C(x) = 3000 + 20x$$

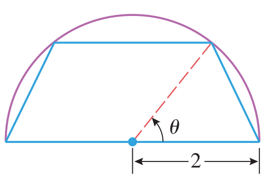
(a) Find the revenue function $R(x)$.
(b) Find the profit function $P(x)$.
(c) Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit.
(d) Find the maximum profit.
(e) What price per unit must be charged to obtain the maximum profit?

45. In a certain chemical manufacturing process, the daily weight y of defective chemical output depends on the total weight x of all output according to the empirical formula
$$y = 0.01x + 0.00003x^2$$
where x and y are in pounds. If the profit is \$100 per pound of nondefective chemical produced and the loss is \$20 per pound of defective chemical produced, how many pounds of chemical should be produced daily to maximize the total daily profit?

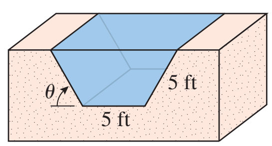
46. An independent truck driver charges a client \$15 for each hour of driving, plus the cost of fuel. At highway speeds of v miles per hour, the trucker's rig gets $10 - 0.07v$ miles per gallon of diesel fuel. If diesel fuel costs \$2.50 per gallon, what speed v will minimize the cost to the client?

47. A trapezoid is inscribed in a semicircle of radius 2 so that one side is along the diameter (Figure Ex-47). Find the maximum possible area for the trapezoid. [*Hint:* Express the area of the trapezoid in terms of θ.]

48. A drainage channel is to be made so that its cross section is a trapezoid with equally sloping sides (Figure Ex-48). If the sides and bottom all have a length of 5 ft, how should the angle θ $(0 \le \theta \le \pi/2)$ be chosen to yield the greatest cross-sectional area of the channel?



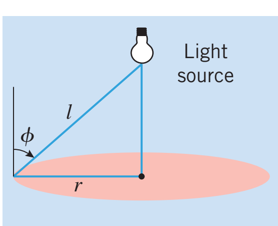
▲ **Figure Ex-47**



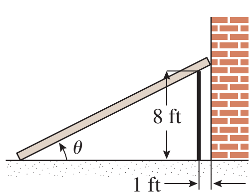
▲ **Figure Ex-48**

49. A lamp is suspended above the center of a round table of radius r. How high above the table should the lamp be placed to achieve maximum illumination at the edge of the table? [Assume that the illumination I is directly proportional to the cosine of the angle of incidence ϕ of the light rays and inversely proportional to the square of the distance l from the light source (Figure Ex-49).]

50. A plank is used to reach over a fence 8 ft high to support a wall that is 1 ft behind the fence (Figure Ex-50). What is the length of the shortest plank that can be used? [*Hint:* Express the length of the plank in terms of the angle θ shown in the figure.]



▲ **Figure Ex-49**



▲ **Figure Ex-50**

51. Find the coordinates of the point P on the curve
$$y = \frac{1}{x^2} \quad (x > 0)$$
where the segment of the tangent line at P that is cut off by the coordinate axes has its shortest length.

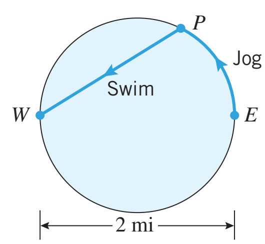
52. Find the x-coordinate of the point P on the parabola
$$y = 1 - x^2 \quad (0 < x \le 1)$$
where the triangle that is enclosed by the tangent line at P and the coordinate axes has the smallest area.

53. Where on the curve $y = (1 + x^2)^{-1}$ does the tangent line have the greatest slope?

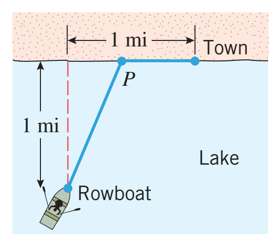
54. Suppose that the number of bacteria in a culture at time t is given by $N = 5000(25 + te^{-t/20})$.
(a) Find the largest and smallest number of bacteria in the culture during the time interval $0 \le t \le 100$.
(b) At what time during the time interval in part (a) is the number of bacteria decreasing most rapidly?

55. The shoreline of Circle Lake is a circle with diameter 2 mi. Nancy's training routine begins at point E on the eastern shore of the lake. She jogs along the north shore to a point P and then swims the straight line distance, if any, from P to point W diametrically opposite E (Figure Ex-55). Nancy swims at a rate of 2 mi/h and jogs at 8 mi/h. How far should Nancy jog in order to complete her training routine in
(a) the least amount of time
(b) the greatest amount of time?

56. A man is floating in a rowboat 1 mile from the (straight) shoreline of a large lake. A town is located on the shoreline 1 mile from the point on the shoreline closest to the man. As suggested in Figure Ex-56, he intends to row in a straight line to some point P on the shoreline and then walk the remaining distance to the town. To what point should he row in order to reach his destination in the least time if
(a) he can walk 5 mi/h and row 3 mi/h
(b) he can walk 5 mi/h and row 4 mi/h?



▲ **Figure Ex-55**

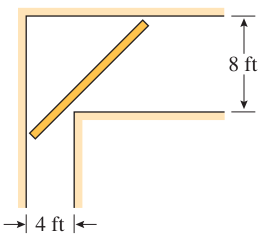


▲ **Figure Ex-56**

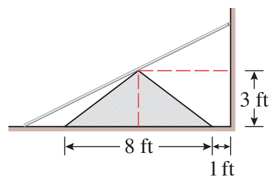
57. A pipe of negligible diameter is to be carried horizontally around a corner from a hallway 8 ft wide into a hallway 4 ft wide (Figure Ex-57 on the next page). What is the maximum length that the pipe can have?

Source: An interesting discussion of this problem in the case where the diameter of the pipe is not neglected is given by Norman Miller in the *American Mathematical Monthly*, Vol. 56, 1949, pp. 177–179.

58. A concrete barrier whose cross section is an isosceles triangle runs parallel to a wall. The height of the barrier is 3 ft, the width of the base of a cross section is 8 ft, and the barrier is positioned on level ground with its base 1 ft from the wall. A straight, stiff metal rod of negligible diameter has one end on the ground, the other end against the wall, and touches the top of the barrier (Figure Ex-58). What is the minimum length the rod can have?



▲ Figure Ex-57



▲ Figure Ex-58

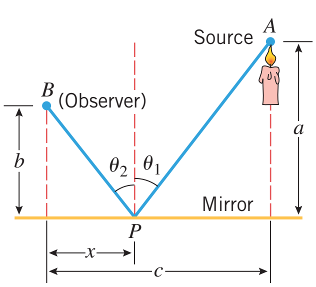
59. Suppose that the intensity of a point light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two point light sources with strengths of S and $8S$ are separated by a distance of 90 cm. Where on the line segment between the two sources is the total intensity a minimum?

60. Given points $A(2, 1)$ and $B(5, 4)$, find the point P in the interval $[2, 5]$ on the x-axis that maximizes angle APB.

61. The lower edge of a painting, 10 ft in height, is 2 ft above an observer's eye level. Assuming that the best view is obtained when the angle subtended at the observer's eye by the painting is maximum, how far from the wall should the observer stand?

FOCUS ON CONCEPTS

62. ***Fermat's principle*** (biography on p. 275) in optics states that light traveling from one point to another follows that path for which the total travel time is minimum. In a uniform medium, the paths of "minimum time" and "shortest distance" turn out to be the same, so that light, if unobstructed, travels along a straight line. Assume that we have a light source, a flat mirror, and an observer in a uniform medium. If a light ray leaves the source, bounces off the mirror, and travels on to the observer, then its path will consist of two line segments, as shown in Figure Ex-62. According to Fermat's principle, the path will be such that the total travel time t is minimum or, since the medium is uniform, the path will be such that the total distance traveled from A to P to B is as small as possible. Assuming the minimum occurs when $dt/dx = 0$, show that the light ray will strike the mirror at the point P where the "angle of incidence" θ_1 equals the "angle of reflection" θ_2.



◀ Figure Ex-62

63. Fermat's principle (Exercise 62) also explains why light rays traveling between air and water undergo bending (refraction). Imagine that we have two uniform media (such as air and water) and a light ray traveling from a source A in one medium to an observer B in the other medium (Figure Ex-63). It is known that light travels at a constant speed in a uniform medium, but more slowly in a dense medium (such as water) than in a thin medium (such as air). Consequently, the path of shortest time from A to B is not necessarily a straight line, but rather some broken line path A to P to B allowing the light to take greatest advantage of its higher speed through the thin medium. ***Snell's law of refraction*** (biography on p. 288) states that the path of the light ray will be such that

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

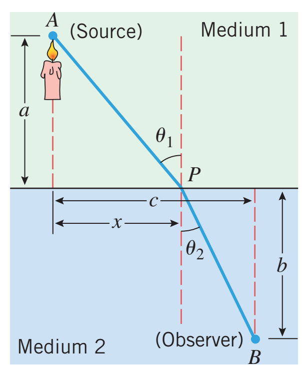
where v_1 is the speed of light in the first medium, v_2 is the speed of light in the second medium, and θ_1 and θ_2 are the angles shown in Figure Ex-63. Show that this follows from the assumption that the path of minimum time occurs when $dt/dx = 0$.

64. A farmer wants to walk at a constant rate from her barn to a straight river, fill her pail, and carry it to her house in the least time.

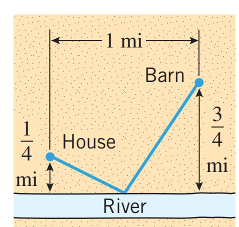
(a) Explain how this problem relates to Fermat's principle and the light-reflection problem in Exercise 62.

(b) Use the result of Exercise 62 to describe geometrically the best path for the farmer to take.

(c) Use part (b) to determine where the farmer should fill her pail if her house and barn are located as in Figure Ex-64.



▲ Figure Ex-63



▲ Figure Ex-64

65. If an unknown physical quantity x is measured n times, the measurements $x_1, x_2, \ldots, x_n$ often vary because of uncontrollable factors such as temperature, atmospheric pressure, and so forth. Thus, a scientist is often faced with the problem of using n different observed measurements to obtain an estimate $\bar{x}$ of an unknown quantity x. One method for making such an estimate is based on the ***least squares principle***, which states that the estimate $\bar{x}$

should be chosen to minimize

$$s = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2$$

which is the sum of the squares of the deviations between the estimate $\bar{x}$ and the measured values. Show that the estimate resulting from the least squares principle is

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

that is, $\bar{x}$ is the arithmetic average of the observed values.

66. Prove: If $f(x) \geq 0$ on an interval and if $f(x)$ has a maximum value on that interval at x_0, then $\sqrt{f(x)}$ also has a maximum value at x_0. Similarly for minimum values. [*Hint:* Use the fact that $\sqrt{x}$ is an increasing function on the interval $[0, +\infty)$.]

67. Writing Discuss the importance of finding intervals of possible values imposed by physical restrictions on variables in an applied maximum or minimum problem.

✔QUICK CHECK ANSWERS 4.5

1. $x + \dfrac{1}{x}$; $(0, +\infty)$ **2.** $x(10 - x)$; $[0, 10]$ **3.** $x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x$; $[0, 3]$
4. $x(20 - 2x)(32 - 2x) = 4x^3 - 104x^2 + 640x$; $[0, 10]$

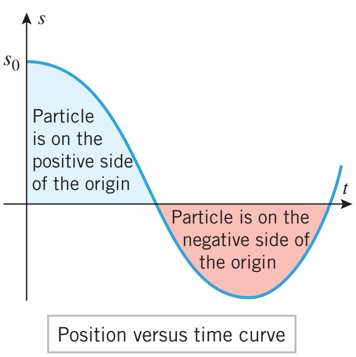
4.6 RECTILINEAR MOTION

In this section we will continue the study of rectilinear motion that we began in Section 2.1. We will define the notion of "acceleration" mathematically, and we will show how the tools of calculus developed earlier in this chapter can be used to analyze rectilinear motion in more depth.

REVIEW OF TERMINOLOGY

Recall from Section 2.1 that a particle that can move in either direction along a coordinate line is said to be in ***rectilinear motion***. The line might be an x-axis, a y-axis, or a coordinate line inclined at some angle. In general discussions we will designate the coordinate line as the s-axis. We will assume that units are chosen for measuring distance and time and that we begin observing the motion of the particle at time $t = 0$. As the particle moves along the s-axis, its coordinate s will be some function of time, say $s = s(t)$. We call $s(t)$ the ***position function*** of the particle,* and we call the graph of s versus t the ***position versus time curve***. If the coordinate of a particle at time t_1 is $s(t_1)$ and the coordinate at a later time t_2 is $s(t_2)$, then $s(t_2) - s(t_1)$ is called the ***displacement*** of the particle over the time interval $[t_1, t_2]$. The displacement describes the change in position of the particle.

Figure 4.6.1 shows a typical position versus time curve for a particle in rectilinear motion. We can tell from that graph that the coordinate of the particle at time $t = 0$ is s_0, and we can tell from the sign of s when the particle is on the negative or the positive side of the origin as it moves along the coordinate line.



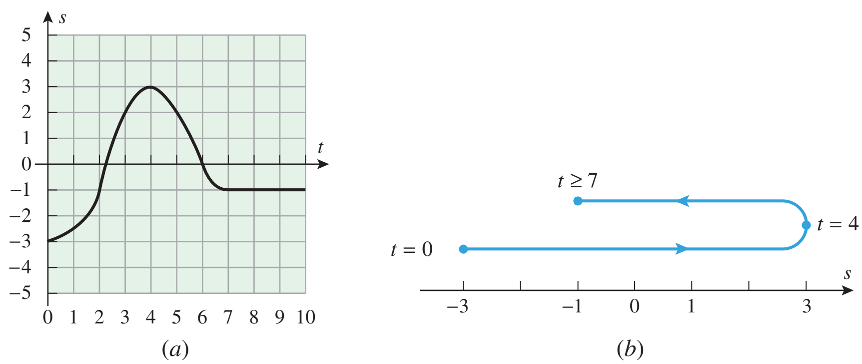
▲ **Figure 4.6.1**

*In writing $s = s(t)$, rather than the more familiar $s = f(t)$, we are using the letter s both as the dependent variable and the name of the function. This is common practice in engineering and physics.

Willebrord van Roijen Snell (1591–1626) Dutch mathematician. Snell, who succeeded his father to the post of Professor of Mathematics at the University of Leiden in 1613, is most famous for the result of light refraction that bears his name. Although this phenomenon was studied as far back as the ancient Greek astronomer Ptolemy, until Snell's work the relationship was incorrectly thought to be $\theta_1/v_1 = \theta_2/v_2$. Snell's law was published by Descartes in 1638 without giving proper credit to Snell. Snell also discovered a method for determining distances by triangulation that founded the modern technique of mapmaking.

▶ **Example 1** Figure 4.6.2a shows the position versus time curve for a particle moving along an s-axis. In words, describe how the position of the particle changes with time.

