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3 TOPICS IN DIFFERENTIATION

The growth and decline of animal populations and natural resources can be modeled using basic functions studied in calculus.

We begin this chapter by extending the process of differentiation to functions that are either difficult or impossible to differentiate directly. We will discuss a combination of direct and indirect methods of differentiation that will allow us to develop a number of new derivative formulas that include the derivatives of logarithmic, exponential, and inverse trigonometric functions. Later in the chapter, we will consider some applications of the derivative. These will include ways in which different rates of change can be related as well as the use of linear functions to approximate nonlinear functions. Finally, we will discuss L'Hôpital's rule, a powerful tool for evaluating limits.

3.1 IMPLICIT DIFFERENTIATION

Up to now we have been concerned with differentiating functions that are given by equations of the form $y = f(x)$. In this section we will consider methods for differentiating functions for which it is inconvenient or impossible to express them in this form.

FUNCTIONS DEFINED EXPLICITLY AND IMPLICITLY

An equation of the form $y = f(x)$ is said to define y ***explicitly*** as a function of x because the variable y appears alone on one side of the equation and does not appear at all on the other side. However, sometimes functions are defined by equations in which y is not alone on one side; for example, the equation

$$yx + y + 1 = x \tag{1}$$

is not of the form $y = f(x)$, but it still defines y as a function of x since it can be rewritten as

$$y = \frac{x-1}{x+1}$$

Thus, we say that (1) defines y ***implicitly*** as a function of x, the function being

$$f(x) = \frac{x-1}{x+1}$$

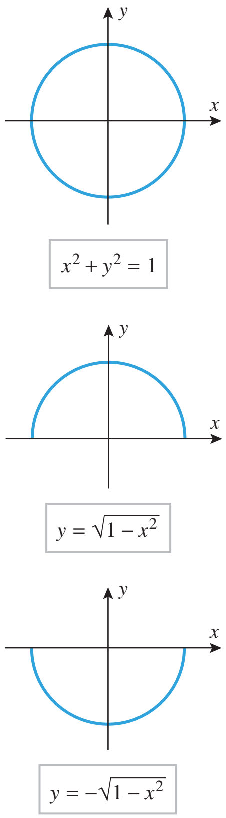
An equation in x and y can implicitly define more than one function of x. This can occur when the graph of the equation fails the vertical line test, so it is not the graph of a function of x. For example, if we solve the equation of the circle

$$x^2 + y^2 = 1 \tag{2}$$

for y in terms of x, we obtain $y = \pm\sqrt{1-x^2}$, so we have found two functions that are defined implicitly by (2), namely,

$$f_1(x) = \sqrt{1-x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1-x^2} \tag{3}$$

The graphs of these functions are the upper and lower semicircles of the circle $x^2 + y^2 = 1$ (Figure 3.1.1). This leads us to the following definition.

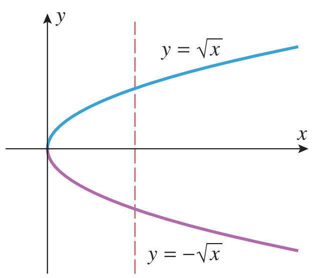


▲ **Figure 3.1.1**

3.1.1 DEFINITION We will say that a given equation in x and y defines the function f ***implicitly*** if the graph of $y = f(x)$ coincides with a portion of the graph of the equation.

▶ **Example 1** The graph of $x = y^2$ is not the graph of a function of x, since it does not pass the vertical line test (Figure 3.1.2). However, if we solve this equation for y in terms of x, we obtain the equations $y = \sqrt{x}$ and $y = -\sqrt{x}$, whose graphs pass the vertical line test and are portions of the graph of $x = y^2$ (Figure 3.1.2). Thus, the equation $x = y^2$ implicitly defines the functions

$$f_1(x) = \sqrt{x} \quad \text{and} \quad f_2(x) = -\sqrt{x} \quad ◀$$



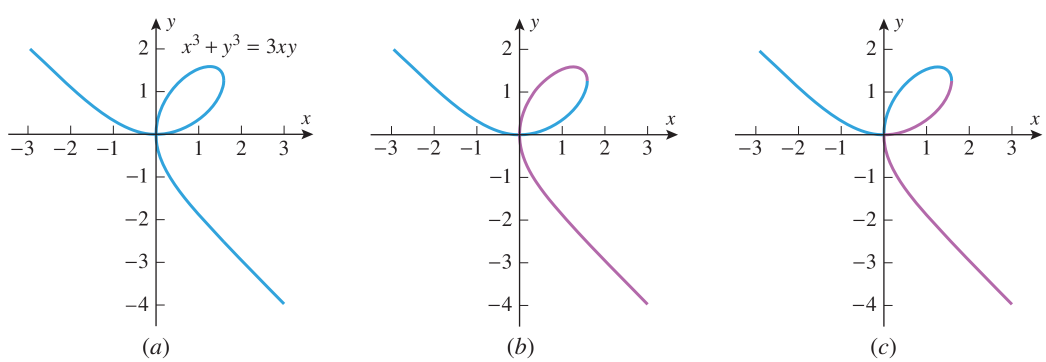
▲ **Figure 3.1.2** The graph of $x = y^2$ does not pass the vertical line test, but the graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ do.

Although it was a trivial matter in the last example to solve the equation $x = y^2$ for y in terms of x, it is difficult or impossible to do this for some equations. For example, the equation

$$x^3 + y^3 = 3xy \tag{4}$$

can be solved for y in terms of x, but the resulting formulas are too complicated to be practical. Other equations, such as $\sin(xy) = y$, cannot be solved for y by any elementary method. Thus, even though an equation may define one or more functions of x, it may not be possible or practical to find explicit formulas for those functions.

Fortunately