Solution. The particle is at $s = -3$ at time $t = 0$. It moves in the positive direction until time $t = 4$, since s is increasing. At time $t = 4$ the particle is at position $s = 3$. At that time it turns around and travels in the negative direction until time $t = 7$, since s is decreasing. At time $t = 7$ the particle is at position $s = -1$, and it remains stationary thereafter, since s is constant for $t > 7$. This is illustrated schematically in Figure 4.6.2b. ◀



▲ **Figure 4.6.2**

VELOCITY AND SPEED

Recall from Formula (5) of Section 2.1 and Formula (4) of Section 2.2 that the instantaneous velocity of a particle in rectilinear motion is the derivative of the position function. Thus, if a particle in rectilinear motion has position function $s(t)$, then we define its ***velocity function*** $v(t)$ to be

$$v(t) = s'(t) = \frac{ds}{dt} \tag{1}$$

We should more properly call $v(t)$ the ***instantaneous velocity function*** to distinguish instantaneous velocity from average velocity. However, we will follow the standard practice of referring to it as the "velocity function," leaving it understood that it describes instantaneous velocity.

The sign of the velocity tells which way the particle is moving—a positive value for $v(t)$ means that s is increasing with time, so the particle is moving in the positive direction, and a negative value for $v(t)$ means that s is decreasing with time, so the particle is moving in the negative direction. If $v(t) = 0$, then the particle has momentarily stopped.

For a particle in rectilinear motion it is important to distinguish between its ***velocity***, which describes how fast and in what direction the particle is moving, and its ***speed***, which describes only how fast the particle is moving. We make this distinction by defining speed to be the absolute value of velocity. Thus a particle with a velocity of 2 m/s has a speed of 2 m/s and is moving in the positive direction, while a particle with a velocity of −2 m/s also has a speed of 2 m/s but is moving in the negative direction.

Since the instantaneous speed of a particle is the absolute value of its instantaneous velocity, we define its ***speed function*** to be

$$|v(t)| = |s'(t)| = \left|\frac{ds}{dt}\right| \tag{2}$$

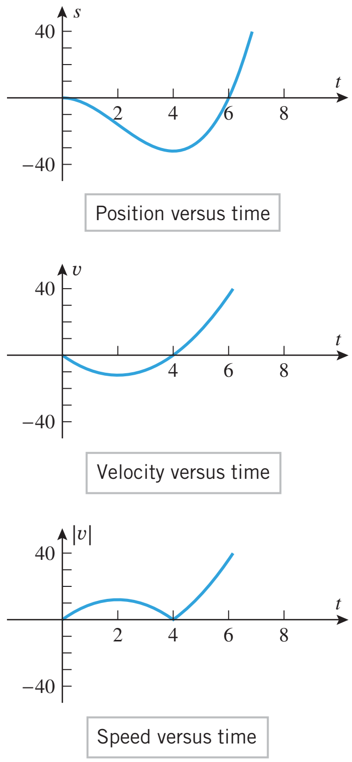
The speed function, which is always nonnegative, tells us how fast the particle is moving but not its direction of motion.

▶ **Example 2** Let $s(t) = t^3 - 6t^2$ be the position function of a particle moving along an s-axis, where s is in meters and t is in seconds. Find the velocity and speed functions, and show the graphs of position, velocity, and speed versus time.

Solution. From (1) and (2), the velocity and speed functions are given by

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t \quad \text{and} \quad |v(t)| = |3t^2 - 12t|$$

The graphs of position, velocity, and speed versus time are shown in Figure 4.6.3. Observe that velocity and speed both have units of meters per second (m/s), since s is in meters (m) and time is in seconds (s). ◀



▲ **Figure 4.6.3**

The graphs in Figure 4.6.3 provide a wealth of visual information about the motion of the particle. For example, the position versus time curve tells us that the particle is on the negative side of the origin for $0 < t < 6$, is on the positive side of the origin for $t > 6$, and is at the origin at times $t = 0$ and $t = 6$. The velocity versus time curve tells us that the particle is moving in the negative direction if $0 < t < 4$, is moving in the positive direction if $t > 4$, and is momentarily stopped at times $t = 0$ and $t = 4$ (the velocity is zero at those times). The speed versus time curve tells us that the speed of the particle is increasing for $0 < t < 2$, decreasing for $2 < t < 4$, and increasing again for $t > 4$.

ACCELERATION

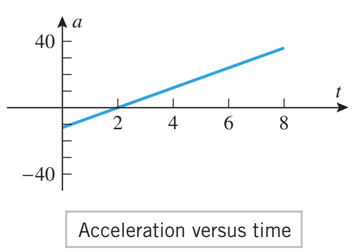
In rectilinear motion, the rate at which the instantaneous velocity of a particle changes with time is called its ***instantaneous acceleration***. Thus, if a particle in rectilinear motion has velocity function $v(t)$, then we define its ***acceleration function*** to be

$$a(t) = v'(t) = \frac{dv}{dt} \tag{3}$$

Alternatively, we can use the fact that $v(t) = s'(t)$ to express the acceleration function in terms of the position function as

$$a(t) = s''(t) = \frac{d^2s}{dt^2} \tag{4}$$

▶ **Example 3** Let $s(t) = t^3 - 6t^2$ be the position function of a particle moving along an s-axis, where s is in meters and t is in seconds. Find the acceleration function $a(t)$, and show the graph of acceleration versus time.



▲ **Figure 4.6.4**

Solution. From Example 2, the velocity function of the particle is $v(t) = 3t^2 - 12t$, so the acceleration function is

$$a(t) = \frac{dv}{dt} = 6t - 12$$

and the acceleration versus time curve is the line shown in Figure 4.6.4. Note that in this example the acceleration has units of m/s^2, since v is in meters per second (m/s) and time is in seconds (s). ◀

SPEEDING UP AND SLOWING DOWN

We will say that a particle in rectilinear motion is ***speeding up*** when its speed is increasing and is ***slowing down*** when its speed is decreasing. In everyday language an object that is speeding up is said to be "accelerating" and an object that is slowing down is said to be "decelerating"; thus, one might expect that a particle in rectilinear motion will be speeding up when its acceleration is positive and slowing down when it is negative. Although this is true for a particle moving in the positive direction, it is *not* true for a particle moving in the negative direction—a particle with negative velocity is speeding up when its acceleration is negative and slowing down when its acceleration is positive. This is because a positive acceleration implies an increasing velocity, and increasing a negative velocity decreases its absolute value; similarly, a negative acceleration implies a decreasing velocity, and decreasing a negative velocity increases its absolute value.

The preceding informal discussion can be summarized as follows (Exercise 41):

If $a(t) = 0$ over a certain time interval, what does this tell you about the motion of the particle during that time?

INTERPRETING THE SIGN OF ACCELERATION *A particle in rectilinear motion is speeding up when its velocity and acceleration have the same sign and slowing down when they have opposite signs.*

▶ **Example 4** In Examples 2 and 3 we found the velocity versus time curve and the acceleration versus time curve for a particle with position function $s(t) = t^3 - 6t^2$. Use those curves to determine when the particle is speeding up and slowing down, and confirm that your results are consistent with the speed versus time curve obtained in Example 2.

Solution. Over the time interval $0 < t < 2$ the velocity and acceleration are negative, so the particle is speeding up. This is consistent with the speed versus time curve, since the speed is increasing over this time interval. Over the time interval $2 < t < 4$ the velocity is negative and the acceleration is positive, so the particle is slowing down. This is also consistent with the speed versus time curve, since the speed is decreasing over this time interval. Finally, on the time interval $t > 4$ the velocity and acceleration are positive, so the particle is speeding up, which again is consistent with the speed versus time curve. ◀

ANALYZING THE POSITION VERSUS TIME CURVE

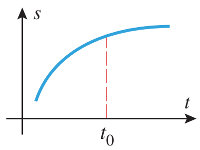
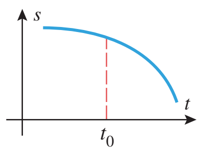
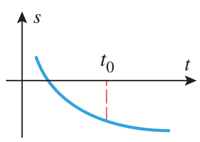
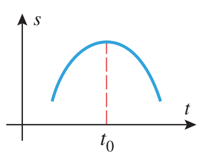
The position versus time curve contains all of the significant information about the position and velocity of a particle in rectilinear motion:

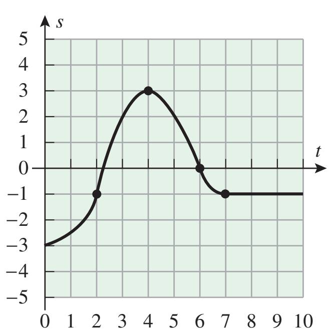
- If $s(t) > 0$, the particle is on the positive side of the s-axis.
- If $s(t) < 0$, the particle is on the negative side of the s-axis.
- The slope of the curve at any time is equal to the instantaneous velocity at that time.
- Where the curve has positive slope, the velocity is positive and the particle is moving in the positive direction.
- Where the curve has negative slope, the velocity is negative and the particle is moving in the negative direction.
- Where the slope of the curve is zero, the velocity is zero, and the particle is momentarily stopped.

Information about the acceleration of a particle in rectilinear motion can also be deduced from the position versus time curve by examining its concavity. For example, we know that the position versus time curve will be concave up on intervals where $s''(t) > 0$ and will be concave down on intervals where $s''(t) < 0$. But we know from (4) that $s''(t)$ is the acceleration, so that on intervals where the position versus time curve is concave up the particle has a positive acceleration, and on intervals where it is concave down the particle has a negative acceleration.

Table 4.6.1 summarizes our observations about the position versus time curve.

Table 4.6.1
ANALYSIS OF PARTICLE MOTION

POSITION VERSUS TIME CURVE	CHARACTERISTICS OF THE CURVE AT $t = t_0$	BEHAVIOR OF THE PARTICLE AT TIME $t = t_0$
	• $s(t_0) > 0$ • Curve has positive slope. • Curve is concave down.	• Particle is on the positive side of the origin. • Particle is moving in the positive direction. • Velocity is decreasing. • Particle is slowing down.
	• $s(t_0) > 0$ • Curve has negative slope. • Curve is concave down.	• Particle is on the positive side of the origin. • Particle is moving in the negative direction. • Velocity is decreasing. • Particle is speeding up.
	• $s(t_0) < 0$ • Curve has negative slope. • Curve is concave up.	• Particle is on the negative side of the origin. • Particle is moving in the negative direction. • Velocity is increasing. • Particle is slowing down.
	• $s(t_0) > 0$ • Curve has zero slope. • Curve is concave down.	• Particle is on the positive side of the origin. • Particle is momentarily stopped. • Velocity is decreasing.



▲ **Figure 4.6.5**

▶ **Example 5** Use the position versus time curve in Figure 4.6.5 to determine when the particle in Example 1 is speeding up and slowing down.

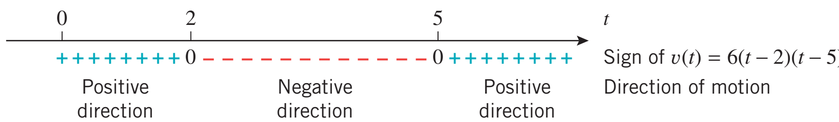
Solution. From $t = 0$ to $t = 2$, the acceleration and velocity are positive, so the particle is speeding up. From $t = 2$ to $t = 4$, the acceleration is negative and the velocity is positive, so the particle is slowing down. At $t = 4$, the velocity is zero, so the particle has momentarily stopped. From $t = 4$ to $t = 6$, the acceleration is negative and the velocity is negative, so the particle is speeding up. From $t = 6$ to $t = 7$, the acceleration is positive and the velocity is negative, so the particle is slowing down. Thereafter, the velocity is zero, so the particle has stopped. ◀

▶ **Example 6** Suppose that the position function of a particle moving on a coordinate line is given by $s(t) = 2t^3 - 21t^2 + 60t + 3$. Analyze the motion of the particle for $t \geq 0$.

Solution. The velocity and acceleration functions are

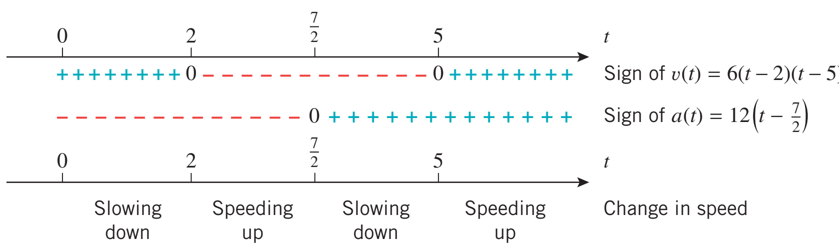
$$v(t) = s'(t) = 6t^2 - 42t + 60 = 6(t-2)(t-5)$$
$$a(t) = v'(t) = 12t - 42 = 12\left(t - \tfrac{7}{2}\right)$$

- *Direction of motion:* The sign analysis of the velocity function in Figure 4.6.6 shows that the particle is moving in the positive direction over the time interval $0 \leq t < 2$, stops momentarily at time $t = 2$, moves in the negative direction over the time interval $2 < t < 5$, stops momentarily at time $t = 5$, and then moves in the positive direction thereafter.



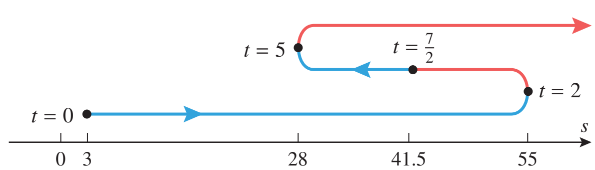
▲ **Figure 4.6.6**

- *Change in speed:* A comparison of the signs of the velocity and acceleration functions is shown in Figure 4.6.7. Since the particle is speeding up when the signs are the same and is slowing down when they are opposite, we see that the particle is slowing down over the time interval $0 \le t < 2$ and stops momentarily at time $t = 2$. It is then speeding up over the time interval $2 < t < \frac{7}{2}$. At time $t = \frac{7}{2}$ the instantaneous acceleration is zero, so the particle is neither speeding up nor slowing down. It is then slowing down over the time interval $\frac{7}{2} < t < 5$ and stops momentarily at time $t = 5$. Thereafter, it is speeding up.



▲ **Figure 4.6.7**

Conclusions: The diagram in Figure 4.6.8 summarizes the above information schematically. The curved line is descriptive only; the actual path is back and forth on the coordinate line. The coordinates of the particle at times $t = 0$, $t = 2$, $t = \frac{7}{2}$, and $t = 5$ were computed from $s(t)$. Segments in red indicate that the particle is speeding up and segments in blue indicate that it is slowing down. ◀



▶ **Figure 4.6.8**

✔QUICK CHECK EXERCISES 4.6 *(See page 296 for answers.)*

1. For a particle in rectilinear motion, the velocity and position functions $v(t)$ and $s(t)$ are related by the equation ________, and the acceleration and velocity functions $a(t)$ and $v(t)$ are related by the equation ________.

2. Suppose that a particle moving along the s-axis has position function $s(t) = 7t - 2t^2$. At time $t = 3$, the particle's position is ________, its velocity is ________, its speed is ________, and its acceleration is ________.

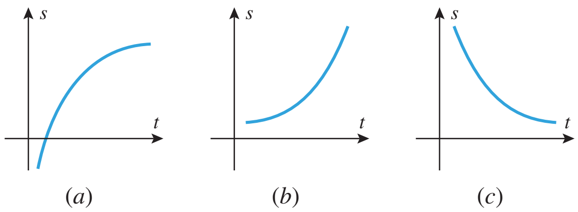
3. A particle in rectilinear motion is speeding up if the signs of its velocity and acceleration are ________, and it is slowing down if these signs are ________.

4. Suppose that a particle moving along the s-axis has position function $s(t) = t^4 - 24t^2$ over the time interval $t \ge 0$. The particle slows down over the time interval(s) ________.

EXERCISE SET 4.6 Graphing Utility

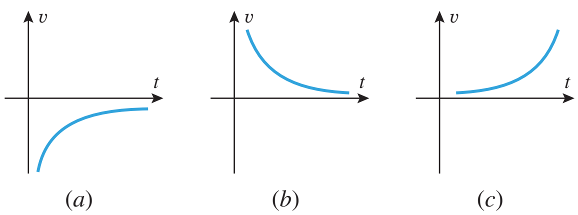
FOCUS ON CONCEPTS

1. The graphs of three position functions are shown in the accompanying figure. In each case determine the signs of the velocity and acceleration, and then determine whether the particle is speeding up or slowing down.



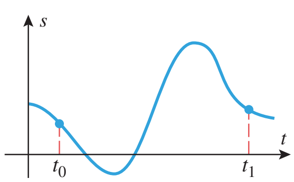
▲ **Figure Ex-1**

2. The graphs of three velocity functions are shown in the accompanying figure. In each case determine the sign of the acceleration, and then determine whether the particle is speeding up or slowing down.



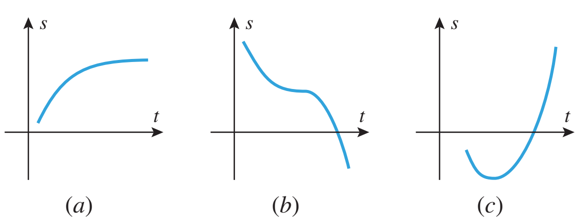
▲ **Figure Ex-2**

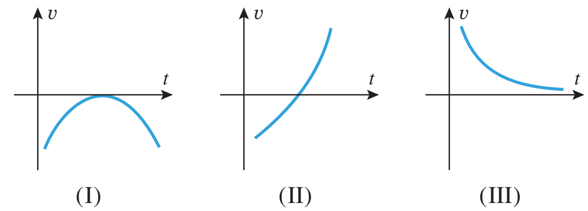
3. The graph of the position function of a particle moving on a horizontal line is shown in the accompanying figure.
(a) Is the particle moving left or right at time t_0?
(b) Is the acceleration positive or negative at time t_0?
(c) Is the particle speeding up or slowing down at time t_0?
(d) Is the particle speeding up or slowing down at time t_1?



◄ **Figure Ex-3**

4. For the graphs in the accompanying figure, match the position functions (a)–(c) with their corresponding velocity functions (I)–(III).

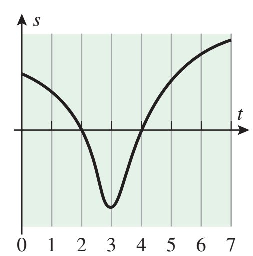




▲ **Figure Ex-4**

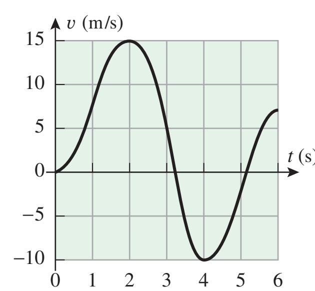
5. Sketch a reasonable graph of s versus t for a mouse that is trapped in a narrow corridor (an s-axis with the positive direction to the right) and scurries back and forth as follows. It runs right with a constant speed of 1.2 m/s for a while, then gradually slows down to 0.6 m/s, then quickly speeds up to 2.0 m/s, then gradually slows to a stop but immediately reverses direction and quickly speeds up to 1.2 m/s.

6. The accompanying figure shows the position versus time curve for an ant that moves along a narrow vertical pipe, where t is measured in seconds and the s-axis is along the pipe with the positive direction up.
(a) When, if ever, is the ant above the origin?
(b) When, if ever, does the ant have velocity zero?
(c) When, if ever, is the ant moving down the pipe?



◄ **Figure Ex-6**

7. The accompanying figure shows the graph of velocity versus time for a particle moving along a coordinate line. Make a rough sketch of the graphs of speed versus time and acceleration versus time.

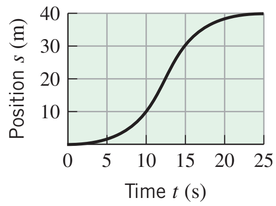


◄ **Figure Ex-7**

8. The accompanying figure (on the next page) shows the position versus time graph for an elevator that ascends 40 m from one stop to the next.
(a) Estimate the velocity when the elevator is halfway up to the top.

(cont.)

(b) Sketch rough graphs of the velocity versus time curve and the acceleration versus time curve.



◀ **Figure Ex-8**

9–12 True–False Determine whether the statement is true or false. Explain your answer. ■

9. A particle is speeding up when its position versus time graph is increasing.

10. Velocity is the derivative of position with respect to time.

11. Acceleration is the absolute value of velocity.

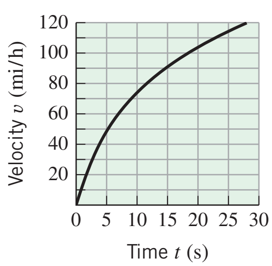
12. If the position versus time curve is increasing and concave down, then the particle is slowing down.

13. The accompanying figure shows the velocity versus time graph for a test run on a Pontiac Grand Prix GTP. Using this graph, estimate
(a) the acceleration at 60 mi/h (in ft/s^2)
(b) the time at which the maximum acceleration occurs.

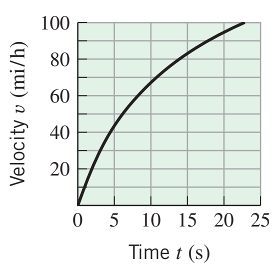
Source: Data from *Car and Driver Magazine*, July 2003.

14. The accompanying figure shows the velocity versus time graph for a test run on a Chevrolet Malibu. Using this graph, estimate
(a) the acceleration at 60 mi/h (in ft/s^2)
(b) the time at which the maximum acceleration occurs.

Source: Data from *Car and Driver Magazine*, November 2003.



▲ **Figure Ex-13**



▲ **Figure Ex-14**

15–16 The function $s(t)$ describes the position of a particle moving along a coordinate line, where s is in meters and t is in seconds.
(a) Make a table showing the position, velocity, and acceleration to two decimal places at times $t = 1, 2, 3, 4, 5$.
(b) At each of the times in part (a), determine whether the particle is stopped; if it is not, state its direction of motion.
(c) At each of the times in part (a), determine whether the particle is speeding up, slowing down, or neither. ■

15. $s(t) = \sin \dfrac{\pi t}{4}$

16. $s(t) = t^4 e^{-t}, \quad t \geq 0$

17–22 The function $s(t)$ describes the position of a particle moving along a coordinate line, where s is in feet and t is in seconds.
(a) Find the velocity and acceleration functions.
(b) Find the position, velocity, speed, and acceleration at time $t = 1$.
(c) At what times is the particle stopped?
(d) When is the particle speeding up? Slowing down?
(e) Find the total distance traveled by the particle from time $t = 0$ to time $t = 5$. ■

17. $s(t) = t^3 - 3t^2, \quad t \geq 0$

18. $s(t) = t^4 - 4t^2 + 4, \quad t \geq 0$

19. $s(t) = 9 - 9\cos(\pi t/3), \quad 0 \leq t \leq 5$

20. $s(t) = \dfrac{t}{t^2 + 4}, \quad t \geq 0$

21. $s(t) = (t^2 + 8)e^{-t/3}, \quad t \geq 0$

22. $s(t) = \frac{1}{4}t^2 - \ln(t + 1), \quad t \geq 0$

23. Let $s(t) = t/(t^2 + 5)$ be the position function of a particle moving along a coordinate line, where s is in meters and t is in seconds. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for $t \geq 0$, and use those graphs where needed.
(a) Use the appropriate graph to make a rough estimate of the time at which the particle first reverses the direction of its motion; and then find the time exactly.
(b) Find the exact position of the particle when it first reverses the direction of its motion.
(c) Use the appropriate graphs to make a rough estimate of the time intervals on which the particle is speeding up and on which it is slowing down; and then find those time intervals exactly.

24. Let $s(t) = t/e^t$ be the position function of a particle moving along a coordinate line, where s is in meters and t is in seconds. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for $t \geq 0$, and use those graphs where needed.
(a) Use the appropriate graph to make a rough estimate of the time at which the particle first reverses the direction of its motion; and then find the time exactly.
(b) Find the exact position of the particle when it first reverses the direction of its motion.
(c) Use the appropriate graphs to make a rough estimate of the time intervals on which the particle is speeding up and on which it is slowing down; and then find those time intervals exactly.

25–32 A position function of a particle moving along a coordinate line is given. Use the method of Example 6 to analyze the motion of the particle for $t \geq 0$, and give a schematic picture of the motion (as in Figure 4.6.8). ■

25. $s = -4t + 3$

26. $s = 5t^2 - 20t$

27. $s = t^3 - 9t^2 + 24t$

28. $s = t^3 - 6t^2 + 9t + 1$

29. $s = 16te^{-(t^2/8)}$ **30.** $s = t + \dfrac{25}{t+2}$

31. $s = \begin{cases} \cos t, & 0 \le t < 2\pi \\ 1, & t \ge 2\pi \end{cases}$

32. $s = \begin{cases} 2t(t-2)^2, & 0 \le t < 3 \\ 13 - 7(t-4)^2, & t \ge 3 \end{cases}$

33. Let $s(t) = 5t^2 - 22t$ be the position function of a particle moving along a coordinate line, where s is in feet and t is in seconds.
(a) Find the maximum speed of the particle during the time interval $1 \le t \le 3$.
(b) When, during the time interval $1 \le t \le 3$, is the particle farthest from the origin? What is its position at that instant?

34. Let $s = 100/(t^2 + 12)$ be the position function of a particle moving along a coordinate line, where s is in feet and t is in seconds. Find the maximum speed of the particle for $t \ge 0$, and find the direction of motion of the particle when it has its maximum speed.

35–36 A position function of a particle moving along a coordinate line is provided. (a) Evaluate s and v when $a = 0$. (b) Evaluate s and a when $v = 0$. ■

35. $s = \ln(3t^2 - 12t + 13)$ **36.** $s = t^3 - 6t^2 + 1$

37. Let $s = \sqrt{2t^2 + 1}$ be the position function of a particle moving along a coordinate line.
(a) Use a graphing utility to generate the graph of v versus t, and make a conjecture about the velocity of the particle as $t \to +\infty$.
(b) Check your conjecture by finding $\lim_{t \to +\infty} v$.

38. (a) Use the chain rule to show that for a particle in rectilinear motion $a = v(dv/ds)$.
(b) Let $s = \sqrt{3t + 7}$, $t \ge 0$. Find a formula for v in terms of s and use the equation in part (a) to find the acceleration when $s = 5$.

39. Suppose that the position functions of two particles, P_1 and P_2, in motion along the same line are

$$s_1 = \tfrac{1}{2}t^2 - t + 3 \quad \text{and} \quad s_2 = -\tfrac{1}{4}t^2 + t + 1$$

respectively, for $t \ge 0$.
(a) Prove that P_1 and P_2 do not collide.
(b) How close do P_1 and P_2 get to each other?
(c) During what intervals of time are they moving in opposite directions?

40. Let $s_A = 15t^2 + 10t + 20$ and $s_B = 5t^2 + 40t$, $t \ge 0$, be the position functions of cars A and B that are moving along parallel straight lanes of a highway.
(a) How far is car A ahead of car B when $t = 0$?
(b) At what instants of time are the cars next to each other?
(c) At what instant of time do they have the same velocity? Which car is ahead at this instant?

41. Prove that a particle is speeding up if the velocity and acceleration have the same sign, and slowing down if they have opposite signs. [*Hint:* Let $r(t) = |v(t)|$ and find $r'(t)$ using the chain rule.]

42. **Writing** A speedometer on a bicycle calculates the bicycle's speed by measuring the time per rotation for one of the bicycle's wheels. Explain how this measurement can be used to calculate an average velocity for the bicycle, and discuss how well it approximates the instantaneous velocity for the bicycle.

43. **Writing** A toy rocket is launched into the air and falls to the ground after its fuel runs out. Describe the rocket's acceleration and when the rocket is speeding up or slowing down during its flight. Accompany your description with a sketch of a graph of the rocket's acceleration versus time.

✔ QUICK CHECK ANSWERS 4.6

1. $v(t) = s'(t)$; $a(t) = v'(t)$ **2.** 3; −5; 5; −4 **3.** the same; opposite **4.** $2 < t < 2\sqrt{3}$

4.7 NEWTON'S METHOD

In Section 1.5 we showed how to approximate the roots of an equation $f(x) = 0$ using the Intermediate-Value Theorem. In this section we will study a technique, called "Newton's Method," that is usually more efficient than that method. Newton's Method is the technique used by many commercial and scientific computer programs for finding roots.

■ NEWTON'S METHOD

In beginning algebra one learns that the solution of a first-degree equation $ax + b = 0$ is given by the formula $x = -b/a$, and the solutions of a second-degree equation

$$ax^2 + bx + c = 0$$

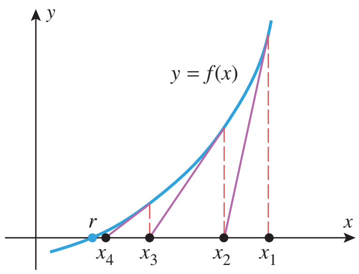
are given by the quadratic formula. Formulas also exist for the solutions of all third- and fourth-degree equations, although they are too complicated to be of practical use. In 1826 it was shown by the Norwegian mathematician Niels Henrik Abel that it is impossible to construct a similar formula for the solutions of a *general* fifth-degree equation or higher. Thus, for a *specific* fifth-degree polynomial equation such as

$$x^5 - 9x^4 + 2x^3 - 5x^2 + 17x - 8 = 0$$

it may be difficult or impossible to find exact values for all of the solutions. Similar difficulties occur for nonpolynomial equations such as

$$x - \cos x = 0$$

For such equations the solutions are generally approximated in some way, often by the method we will now discuss.



▲ **Figure 4.7.1**

Suppose that we are trying to find a root r of the equation $f(x) = 0$, and suppose that by some method we are able to obtain an initial rough estimate, x_1, of r, say by generating the graph of $y = f(x)$ with a graphing utility and examining the x-intercept. If $f(x_1) = 0$, then $r = x_1$. If $f(x_1) \neq 0$, then we consider an easier problem, that of finding a root to a linear equation. The best linear approximation to $y = f(x)$ near $x = x_1$ is given by the tangent line to the graph of f at x_1, so it might be reasonable to expect that the x-intercept to this tangent line provides an improved approximation to r. Call this intercept x_2 (Figure 4.7.1). We can now treat x_2 in the same way we did x_1. If $f(x_2) = 0$, then $r = x_2$. If $f(x_2) \neq 0$, then construct the tangent line to the graph of f at x_2, and take x_3 to be the x-intercept of this tangent line. Continuing in this way we can generate a succession of values $x_1, x_2, x_3, x_4, \ldots$ that will usually approach r. This procedure for approximating r is called ***Newton's Method***.

To implement Newton's Method analytically, we must derive a formula that will tell us how to calculate each improved approximation from the preceding approximation. For this purpose, we note that the point-slope form of the tangent line to $y = f(x)$ at the initial

Niels Henrik Abel (1802–1829) Norwegian mathematician. Abel was the son of a poor Lutheran minister and a remarkably beautiful mother from whom he inherited strikingly good looks. In his brief life of 26 years Abel lived in virtual poverty and suffered a succession of adversities, yet he managed to prove major results that altered the mathematical landscape forever. At the age of thirteen he was sent away from home to a school whose better days had long passed. By a stroke of luck the school had just hired a teacher named Bernt Michael Holmboe, who quickly discovered that Abel had extraordinary mathematical ability. Together, they studied the calculus texts of Euler and works of Newton and the later French mathematicians. By the time he graduated, Abel was familar with most of the great mathematical literature. In 1820 his father died, leaving the family in dire financial straits. Abel was able to enter the University of Christiania in Oslo only because he was granted a free room and several professors supported him directly from their salaries. The University had no advanced courses in mathematics, so Abel took a preliminary degree in 1822 and then continued to study mathematics on his own. In 1824 he published at his own expense the proof that it is impossible to solve the general fifth-degree polynomial equation algebraically. With the hope that this landmark paper would lead to his recognition and acceptance by the European mathematical community, Abel sent the paper to the great German mathematician Gauss, who casually declared it to be a "monstrosity" and tossed it aside. However, in 1826 Abel's paper on the fifth-degree equation and other work was published in the first issue of a new journal, founded by his friend, Leopold Crelle. In the summer of 1826 he completed a landmark work on transcendental functions, which he submitted to the French Academy of Sciences. He hoped to establish himself as a major mathematician, for many young mathematicians had gained quick distinction by having their work accepted by the Academy. However, Abel waited in vain because the paper was either ignored or misplaced by one of the referees, and it did not surface again until two years after his death. That paper was later described by one major mathematician as "…the most important mathematical discovery that has been made in our century…." After submitting his paper, Abel returned to Norway, ill with tuberculosis and in heavy debt. While eking out a meager living as a tutor, he continued to produce great work and his fame spread. Soon great efforts were being made to secure a suitable mathematical position for him. Fearing that his great work had been lost by the Academy, he mailed a proof of the main results to Crelle in January of 1829. In April he suffered a violent hemorrhage and died. Two days later Crelle wrote to inform him that an appointment had been secured for him in Berlin and his days of poverty were over! Abel's great paper was finally published by the Academy twelve years after his death.

approximation x_1 is

$$y - f(x_1) = f'(x_1)(x - x_1) \tag{1}$$

If $f'(x_1) \neq 0$, then this line is not parallel to the x-axis and consequently it crosses the x-axis at some point $(x_2, 0)$. Substituting the coordinates of this point in (1) yields

$$-f(x_1) = f'(x_1)(x_2 - x_1)$$

Solving for x_2 we obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{2}$$

The next approximation can be obtained more easily. If we view x_2 as the starting approximation and x_3 the new approximation, we can simply apply (2) with x_2 in place of x_1 and x_3 in place of x_2. This yields

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \tag{3}$$

provided $f'(x_2) \neq 0$. In general, if x_n is the nth approximation, then it is evident from the pattern in (2) and (3) that the improved approximation x_{n+1} is given by

Newton's Method

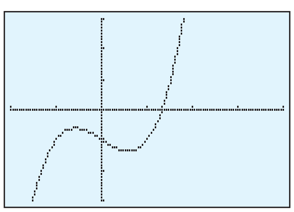
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 1, 2, 3, \ldots \tag{4}$$

▶ **Example 1** Use Newton's Method to approximate the real solutions of

$$x^3 - x - 1 = 0$$

Solution. Let $f(x) = x^3 - x - 1$, so $f'(x) = 3x^2 - 1$ and (4) becomes

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} \tag{5}$$



[−2, 4] × [−3, 3]
xScl = 1, yScl = 1

$y = x^3 - x - 1$

▲ **Figure 4.7.2**

From the graph of f in Figure 4.7.2, we see that the given equation has only one real solution. This solution lies between 1 and 2 because $f(1) = -1 < 0$ and $f(2) = 5 > 0$. We will use $x_1 = 1.5$ as our first approximation ($x_1 = 1$ or $x_1 = 2$ would also be reasonable choices).

Letting $n = 1$ in (5) and substituting $x_1 = 1.5$ yields

$$x_2 = 1.5 - \frac{(1.5)^3 - 1.5 - 1}{3(1.5)^2 - 1} \approx 1.34782609 \tag{6}$$

(We used a calculator that displays nine digits.) Next, we let $n = 2$ in (5) and substitute x_2 to obtain

$$x_3 = x_2 - \frac{x_2^3 - x_2 - 1}{3x_2^2 - 1} \approx 1.32520040 \tag{7}$$

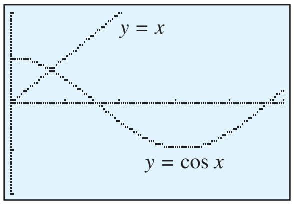
If we continue this process until two identical approximations are generated in succession, we obtain

$$\begin{aligned} x_1 &= 1.5 \\ x_2 &\approx 1.34782609 \\ x_3 &\approx 1.32520040 \\ x_4 &\approx 1.32471817 \\ x_5 &\approx 1.32471796 \\ x_6 &\approx 1.32471796 \end{aligned}$$

TECHNOLOGY MASTERY

Many calculators and computer programs calculate internally with more digits than they display. Where possible, you should use stored calculated values rather than values displayed from earlier calculations. Thus, in Example 1 the value of x_2 used in (7) should be the stored value, not the value in (6).

At this stage there is no need to continue further because we have reached the display accuracy limit of our calculator, and all subsequent approximations that the calculator generates will likely be the same. Thus, the solution is approximately $x \approx 1.32471796$. ◀



$[0, 5] \times [-2, 2]$
xScl = 1, yScl = 1

▲ **Figure 4.7.3**

▶ **Example 2** It is evident from Figure 4.7.3 that if x is in radians, then the equation

$$\cos x = x$$

has a solution between 0 and 1. Use Newton's Method to approximate it.

Solution. Rewrite the equation as

$$x - \cos x = 0$$

and apply (4) with $f(x) = x - \cos x$. Since $f'(x) = 1 + \sin x$, (4) becomes

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n} \tag{8}$$

From Figure 4.7.3, the solution seems closer to $x = 1$ than $x = 0$, so we will use $x_1 = 1$ (radian) as our initial approximation. Letting $n = 1$ in (8) and substituting $x_1 = 1$ yields

$$x_2 = 1 - \frac{1 - \cos 1}{1 + \sin 1} \approx 0.750363868$$

Next, letting $n = 2$ in (8) and substituting this value of x_2 yields

$$x_3 = x_2 - \frac{x_2 - \cos x_2}{1 + \sin x_2} \approx 0.739112891$$

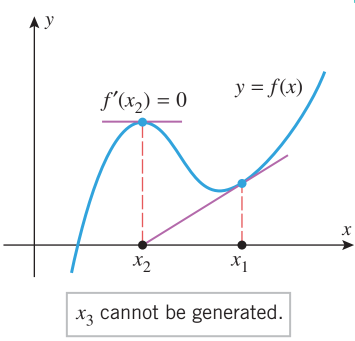
If we continue this process until two identical approximations are generated in succession, we obtain

$$\begin{aligned} x_1 &= 1 \\ x_2 &\approx 0.750363868 \\ x_3 &\approx 0.739112891 \\ x_4 &\approx 0.739085133 \\ x_5 &\approx 0.739085133 \end{aligned}$$

Thus, to the accuracy limit of our calculator, the solution of the equation $\cos x = x$ is $x \approx 0.739085133$. ◀

SOME DIFFICULTIES WITH NEWTON'S METHOD

When Newton's Method works, the approximations usually converge toward the solution with dramatic speed. However, there are situations in which the method fails. For example, if $f'(x_n) = 0$ for some n, then (4) involves a division by zero, making it impossible to generate x_{n+1}. However, this is to be expected because the tangent line to $y = f(x)$ is parallel to the x-axis where $f'(x_n) = 0$, and hence this tangent line does not cross the x-axis to generate the next approximation (Figure 4.7.4).



▲ **Figure 4.7.4**

Newton's Method can fail for other reasons as well; sometimes it may overlook the root you are trying to find and converge to a different root, and sometimes it may fail to converge altogether. For example, consider the equation

$$x^{1/3} = 0$$

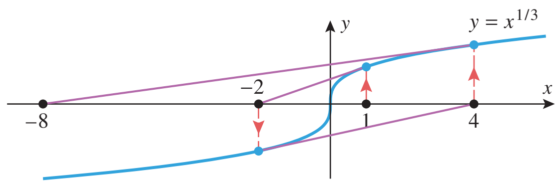
which has $x = 0$ as its only solution, and try to approximate this solution by Newton's Method with a starting value of $x_0 = 1$. Letting $f(x) = x^{1/3}$, Formula (4) becomes

$$x_{n+1} = x_n - \frac{(x_n)^{1/3}}{\frac{1}{3}(x_n)^{-2/3}} = x_n - 3x_n = -2x_n$$

Beginning with $x_1 = 1$, the successive values generated by this formula are

$$x_1 = 1, \quad x_2 = -2, \quad x_3 = 4, \quad x_4 = -8, \ldots$$

which obviously do not converge to $x = 0$. Figure 4.7.5 illustrates what is happening geometrically in this situation.

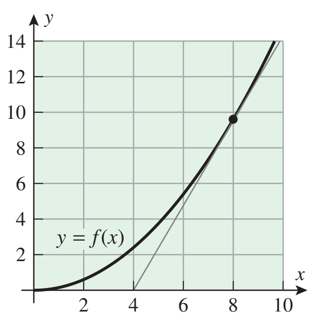


▶ **Figure 4.7.5**

To learn more about the conditions under which Newton's Method converges and for a discussion of error questions, you should consult a book on numerical analysis. For a more in-depth discussion of Newton's Method and its relationship to contemporary studies of chaos and fractals, you may want to read the article, "Newton's Method and Fractal Patterns," by Philip Straffin, which appears in *Applications of Calculus*, MAA Notes, Vol. 3, No. 29, 1993, published by the Mathematical Association of America.

✔ QUICK CHECK EXERCISES 4.7 (See page 302 for answers.)

1. Use the accompanying graph to estimate x_2 and x_3 if Newton's Method is applied to the equation $y = f(x)$ with $x_1 = 8$.
2. Suppose that $f(1) = 2$ and $f'(1) = 4$. If Newton's Method is applied to $y = f(x)$ with $x_1 = 1$, then $x_2 =$ ________.
3. Suppose we are given that $f(0) = 3$ and that $x_2 = 3$ when Newton's Method is applied to $y = f(x)$ with $x_1 = 0$. Then $f'(0) =$ ________.
4. If Newton's Method is applied to $y = e^x - 1$ with $x_1 = \ln 2$, then $x_2 =$ ________.



◀ **Figure Ex-1**

EXERCISE SET 4.7 Graphing Utility

In this exercise set, express your answers with as many decimal digits as your calculating utility can display, but use the procedure in the Technology Mastery on p. 298. ■

1. Approximate $\sqrt{2}$ by applying Newton's Method to the equation $x^2 - 2 = 0$.
2. Approximate $\sqrt{5}$ by applying Newton's Method to the equation $x^2 - 5 = 0$.
3. Approximate $\sqrt[3]{6}$ by applying Newton's Method to the equation $x^3 - 6 = 0$.
4. To what equation would you apply Newton's Method to approximate the nth root of a?

5–8 The given equation has one real solution. Approximate it by Newton's Method. ■

5. $x^3 - 2x - 2 = 0$ **6.** $x^3 + x - 1 = 0$

7. $x^5 + x^4 - 5 = 0$ **8.** $x^5 - 3x + 3 = 0$

9–14 Use a graphing utility to determine how many solutions the equation has, and then use Newton's Method to approximate the solution that satisfies the stated condition. ■

9. $x^4 + x^2 - 4 = 0$; $x < 0$

10. $x^5 - 5x^3 - 2 = 0$; $x > 0$

11. $2\cos x = x$; $x > 0$ **12.** $\sin x = x^2$; $x > 0$

13. $x - \tan x = 0;\ \pi/2 < x < 3\pi/2$

14. $1 + e^x \sin x = 0;\ \pi/2 < x < 3\pi/2$

15–20 Use a graphing utility to determine the number of times the curves intersect and then apply Newton's Method, where needed, to approximate the x-coordinates of all intersections.

15. $y = x^3$ and $y = 1 - x$

16. $y = \sin x$ and $y = x^3 - 2x^2 + 1$

17. $y = x^2$ and $y = \sqrt{2x + 1}$

18. $y = \frac{1}{8}x^3 - 1$ and $y = \cos x - 2$

19. $y = 1$ and $y = e^x \sin x;\ 0 < x < \pi$

20. $y = e^{-x}$ and $y = \ln x$

21–24 True–False Determine whether the statement is true or false. Explain your answer.

21. Newton's Method uses the tangent line to $y = f(x)$ at $x = x_n$ to compute x_{n+1}.

22. Newton's Method is a process to find exact solutions to $f(x) = 0$.

23. If $f(x) = 0$ has a root, then Newton's Method starting at $x = x_1$ will approximate the root nearest x_1.

24. Newton's Method can be used to appoximate a point of intersection of two curves.

25. The ***mechanic's rule*** for approximating square roots states that $\sqrt{a} \approx x_{n+1}$, where

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 1, 2, 3, \ldots$$

and x_1 is any positive approximation to $\sqrt{a}$.

(a) Apply Newton's Method to

$$f(x) = x^2 - a$$

to derive the mechanic's rule.

(b) Use the mechanic's rule to approximate $\sqrt{10}$.

26. Many calculators compute reciprocals using the approximation $1/a \approx x_{n+1}$, where

$$x_{n+1} = x_n(2 - ax_n), \quad n = 1, 2, 3, \ldots$$

and x_1 is an initial approximation to $1/a$. This formula makes it possible to perform divisions using multiplications and subtractions, which is a faster procedure than dividing directly.

(a) Apply Newton's Method to

$$f(x) = \frac{1}{x} - a$$

to derive this approximation.

(b) Use the formula to approximate $\frac{1}{17}$.

27. Use Newton's Method to approximate the absolute minimum of $f(x) = \frac{1}{4}x^4 + x^2 - 5x$.

28. Use Newton's Method to approximate the absolute maximum of $f(x) = x \sin x$ on the interval $[0, \pi]$.

29. For the function

$$f(x) = \frac{e^{-x}}{1 + x^2}$$

use Newton's Method to approximate the x-coordinates of the inflection points to two decimal places.

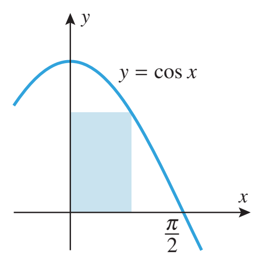
30. Use Newton's Method to approximate the absolute maximum of $f(x) = (1 - 2x)\tan^{-1} x$.

31. Use Newton's Method to approximate the coordinates of the point on the parabola $y = x^2$ that is closest to the point $(1, 0)$.

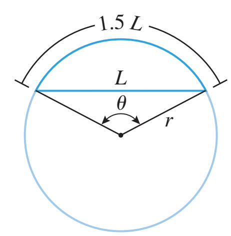
32. Use Newton's Method to approximate the dimensions of the rectangle of largest area that can be inscribed under the curve $y = \cos x$ for $0 \le x \le \pi/2$ (Figure Ex-32).

33. (a) Show that on a circle of radius r, the central angle θ that subtends an arc whose length is 1.5 times the length L of its chord satisfies the equation $\theta = 3\sin(\theta/2)$ (Figure Ex-33).

(b) Use Newton's Method to approximate θ.

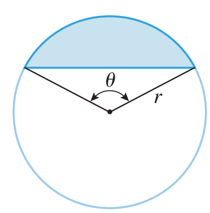


▲ **Figure Ex-32**



▲ **Figure Ex-33**

34. A ***segment*** of a circle is the region enclosed by an arc and its chord (Figure Ex-34). If r is the radius of the circle and θ the angle subtended at the center of the circle, then it can be shown that the area A of the segment is $A = \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is in radians. Find the value of θ for which the area of the segment is one-fourth the area of the circle. Give θ to the nearest degree.



◀ **Figure Ex-34**

35–36 Use Newton's Method to approximate all real values of y satisfying the given equation for the indicated value of x.

35. $xy^4 + x^3y = 1;\ x = 1$

36. $xy - \cos\left(\frac{1}{2}xy\right) = 0;\ x = 2$

37. An ***annuity*** is a sequence of equal payments that are paid or received at regular time intervals. For example, you may want to deposit equal amounts at the end of each year into an interest-bearing account for the purpose of accumulating a lump sum at some future time. If, at the end of each year, interest of $i \times 100\%$ on the account balance for that year is added to the account, then the account is said to pay $i \times 100\%$ interest, ***compounded annually***. It can be shown that if payments of Q dollars are deposited at the end of each year into an account that pays $i \times 100\%$ compounded annually, then at the time when the nth payment and the accrued interest for the past year are deposited, the amount $S(n)$ in the account is given by the formula

$$S(n) = \frac{Q}{i}[(1+i)^n - 1]$$

Suppose that you can invest \$5000 in an interest-bearing account at the end of each year, and your objective is to have \$250,000 on the 25th payment. Approximately what annual compound interest rate must the account pay for you to achieve your goal? [*Hint:* Show that the interest rate i satisfies the equation $50i = (1+i)^{25} - 1$, and solve it using Newton's Method.]

FOCUS ON CONCEPTS

38. (a) Use a graphing utility to generate the graph of

$$f(x) = \frac{x}{x^2+1}$$

and use it to explain what happens if you apply Newton's Method with a starting value of $x_1 = 2$. Check your conclusion by computing x_2, x_3, x_4, and x_5.

(b) Use the graph generated in part (a) to explain what happens if you apply Newton's Method with a starting value of $x_1 = 0.5$. Check your conclusion by computing x_2, x_3, x_4, and x_5.

39. (a) Apply Newton's Method to $f(x) = x^2 + 1$ with a starting value of $x_1 = 0.5$, and determine if the values of $x_2, \ldots, x_{10}$ appear to converge.

(b) Explain what is happening.

40. In each part, explain what happens if you apply Newton's Method to a function f when the given condition is satisfied for some value of n.

(a) $f(x_n) = 0$ (b) $x_{n+1} = x_n$

(c) $x_{n+2} = x_n \neq x_{n+1}$

41. Writing Compare Newton's Method and the Intermediate-Value Theorem (1.5.7; see Example 5 in Section 1.5) as methods to locate solutions to $f(x) = 0$.

42. Writing Newton's Method uses a local linear approximation to $y = f(x)$ at $x = x_n$ to find an "improved" approximation x_{n+1} to a zero of f. Your friend proposes a process that uses a local quadratic approximation to $y = f(x)$ at $x = x_n$ (that is, matching values for the function and its first two derivatives) to obtain x_{n+1}. Discuss the pros and cons of this proposal. Support your statements with some examples.

✔ QUICK CHECK ANSWERS 4.7

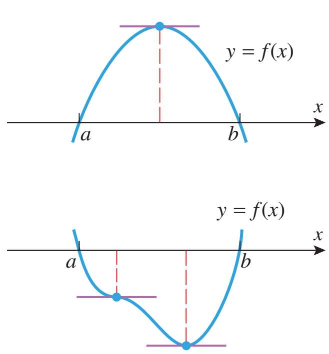
1. $x_2 \approx 4, x_3 \approx 2$ **2.** $\frac{1}{2}$ **3.** -1 **4.** $\ln 2 - \frac{1}{2} \approx 0.193147$

4.8 ROLLE'S THEOREM; MEAN-VALUE THEOREM

In this section we will discuss a result called the Mean-Value Theorem. This theorem has so many important consequences that it is regarded as one of the major principles in calculus.

ROLLE'S THEOREM

We will begin with a special case of the Mean-Value Theorem, called Rolle's Theorem, in honor of the mathematician Michel Rolle. This theorem states the geometrically obvious fact that if the graph of a differentiable function intersects the x-axis at two places, a and b, then somewhere between a and b there must be at least one place where the tangent line is horizontal (Figure 4.8.1). The precise statement of the theorem is as follows.



▲ **Figure 4.8.1**

4.8.1 THEOREM (*Rolle's Theorem*) *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If*

$$f(a) = 0 \quad \text{and} \quad f(b) = 0$$

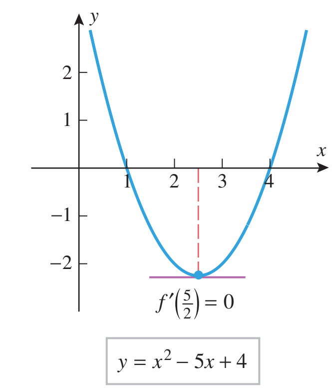
then there is at least one point c in the interval (a, b) such that $f'(c) = 0$.

PROOF We will divide the proof into three cases: the case where $f(x) = 0$ for all x in (a, b), the case where $f(x) > 0$ at some point in (a, b), and the case where $f(x) < 0$ at some point in (a, b).

CASE 1 If $f(x) = 0$ for all x in (a, b), then $f'(c) = 0$ at every point c in (a, b) because f is a constant function on that interval.

CASE 2 Assume that $f(x) > 0$ at some point in (a, b). Since f is continuous on $[a, b]$, it follows from the Extreme-Value Theorem (4.4.2) that f has an absolute maximum on $[a, b]$. The absolute maximum value cannot occur at an endpoint of $[a, b]$ because we have assumed that $f(a) = f(b) = 0$, and that $f(x) > 0$ at some point in (a, b). Thus, the absolute maximum must occur at some point c in (a, b). It follows from Theorem 4.4.3 that c is a critical point of f, and since f is differentiable on (a, b), this critical point must be a stationary point; that is, $f'(c) = 0$.

CASE 3 Assume that $f(x) < 0$ at some point in (a, b). The proof of this case is similar to Case 2 and will be omitted. ■



▲ **Figure 4.8.2**

▶ **Example 1** Find the two x-intercepts of the function $f(x) = x^2 - 5x + 4$ and confirm that $f'(c) = 0$ at some point c between those intercepts.

Solution. The function f can be factored as

$$x^2 - 5x + 4 = (x - 1)(x - 4)$$

so the x-intercepts are $x = 1$ and $x = 4$. Since the polynomial f is continuous and differentiable everywhere, the hypotheses of Rolle's Theorem are satisfied on the interval $[1, 4]$. Thus, we are guaranteed the existence of at least one point c in the interval $(1, 4)$ such that $f'(c) = 0$. Differentiating f yields

$$f'(x) = 2x - 5$$

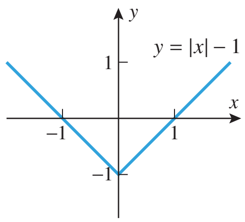
Solving the equation $f'(x) = 0$ yields $x = \frac{5}{2}$, so $c = \frac{5}{2}$ is a point in the interval $(1, 4)$ at which $f'(c) = 0$ (Figure 4.8.2). ◀

▶ **Example 2** The differentiability requirement in Rolle's Theorem is critical. If f fails to be differentiable at even one place in the interval (a, b), then the conclusion of the

Michel Rolle (1652–1719) French mathematician. Rolle, the son of a shopkeeper, received only an elementary education. He married early and as a young man struggled hard to support his family on the meager wages of a transcriber for notaries and attorneys. In spite of his financial problems and minimal education, Rolle studied algebra and Diophantine analysis (a branch of number theory) on his own. Rolle's fortune changed dramatically in 1682 when he published an elegant solution of a difficult, unsolved problem in Diophantine analysis. The public recognition of his achievement led to a patronage under minister Louvois, a job as an elementary mathematics teacher, and eventually to a short-term administrative post in the Ministry of War. In 1685 he joined the Académie des Sciences in a low-level position for which he received no regular salary until 1699. He stayed at the Académie until he died of apoplexy in 1719.

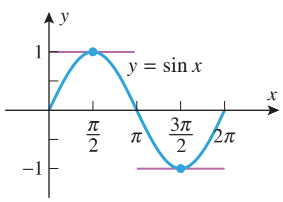
While Rolle's forte was always Diophantine analysis, his most important work was a book on the algebra of equations, called *Traité d'algèbre*, published in 1690. In that book Rolle firmly established the notation $\sqrt[n]{a}$ [earlier written as $\sqrt{n\ a}$] for the nth root of a, and proved a polynomial version of the theorem that today bears his name. (Rolle's Theorem was named by Giusto Bellavitis in 1846.) Ironically, Rolle was one of the most vocal early antagonists of calculus. He strove intently to demonstrate that it gave erroneous results and was based on unsound reasoning. He quarreled so vigorously on the subject that the Académie des Sciences was forced to intervene on several occasions. Among his several achievements, Rolle helped advance the currently accepted size order for negative numbers. Descartes, for example, viewed −2 as smaller than −5. Rolle preceded most of his contemporaries by adopting the current convention in 1691.

theorem may not hold. For example, the function $f(x) = |x| - 1$ graphed in Figure 4.8.3 has roots at $x = -1$ and $x = 1$, yet there is no horizontal tangent to the graph of f over the interval $(-1, 1)$. ◀



▲ **Figure 4.8.3**

▶ **Example 3** If f satisfies the conditions of Rolle's Theorem on $[a, b]$, then the theorem guarantees the existence of *at least* one point c in (a, b) at which $f'(c) = 0$. There may, however, be more than one such c. For example, the function $f(x) = \sin x$ is continuous and differentiable everywhere, so the hypotheses of Rolle's Theorem are satisfied on the interval $[0, 2\pi]$ whose endpoints are roots of f. As indicated in Figure 4.8.4, there are two points in the interval $[0, 2\pi]$ at which the graph of f has a horizontal tangent, $c_1 = \pi/2$ and $c_2 = 3\pi/2$. ◀

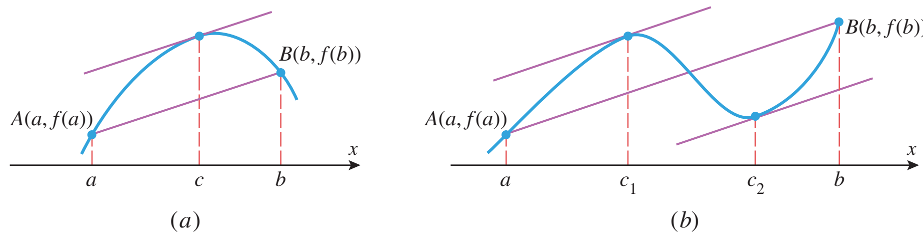


▲ **Figure 4.8.4**

In Examples 1 and 3 we were able to find exact values of c because the equation $f'(x) = 0$ was easy to solve. However, in the applications of Rolle's Theorem it is usually the *existence of* c that is important and not its actual value.

THE MEAN-VALUE THEOREM

Rolle's Theorem is a special case of a more general result, called the ***Mean-Value Theorem***. Geometrically, this theorem states that between any two points $A(a, f(a))$ and $B(b, f(b))$ on the graph of a differentiable function f, there is at least one place where the tangent line to the graph is parallel to the secant line joining A and B (Figure 4.8.5).



▲ **Figure 4.8.5**

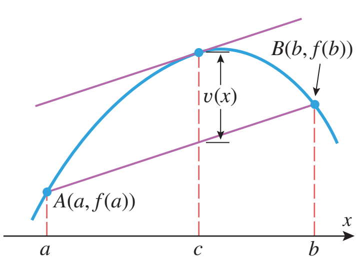
Note that the slope of the secant line joining $A(a, f(a))$ and $B(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a}$$

and that the slope of the tangent line at c in Figure 4.8.5a is $f'(c)$. Similarly, in Figure 4.8.5b the slopes of the tangent lines at c_1 and c_2 are $f'(c_1)$ and $f'(c_2)$, respectively. Since nonvertical parallel lines have the same slope, the Mean-Value Theorem can be stated precisely as follows.

4.8.2 THEOREM (*Mean-Value Theorem*) *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$



The tangent line is parallel to the secant line where the vertical distance $v(x)$ between the secant line and the graph of f is maximum.

▲ **Figure 4.8.6**

MOTIVATION FOR THE PROOF OF THEOREM 4.8.2 Figure 4.8.6 suggests that (1) will hold (i.e., the tangent line will be parallel to the secant line) at a point c where the vertical distance between the curve and the secant line is maximum. Thus, to prove the Mean-Value Theorem it is natural to begin by looking for a formula for the vertical distance $v(x)$ between the curve $y = f(x)$ and the secant line joining $(a, f(a))$ and $(b, f(b))$.

PROOF OF THEOREM 4.8.2 Since the two-point form of the equation of the secant line joining $(a, f(a))$ and $(b, f(b))$ is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or, equivalently,

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

the difference $v(x)$ between the height of the graph of f and the height of the secant line is

$$v(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right] \tag{2}$$

Since $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), so is $v(x)$. Moreover,

$$v(a) = 0 \quad \text{and} \quad v(b) = 0$$

so that $v(x)$ satisfies the hypotheses of Rolle's Theorem on the interval $[a, b]$. Thus, there is a point c in (a, b) such that $v'(c) = 0$. But from Equation (2)

$$v'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

so

$$v'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Since $v'(c) = 0$, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \blacksquare$$

▶ **Example 4** Show that the function $f(x) = \frac{1}{4}x^3 + 1$ satisfies the hypotheses of the Mean-Value Theorem over the interval $[0, 2]$, and find all values of c in the interval $(0, 2)$ at which the tangent line to the graph of f is parallel to the secant line joining the points $(0, f(0))$ and $(2, f(2))$.

Solution. The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[0, 2]$ and differentiable on $(0, 2)$, so the hypotheses of the Mean-Value Theorem are satisfied with $a = 0$ and $b = 2$. But

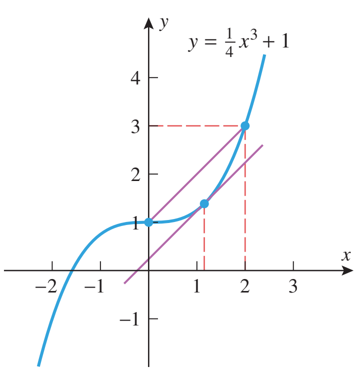
$$f(a) = f(0) = 1, \qquad f(b) = f(2) = 3$$

$$f'(x) = \frac{3x^2}{4}, \qquad f'(c) = \frac{3c^2}{4}$$

so in this case Equation (1) becomes

$$\frac{3c^2}{4} = \frac{3 - 1}{2 - 0} \quad \text{or} \quad 3c^2 = 4$$

which has the two solutions $c = \pm 2/\sqrt{3} \approx \pm 1.15$. However, only the positive solution lies in the interval $(0, 2)$; this value of c is consistent with Figure 4.8.7. ◀



▲ **Figure 4.8.7**

VELOCITY INTERPRETATION OF THE MEAN-VALUE THEOREM

There is a nice interpretation of the Mean-Value Theorem in the situation where $x = f(t)$ is the position versus time curve for a car moving along a straight road. In this case, the right side of (1) is the average velocity of the car over the time interval from $a \le t \le b$, and the left side is the instantaneous velocity at time $t = c$. Thus, the Mean-Value Theorem implies that at least once during the time interval the instantaneous velocity must equal the average velocity. This agrees with our real-world experience—if the average velocity for a trip is 40 mi/h, then sometime during the trip the speedometer has to read 40 mi/h.

▶ **Example 5** You are driving on a straight highway on which the speed limit is 55 mi/h. At 8:05 A.M. a police car clocks your velocity at 50 mi/h and at 8:10 A.M. a second police car posted 5 mi down the road clocks your velocity at 55 mi/h. Explain why the police have a right to charge you with a speeding violation.

Solution. You traveled 5 mi in 5 min $\left(=\frac{1}{12}\text{ h}\right)$, so your average velocity was 60 mi/h. Therefore, the Mean-Value Theorem guarantees the police that your instantaneous velocity was 60 mi/h at least once over the 5 mi section of highway. ◀

CONSEQUENCES OF THE MEAN-VALUE THEOREM

We stated at the beginning of this section that the Mean-Value Theorem is the starting point for many important results in calculus. As an example of this, we will use it to prove Theorem 4.1.2, which was one of our fundamental tools for analyzing graphs of functions.

4.1.2 THEOREM (*Revisited*) *Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).*

(a) If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.

(c) If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

PROOF (*a*) Suppose that x_1 and x_2 are points in $[a, b]$ such that $x_1 < x_2$. We must show that $f(x_1) < f(x_2)$. Because the hypotheses of the Mean-Value Theorem are satisfied on the entire interval $[a, b]$, they are satisfied on the subinterval $[x_1, x_2]$. Thus, there is some point c in the open interval (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

or, equivalently,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \tag{3}$$

Since c is in the open interval (x_1, x_2), it follows that $a < c < b$; thus, $f'(c) > 0$. However, $x_2 - x_1 > 0$ since we assumed that $x_1 < x_2$. It follows from (3) that $f(x_2) - f(x_1) > 0$ or, equivalently, $f(x_1) < f(x_2)$, which is what we were to prove. The proofs of parts (b) and (c) are similar and are left as exercises. ■

THE CONSTANT DIFFERENCE THEOREM

We know from our earliest study of derivatives that the derivative of a constant is zero. Part (c) of Theorem 4.1.2 is the converse of that result; that is, a function whose derivative is zero on an interval must be constant on that interval. If we apply this to the difference of two functions, we obtain the following useful theorem.

4.8.3 THEOREM (*Constant Difference Theorem*) *If f and g are differentiable on an interval, and if $f'(x) = g'(x)$ for all x in that interval, then $f - g$ is constant on the interval; that is, there is a constant k such that $f(x) - g(x) = k$ or, equivalently,*

$$f(x) = g(x) + k$$

for all x in the interval.

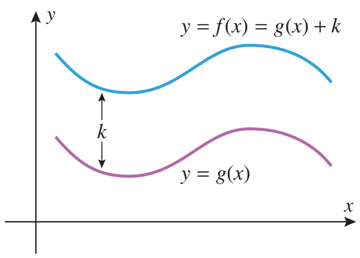
PROOF Let x_1 and x_2 be any points in the interval such that $x_1 < x_2$. Since the functions f and g are differentiable on the interval, they are continuous on the interval. Since $[x_1, x_2]$ is a subinterval, it follows that f and g are continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Moreover, it follows from the basic properties of derivatives and continuity that the same is true of the function

$$F(x) = f(x) - g(x)$$

Since

$$F'(x) = f'(x) - g'(x) = 0$$

it follows from part (c) of Theorem 4.1.2 that $F(x) = f(x) - g(x)$ is constant on the interval $[x_1, x_2]$. This means that $f(x) - g(x)$ has the same value at any two points x_1 and x_2 in the interval, and this implies that $f - g$ is constant on the interval. ■



If $f'(x) = g'(x)$ on an interval, then the graphs of f and g are vertical translations of each other.

▲ **Figure 4.8.8**

Geometrically, the Constant Difference Theorem tells us that if f and g have the same derivative on an interval, then the graphs of f and g are vertical translations of each other over that interval (Figure 4.8.8).

▶ **Example 6** Part (c) of Theorem 4.1.2 is sometimes useful for establishing identities. For example, although we do not need calculus to prove the identity

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \qquad (-1 \le x \le 1) \tag{4}$$

it can be done by letting $f(x) = \sin^{-1} x + \cos^{-1} x$. It follows from Formulas (9) and (10) of Section 3.3 that

$$f'(x) = \frac{d}{dx}[\sin^{-1} x] + \frac{d}{dx}[\cos^{-1} x] = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

so $f(x) = \sin^{-1} x + \cos^{-1} x$ is constant on the interval $[-1, 1]$. We can find this constant by evaluating f at any convenient point in this interval. For example, using $x = 0$ we obtain

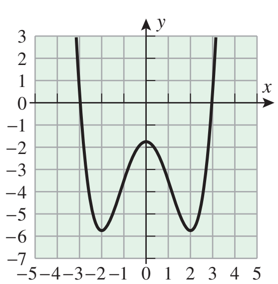
$$f(0) = \sin^{-1} 0 + \cos^{-1} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

which proves (4). ◀

✔QUICK CHECK EXERCISES 4.8 *(See page 310 for answers.)*

1. Let $f(x) = x^2 - x$.
   (a) An interval on which f satisfies the hypotheses of Rolle's Theorem is ______.
   (b) Find all values of c that satisfy the conclusion of Rolle's Theorem for the function f on the interval in part (a).

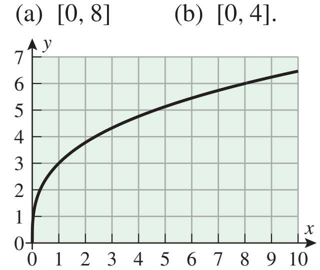
2. Use the accompanying graph of f to find an interval $[a, b]$ on which Rolle's Theorem applies, and find all values of c in that interval that satisfy the conclusion of the theorem.



◀ **Figure Ex-2**

3. Let $f(x) = x^2 - x$.
   (a) Find a point b such that the slope of the secant line through $(0, 0)$ and $(b, f(b))$ is 1.
   (b) Find all values of c that satisfy the conclusion of the Mean-Value Theorem for the function f on the interval $[0, b]$, where b is the point found in part (a).

4. Use the graph of f in the accompanying figure to estimate all values of c that satisfy the conclusion of the Mean-Value Theorem on the interval
   (a) $[0, 8]$ (b) $[0, 4]$.



◀ **Figure Ex-4**

5. Find a function f such that the graph of f contains the point $(1, 5)$ and such that for every value of x_0 the tangent line to the graph of f at x_0 is parallel to the tangent line to the graph of $y = x^2$ at x_0.

EXERCISE SET 4.8 Graphing Utility

1–4 Verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

1. $f(x) = x^2 - 8x + 15$; $[3, 5]$

2. $f(x) = x^3 - 3x^2 + 2x$; $[0, 2]$

3. $f(x) = \cos x$; $[\pi/2, 3\pi/2]$

4. $f(x) = \ln(4 + 2x - x^2)$; $[-1, 3]$

5–8 Verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

5. $f(x) = x^2 - x$; $[-3, 5]$

6. $f(x) = x^3 + x - 4$; $[-1, 2]$

7. $f(x) = \sqrt{x+1}$; $[0, 3]$

8. $f(x) = x - \dfrac{1}{x}$; $[3, 4]$

9. (a) Find an interval $[a, b]$ on which
$$f(x) = x^4 + x^3 - x^2 + x - 2$$
satisfies the hypotheses of Rolle's Theorem.
(b) Generate the graph of $f'(x)$, and use it to make rough estimates of all values of c in the interval obtained in part (a) that satisfy the conclusion of Rolle's Theorem.
(c) Use Newton's Method to improve on the rough estimates obtained in part (b).

10. Let $f(x) = x^3 - 4x$.
(a) Find the equation of the secant line through the points $(-2, f(-2))$ and $(1, f(1))$.
(b) Show that there is only one point c in the interval $(-2, 1)$ that satisfies the conclusion of the Mean-Value Theorem for the secant line in part (a).
(c) Find the equation of the tangent line to the graph of f at the point $(c, f(c))$.
(d) Use a graphing utility to generate the secant line in part (a) and the tangent line in part (c) in the same coordinate system, and confirm visually that the two lines seem parallel.

11–14 True–False Determine whether the statement is true or false. Explain your answer.

11. Rolle's Theorem says that if f is a continuous function on $[a, b]$ and $f(a) = f(b)$, then there is a point between a and b at which the curve $y = f(x)$ has a horizontal tangent line.

12. If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is a point between a and b at which the instantaneous rate of change of f matches the average rate of change of f over $[a, b]$.

13. The Constant Difference Theorem says that if two functions have derivatives that differ by a constant on an interval, then the functions are equal on the interval.

14. One application of the Mean-Value Theorem is to prove that a function with positive derivative on an interval must be increasing on that interval.

FOCUS ON CONCEPTS

15. Let $f(x) = \tan x$.
(a) Show that there is no point c in the interval $(0, \pi)$ such that $f'(c) = 0$, even though $f(0) = f(\pi) = 0$.
(b) Explain why the result in part (a) does not contradict Rolle's Theorem.

16. Let $f(x) = x^{2/3}$, $a = -1$, and $b = 8$.
(a) Show that there is no point c in (a, b) such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
(b) Explain why the result in part (a) does not contradict the Mean-Value Theorem.

17. (a) Show that if f is differentiable on $(-\infty, +\infty)$, and if $y = f(x)$ and $y = f'(x)$ are graphed in the same coordinate system, then between any two x-intercepts of f there is at least one x-intercept of f'.
(b) Give some examples that illustrate this.

18. Review Formulas (8) and (9) in Section 2.1 and use the Mean-Value Theorem to show that if f is differentiable on $(-\infty, +\infty)$, then for any interval $[x_0, x_1]$ there is at least one point in (x_0, x_1) where the instantaneous rate of change of y with respect to x is equal to the average rate of change over the interval.

19–21 Use the result of Exercise 18 in these exercises.

19. An automobile travels 4 mi along a straight road in 5 min. Show that the speedometer reads exactly 48 mi/h at least once during the trip.

20. At 11 A.M. on a certain morning the outside temperature was 76°F. At 11 P.M. that evening it had dropped to 52°F.
(a) Show that at some instant during this period the temperature was decreasing at the rate of 2°F/h.
(b) Suppose that you know the temperature reached a high of 88°F sometime between 11 A.M. and 11 P.M. Show that at some instant during this period the temperature was decreasing at a rate greater than 3°F/h.

21. Suppose that two runners in a 100 m dash finish in a tie. Show that they had the same velocity at least once during the race.

22. Use the fact that

$$\frac{d}{dx}[x \ln(2-x)] = \ln(2-x) - \frac{x}{2-x}$$

to show that the equation $x = (2-x)\ln(2-x)$ has at least one solution in the interval $(0, 1)$.

23. (a) Use the Constant Difference Theorem (4.8.3) to show that if $f'(x) = g'(x)$ for all x in the interval $(-\infty, +\infty)$, and if f and g have the same value at some point x_0, then $f(x) = g(x)$ for all x in $(-\infty, +\infty)$.
(b) Use the result in part (a) to confirm the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

24. (a) Use the Constant Difference Theorem (4.8.3) to show that if $f'(x) = g'(x)$ for all x in $(-\infty, +\infty)$, and if $f(x_0) - g(x_0) = c$ at some point x_0, then

$$f(x) - g(x) = c$$

for all x in $(-\infty, +\infty)$.
(b) Use the result in part (a) to show that the function

$$h(x) = (x-1)^3 - (x^2+3)(x-3)$$

is constant for all x in $(-\infty, +\infty)$, and find the constant.
(c) Check the result in part (b) by multiplying out and simplifying the formula for $h(x)$.

25. Let $g(x) = xe^x - e^x$. Find $f(x)$ so that $f'(x) = g'(x)$ and $f(1) = 2$.

26. Let $g(x) = \tan^{-1} x$. Find $f(x)$ so that $f'(x) = g'(x)$ and $f(1) = 2$.

FOCUS ON CONCEPTS

27. (a) Use the Mean-Value Theorem to show that if f is differentiable on an interval, and if $|f'(x)| \le M$ for all values of x in the interval, then

$$|f(x) - f(y)| \le M|x - y|$$

for all values of x and y in the interval.
(b) Use the result in part (a) to show that

$$|\sin x - \sin y| \le |x - y|$$

for all real values of x and y.

28. (a) Use the Mean-Value Theorem to show that if f is differentiable on an open interval, and if $|f'(x)| \ge M$ for all values of x in the interval, then

$$|f(x) - f(y)| \ge M|x - y|$$

for all values of x and y in the interval.
(b) Use the result in part (a) to show that

$$|\tan x - \tan y| \ge |x - y|$$

for all values of x and y in the interval $(-\pi/2, \pi/2)$.
(c) Use the result in part (b) to show that

$$|\tan x + \tan y| \ge |x + y|$$

for all values of x and y in the interval $(-\pi/2, \pi/2)$.

29. (a) Use the Mean-Value Theorem to show that

$$\sqrt{y} - \sqrt{x} < \frac{y - x}{2\sqrt{x}}$$

if $0 < x < y$.
(b) Use the result in part (a) to show that if $0 < x < y$, then $\sqrt{xy} < \frac{1}{2}(x + y)$.

30. Show that if f is differentiable on an open interval and $f'(x) \ne 0$ on the interval, the equation $f(x) = 0$ can have at most one real root in the interval.

31. Use the result in Exercise 30 to show the following:
(a) The equation $x^3 + 4x - 1 = 0$ has exactly one real root.
(b) If $b^2 - 3ac < 0$ and if $a \ne 0$, then the equation

$$ax^3 + bx^2 + cx + d = 0$$

has exactly one real root.

32. Use the inequality $\sqrt{3} < 1.8$ to prove that

$$1.7 < \sqrt{3} < 1.75$$

[*Hint:* Let $f(x) = \sqrt{x}$, $a = 3$, and $b = 4$ in the Mean-Value Theorem.]

33. Use the Mean-Value Theorem to prove that

$$\frac{x}{1+x^2} < \tan^{-1} x < x \quad (x > 0)$$

34. (a) Show that if f and g are functions for which

$$f'(x) = g(x) \quad \text{and} \quad g'(x) = f(x)$$

for all x, then $f^2(x) - g^2(x)$ is a constant.
(b) Show that the function $f(x) = \frac{1}{2}(e^x + e^{-x})$ and the function $g(x) = \frac{1}{2}(e^x - e^{-x})$ have this property.

35. (a) Show that if f and g are functions for which

$$f'(x) = g(x) \quad \text{and} \quad g'(x) = -f(x)$$

for all x, then $f^2(x) + g^2(x)$ is a constant.
(b) Give an example of functions f and g with this property.

FOCUS ON CONCEPTS

36. Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Prove: If $f(a) = g(a)$ and $f(b) = g(b)$, then there is a point c in (a, b) such that $f'(c) = g'(c)$.

37. Illustrate the result in Exercise 36 by drawing an appropriate picture.

38. (a) Prove that if $f''(x) > 0$ for all x in (a, b), then $f'(x) = 0$ at most once in (a, b).
(b) Give a geometric interpretation of the result in (a).

39. (a) Prove part (*b*) of Theorem 4.1.2.
(b) Prove part (*c*) of Theorem 4.1.2.

40. Use the Mean-Value Theorem to prove the following result: Let f be continuous at x_0 and suppose that $\lim_{x \to x_0} f'(x)$ exists. Then f is differentiable at x_0, and

$$f'(x_0) = \lim_{x \to x_0} f'(x)$$

[*Hint:* The derivative $f'(x_0)$ is given by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided this limit exists.]

FOCUS ON CONCEPTS

41. Let
$$f(x) = \begin{cases} 3x^2, & x \le 1 \\ ax + b, & x > 1 \end{cases}$$
Find the values of a and b so that f will be differentiable at $x = 1$.

42. (a) Let
$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$$
Show that
$$\lim_{x\to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$$
but that $f'(0)$ does not exist.

(b) Let
$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^3, & x > 0 \end{cases}$$
Show that $f'(0)$ exists but $f''(0)$ does not.

43. Use the Mean-Value Theorem to prove the following result: The graph of a function f has a point of vertical tangency at $(x_0, f(x_0))$ if f is continuous at x_0 and $f'(x)$ approaches either $+\infty$ or $-\infty$ as $x \to x_0^+$ and as $x \to x_0^-$.

44. **Writing** Suppose that $p(x)$ is a nonconstant polynomial with zeros at $x = a$ and $x = b$. Explain how both the Extreme-Value Theorem (4.4.2) and Rolle's Theorem can be used to show that p has a critical point between a and b.

45. **Writing** Find and describe a physical situation that illustrates the Mean-Value Theorem.

✔ QUICK CHECK ANSWERS 4.8

1. (a) $[0, 1]$ (b) $c = \frac{1}{2}$ **2.** $[-3, 3]$; $c = -2, 0, 2$ **3.** (a) $b = 2$ (b) $c = 1$ **4.** (a) 1.5 (b) 0.8 **5.** $f(x) = x^2 + 4$

CHAPTER 4 REVIEW EXERCISES

Graphing Utility C CAS

1. (a) If $x_1 < x_2$, what relationship must hold between $f(x_1)$ and $f(x_2)$ if f is increasing on an interval containing x_1 and x_2? Decreasing? Constant?
(b) What condition on f' ensures that f is increasing on an interval $[a, b]$? Decreasing? Constant?

2. (a) What condition on f' ensures that f is concave up on an open interval? Concave down?
(b) What condition on f'' ensures that f is concave up on an open interval? Concave down?
(c) In words, what is an inflection point of f?

3–10 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

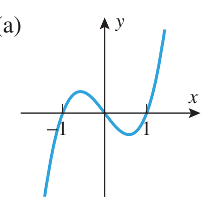
3. $f(x) = x^2 - 5x + 6$
4. $f(x) = x^4 - 8x^2 + 16$
5. $f(x) = \dfrac{x^2}{x^2 + 2}$
6. $f(x) = \sqrt[3]{x + 2}$
7. $f(x) = x^{1/3}(x + 4)$
8. $f(x) = x^{4/3} - x^{1/3}$
9. $f(x) = 1/e^{x^2}$
10. $f(x) = \tan^{-1} x^2$

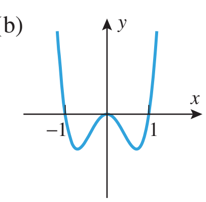
11–14 Analyze the trigonometric function f over the specified interval, stating where f is increasing, decreasing, concave up, and concave down, and stating the x-coordinates of all inflection points. Confirm that your results are consistent with the graph of f generated with a graphing utility.

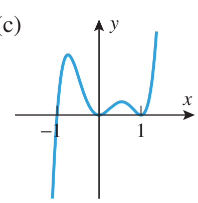
11. $f(x) = \cos x$; $[0, 2\pi]$
12. $f(x) = \tan x$; $(-\pi/2, \pi/2)$
13. $f(x) = \sin x \cos x$; $[0, \pi]$
14. $f(x) = \cos^2 x - 2 \sin x$; $[0, 2\pi]$

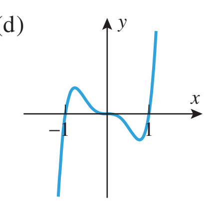
15. In each part, sketch a continuous curve $y = f(x)$ with the stated properties.
(a) $f(2) = 4$, $f'(2) = 1$, $f''(x) < 0$ for $x < 2$, $f''(x) > 0$ for $x > 2$
(b) $f(2) = 4$, $f''(x) > 0$ for $x < 2$, $f''(x) < 0$ for $x > 2$, $\lim_{x\to 2^-} f'(x) = +\infty$, $\lim_{x \to 2^+} f'(x) = +\infty$
(c) $f(2) = 4$, $f''(x) < 0$ for $x \ne 2$, $\lim_{x \to 2^-} f'(x) = 1$, $\lim_{x\to 2^+} f'(x) = -1$

16. In parts (a)–(d), the graph of a polynomial with degree at most 6 is given. Find equations for polynomials that produce graphs with these shapes, and check your answers with a graphing utility.

(a)


(b)


(c)


(d)


17. For a general quadratic polynomial
$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$
find conditions on a, b, and c to ensure that f is always increasing or always decreasing on $[0, +\infty)$.

18. For the general cubic polynomial
$$f(x) = ax^3 + bx^2 + cx + d \quad (a \neq 0)$$
find conditions on a, b, c, and d to ensure that f is always increasing or always decreasing on $(-\infty, +\infty)$.

19. Use a graphing utility to estimate the value of x at which
$$f(x) = \frac{2^x}{1 + 2^{x+1}}$$
is increasing most rapidly.

20. Prove that for any positive constants a and k, the graph of
$$y = \frac{a^x}{1 + a^{x+k}}$$
has an inflection point at $x = -k$.

21. (a) Where on the graph of $y = f(x)$ would you expect y to be increasing or decreasing most rapidly with respect to x?
(b) In words, what is a relative extremum?
(c) State a procedure for determining where the relative extrema of f occur.

22. Determine whether the statement is true or false. If it is false, give an example for which the statement fails.
(a) If f has a relative maximum at x_0, then $f(x_0)$ is the largest value that $f(x)$ can have.
(b) If the largest value for f on the interval (a, b) is at x_0, then f has a relative maximum at x_0.
(c) A function f has a relative extremum at each of its critical points.

23. (a) According to the first derivative test, what conditions ensure that f has a relative maximum at x_0? A relative minimum?
(b) According to the second derivative test, what conditions ensure that f has a relative maximum at x_0? A relative minimum?

24–26 Locate the critical points and identify which critical points correspond to stationary points. ■

24. (a) $f(x) = x^3 + 3x^2 - 9x + 1$
(b) $f(x) = x^4 - 6x^2 - 3$

25. (a) $f(x) = \dfrac{x}{x^2 + 2}$ (b) $f(x) = \dfrac{x^2 - 3}{x^2 + 1}$

26. (a) $f(x) = x^{1/3}(x - 4)$ (b) $f(x) = x^{4/3} - 6x^{1/3}$

27. In each part, find all critical points, and use the first derivative test to classify them as relative maxima, relative minima, or neither.
(a) $f(x) = x^{1/3}(x - 7)^2$
(b) $f(x) = 2\sin x - \cos 2x, \quad 0 \le x \le 2\pi$
(c) $f(x) = 3x - (x - 1)^{3/2}$

28. In each part, find all critical points, and use the second derivative test (where possible) to classify them as relative maxima, relative minima, or neither.
(a) $f(x) = x^{-1/2} + \frac{1}{9}x^{1/2}$
(b) $f(x) = x^2 + 8/x$
(c) $f(x) = \sin^2 x - \cos x, \quad 0 \le x \le 2\pi$

29–36 Give a graph of the function f, and identify the limits as $x \to \pm\infty$, as well as locations of all relative extrema, inflection points, and asymptotes (as appropriate). ■

29. $f(x) = x^4 - 3x^3 + 3x^2 + 1$

30. $f(x) = x^5 - 4x^4 + 4x^3$

31. $f(x) = \tan(x^2 + 1)$

32. $f(x) = x - \cos x$

33. $f(x) = \dfrac{x^2}{x^2 + 2x + 5}$

34. $f(x) = \dfrac{25 - 9x^2}{x^3}$

35. $f(x) = \begin{cases} \frac{1}{2}x^2, & x \le 0 \\ -x^2, & x > 0 \end{cases}$

36. $f(x) = (1 + x)^{2/3}(3 - x)^{1/3}$

37–44 Use any method to find the relative extrema of the function f. ■

37. $f(x) = x^3 + 5x - 2$

38. $f(x) = x^4 - 2x^2 + 7$

39. $f(x) = x^{4/5}$

40. $f(x) = 2x + x^{2/3}$

41. $f(x) = \dfrac{x^2}{x^2 + 1}$

42. $f(x) = \dfrac{x}{x + 2}$

43. $f(x) = \ln(1 + x^2)$

44. $f(x) = x^2 e^x$

45–46 When using a graphing utility, important features of a graph may be missed if the viewing window is not chosen appropriately. This is illustrated in Exercises 45 and 46. ■

45. (a) Generate the graph of $f(x) = \frac{1}{3}x^3 - \frac{1}{400}x$ over the interval $[-5, 5]$, and make a conjecture about the locations and nature of all critical points.
(b) Find the exact locations of all the critical points, and classify them as relative maxima, relative minima, or neither.
(c) Confirm the results in part (b) by graphing f over an appropriate interval.

46. (a) Generate the graph of
$$f(x) = \tfrac{1}{5}x^5 - \tfrac{7}{8}x^4 + \tfrac{1}{3}x^3 + \tfrac{7}{2}x^2 - 6x$$
over the interval $[-5, 5]$, and make a conjecture about the locations and nature of all critical points.
(b) Find the exact locations of all the critical points, and classify them as relative maxima, relative minima, or neither.
(c) Confirm the results in part (b) by graphing portions of f over appropriate intervals. [*Note:* It will not be possible to find a single window in which all of the critical points are discernible.]

47. (a) Use a graphing utility to generate the graphs of $y = x$ and $y = (x^3 - 8)/(x^2 + 1)$ together over the interval $[-5, 5]$, and make a conjecture about the relationship between the two graphs.
(b) Confirm your conjecture in part (a).

48. Use implicit differentiation to show that a function defined implicitly by $\sin x + \cos y = 2y$ has a critical point whenever $\cos x = 0$. Then use either the first or second derivative test to classify these critical points as relative maxima or minima.

49. Let
$$f(x) = \frac{2x^3 + x^2 - 15x + 7}{(2x - 1)(3x^2 + x - 1)}$$
Graph $y = f(x)$, and find the equations of all horizontal and vertical asymptotes. Explain why there is no vertical asymptote at $x = \frac{1}{2}$, even though the denominator of f is zero at that point.

50. Let
$$f(x) = \frac{x^5 - x^4 - 3x^3 + 2x + 4}{x^7 - 2x^6 - 3x^5 + 6x^4 + 4x - 8}$$
(a) Use a CAS to factor the numerator and denominator of f, and use the results to determine the locations of all vertical asymptotes.
(b) Confirm that your answer is consistent with the graph of f.

51. (a) What inequality must $f(x)$ satisfy for the function f to have an absolute maximum on an interval I at x_0?
(b) What inequality must $f(x)$ satisfy for f to have an absolute minimum on an interval I at x_0?
(c) What is the difference between an absolute extremum and a relative extremum?

52. According to the Extreme-Value Theorem, what conditions on a function f and a given interval guarantee that f will have both an absolute maximum and an absolute minimum on the interval?

53. In each part, determine whether the statement is true or false, and justify your answer.
(a) If f is differentiable on the open interval (a, b), and if f has an absolute extremum on that interval, then it must occur at a stationary point of f.
(b) If f is continuous on the open interval (a, b), and if f has an absolute extremum on that interval, then it must occur at a stationary point of f.

54–56 In each part, find the absolute minimum m and the absolute maximum M of f on the given interval (if they exist), and state where the absolute extrema occur.

54. (a) $f(x) = 1/x$; $[-2, -1]$
(b) $f(x) = x^3 - x^4$; $\left[-1, \frac{3}{2}\right]$
(c) $f(x) = x - \tan x$; $[-\pi/4, \pi/4]$
(d) $f(x) = -|x^2 - 2x|$; $[1, 3]$

55. (a) $f(x) = x^2 - 3x - 1$; $(-\infty, +\infty)$
(b) $f(x) = x^3 - 3x - 2$; $(-\infty, +\infty)$
(c) $f(x) = e^x/x^2$; $(0, +\infty)$
(d) $f(x) = x^x$; $(0, +\infty)$

56. (a) $f(x) = 2x^5 - 5x^4 + 7$; $(-1, 3)$
(b) $f(x) = (3 - x)/(2 - x)$; $(0, 2)$
(c) $f(x) = 2x/(x^2 + 3)$; $(0, 2]$
(d) $f(x) = x^2(x - 2)^{1/3}$; $(0, 3]$

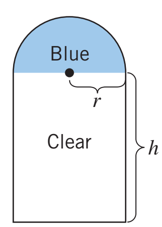
57. In each part, use a graphing utility to estimate the absolute maximum and minimum values of f, if any, on the stated interval, and then use calculus methods to find the exact values.
(a) $f(x) = (x^2 - 1)^2$; $(-\infty, +\infty)$
(b) $f(x) = x/(x^2 + 1)$; $[0, +\infty)$
(c) $f(x) = 2\sec x - \tan x$; $[0, \pi/4]$
(d) $f(x) = x/2 + \ln(x^2 + 1)$; $[-4, 0]$

58. Prove that $x \le \sin^{-1} x$ for all x in $[0, 1]$.

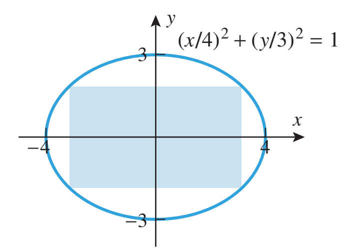
59. Let
$$f(x) = \frac{x^3 + 2}{x^4 + 1}$$
(a) Generate the graph of $y = f(x)$, and use the graph to make rough estimates of the coordinates of the absolute extrema.
(b) Use a CAS to solve the equation $f'(x) = 0$ and then use it to make more accurate approximations of the coordinates in part (a).

60. A church window consists of a blue semicircular section surmounting a clear rectangular section as shown in the accompanying figure. The blue glass lets through half as much light per unit area as the clear glass. Find the radius r of the window that admits the most light if the perimeter of the entire window is to be P feet.

61. Find the dimensions of the rectangle of maximum area that can be inscribed inside the ellipse $(x/4)^2 + (y/3)^2 = 1$ (see the accompanying figure).

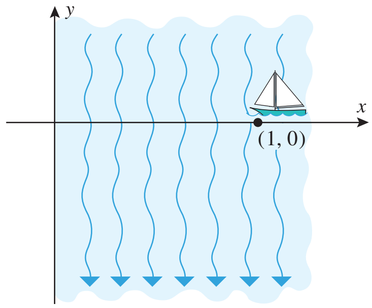


▲ **Figure Ex-60**



▲ **Figure Ex-61**

62. As shown in the accompanying figure on the next page, suppose that a boat enters the river at the point $(1, 0)$ and maintains a heading toward the origin. As a result of the strong current, the boat follows the path
$$y = \frac{x^{10/3} - 1}{2x^{2/3}}$$
where x and y are in miles.
(a) Graph the path taken by the boat.
(b) Can the boat reach the origin? If not, discuss its fate and find how close it comes to the origin.



◀ Figure Ex-62

63. A sheet of cardboard 12 in square is used to make an open box by cutting squares of equal size from the four corners and folding up the sides. What size squares should be cut to obtain a box with largest possible volume?

64. Is it true or false that a particle in rectilinear motion is speeding up when its velocity is increasing and slowing down when its velocity is decreasing? Justify your answer.

65. (a) Can an object in rectilinear motion reverse direction if its acceleration is constant? Justify your answer using a velocity versus time curve.
(b) Can an object in rectilinear motion have increasing speed and decreasing acceleration? Justify your answer using a velocity versus time curve.

66. Suppose that the position function of a particle in rectilinear motion is given by the formula $s(t) = t/(2t^2 + 8)$ for $t \geq 0$.
(a) Use a graphing utility to generate the position, velocity, and acceleration versus time curves.
(b) Use the appropriate graph to make a rough estimate of the time when the particle reverses direction, and then find that time exactly.
(c) Find the position, velocity, and acceleration at the instant when the particle reverses direction.
(d) Use the appropriate graphs to make rough estimates of the time intervals on which the particle is speeding up and the time intervals on which it is slowing down, and then find those time intervals exactly.
(e) When does the particle have its maximum and minimum velocities?

C **67.** For parts (a)–(f), suppose that the position function of a particle in rectilinear motion is given by the formula

$$s(t) = \frac{t^2 + 1}{t^4 + 1}, \quad t \geq 0$$

(a) Use a CAS to find simplified formulas for the velocity function $v(t)$ and the acceleration function $a(t)$.
(b) Graph the position, velocity, and acceleration versus time curves.
(c) Use the appropriate graph to make a rough estimate of the time at which the particle is farthest from the origin and its distance from the origin at that time.
(d) Use the appropriate graph to make a rough estimate of the time interval during which the particle is moving in the positive direction.
(e) Use the appropriate graphs to make rough estimates of the time intervals during which the particle is speeding up and the time intervals during which it is slowing down.
(f) Use the appropriate graph to make a rough estimate of the maximum speed of the particle and the time at which the maximum speed occurs.

68. Draw an appropriate picture, and describe the basic idea of Newton's Method without using any formulas.

69. Use Newton's Method to approximate all three solutions of $x^3 - 4x + 1 = 0$.

70. Use Newton's Method to approximate the smallest positive solution of $\sin x + \cos x = 0$.

71. Use a graphing utility to determine the number of times the curve $y = x^3$ intersects the curve $y = (x/2) - 1$. Then apply Newton's Method to approximate the x-coordinates of all intersections.

72. According to ***Kepler's law***, the planets in our solar system move in elliptical orbits around the Sun. If a planet's closest approach to the Sun occurs at time $t = 0$, then the distance r from the center of the planet to the center of the Sun at some later time t can be determined from the equation

$$r = a(1 - e\cos\phi)$$

where a is the average distance between centers, e is a positive constant that measures the "flatness" of the elliptical orbit, and ϕ is the solution of *Kepler's equation*

$$\frac{2\pi t}{T} = \phi - e\sin\phi$$

in which T is the time it takes for one complete orbit of the planet. Estimate the distance from the Earth to the Sun when $t = 90$ days. [First find ϕ from Kepler's equation, and then use this value of ϕ to find the distance. Use $a = 150 \times 10^6$ km, $e = 0.0167$, and $T = 365$ days.]

73. Using the formulas in Exercise 72, find the distance from the planet Mars to the Sun when $t = 1$ year. For Mars use $a = 228 \times 10^6$ km, $e = 0.0934$, and $T = 1.88$ years.

74. Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and suppose that $f(a) = f(b)$. Is it true or false that f must have at least one stationary point in (a, b)? Justify your answer.

75. In each part, determine whether all of the hypotheses of Rolle's Theorem are satisfied on the stated interval. If not, state which hypotheses fail; if so, find all values of c guaranteed in the conclusion of the theorem.
(a) $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$
(b) $f(x) = x^{2/3} - 1$ on $[-1, 1]$
(c) $f(x) = \sin(x^2)$ on $[0, \sqrt{\pi}\,]$

76. In each part, determine whether all of the hypotheses of the Mean-Value Theorem are satisfied on the stated interval. If not, state which hypotheses fail; if so, find all values of c guaranteed in the conclusion of the theorem.
(a) $f(x) = |x - 1|$ on $[-2, 2]$

(cont.)

(b) $f(x) = \dfrac{x+1}{x-1}$ on $[2, 3]$

(c) $f(x) = \begin{cases} 3 - x^2 & \text{if } x \le 1 \\ 2/x & \text{if } x > 1 \end{cases}$ on $[0, 2]$

77. Use the fact that

$$\frac{d}{dx}(x^6 - 2x^2 + x) = 6x^5 - 4x + 1$$

to show that the equation $6x^5 - 4x + 1 = 0$ has at least one solution in the interval $(0, 1)$.

78. Let $g(x) = x^3 - 4x + 6$. Find $f(x)$ so that $f'(x) = g'(x)$ and $f(1) = 2$.

CHAPTER 4 MAKING CONNECTIONS

1. Suppose that $g(x)$ is a function that is defined and differentiable for all real numbers x and that $g(x)$ has the following properties:

(i) $g(0) = 2$ and $g'(0) = -\frac{2}{3}$.
(ii) $g(4) = 3$ and $g'(4) = 3$.
(iii) $g(x)$ is concave up for $x < 4$ and concave down for $x > 4$.
(iv) $g(x) \ge -10$ for all x.

Use these properties to answer the following questions.

(a) How many zeros does g have?
(b) How many zeros does g' have?
(c) Exactly one of the following limits is possible:

$$\lim_{x\to+\infty} g'(x) = -5, \quad \lim_{x\to+\infty} g'(x) = 0, \quad \lim_{x\to+\infty} g'(x) = 5$$

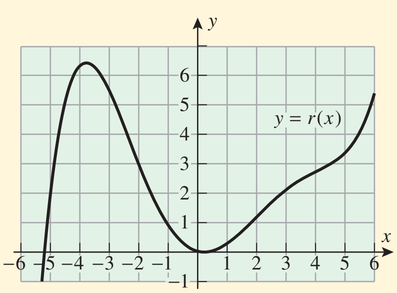
Identify which of these results is possible and draw a rough sketch of the graph of such a function $g(x)$. Explain why the other two results are impossible.

2. The two graphs in the accompanying figure depict a function $r(x)$ and its derivative $r'(x)$.

(a) Approximate the coordinates of each inflection point on the graph of $y = r(x)$.
(b) Suppose that $f(x)$ is a function that is continuous everywhere and whose *derivative* satisfies

$$f'(x) = (x^2 - 4) \cdot r(x)$$

What are the critical points for $f(x)$? At each critical point, identify whether $f(x)$ has a (relative) maximum, minimum, or neither a maximum or minimum. Approximate $f''(1)$.



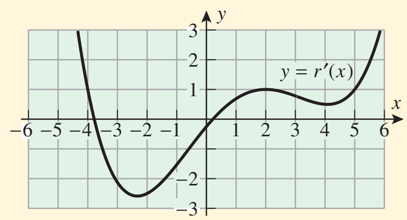


Figure Ex-2

3. With the function $r(x)$ as provided in Exercise 2, let $g(x)$ be a function that is continuous everywhere such that $g'(x) = x - r(x)$. For which values of x does $g(x)$ have an inflection point?

4. Suppose that f is a function whose derivative is continuous everywhere. Assume that there exists a real number c such that when Newton's Method is applied to f, the inequality

$$|x_n - c| < \frac{1}{n}$$

is satisfied for all values of $n = 1, 2, 3, \ldots$.

(a) Explain why

$$|x_{n+1} - x_n| < \frac{2}{n}$$

for all values of $n = 1, 2, 3, \ldots$.

(b) Show that there exists a positive constant M such that

$$|f(x_n)| \le M|x_{n+1} - x_n| < \frac{2M}{n}$$

for all values of $n = 1, 2, 3, \ldots$.

(c) Prove that if $f(c) \ne 0$, then there exists a positive integer N such that

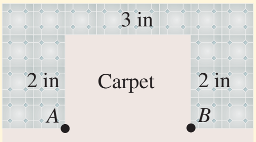
$$\frac{|f(c)|}{2} < |f(x_n)|$$

if $n > N$. [*Hint:* Argue that $f(x) \to f(c)$ as $x \to c$ and then apply Definition 1.4.1 with $\epsilon = \frac{1}{2}|f(c)|$.]

(d) What can you conclude from parts (b) and (c)?

5. What are the important elements in the argument suggested by Exercise 4? Can you extend this argument to a wider collection of functions?

6. A bug crawling on a linoleum floor along the edge of a plush carpet encounters an irregularity in the form of a 2 in by 3 in rectangular section of carpet that juts out into the linoleum as illustrated in Figure Ex-6a on the next page.



◀ **Figure Ex-6a**

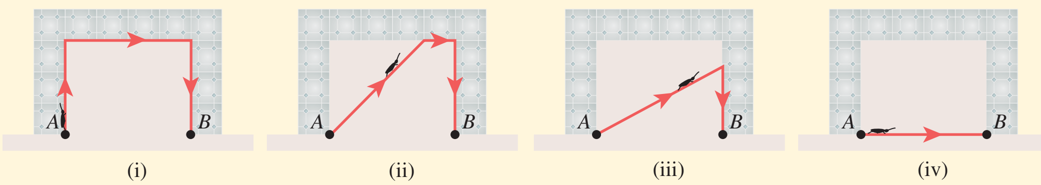
The bug crawls at 0.7 in/s on the linoleum, but only at 0.3 in/s through the carpet, and its goal is to travel from point A to point B. Four possible routes from A to B are as follows: (i) crawl on linoleum along the edge of the carpet; (ii) crawl through the carpet to a point on the wider side of the rectangle, and finish the journey on linoleum along the edge of the carpet; (iii) crawl through the carpet to a point on the shorter side of the rectangle, and finish the journey on linoleum along the edge of the carpet; or (iv) crawl through the carpet directly to point B. (See Figure Ex-6b.)

(a) Calculate the times it would take the bug to crawl from A to B via routes (i) and (iv).

(b) Suppose the bug follows route (ii) and use x to represent the total distance the bug crawls on linoleum. Identify the appropriate interval for x in this case, and determine the shortest time for the bug to complete the journey using route (ii).

(c) Suppose the bug follows route (iii) and again use x to represent the total distance the bug crawls on linoleum. Identify the appropriate interval for x in this case, and determine the shortest time for the bug to complete the journey using route (iii).

(d) Which of routes (i), (ii), (iii), or (iv) is quickest? What is the shortest time for the bug to complete the journey?



▲ **Figure Ex-6b**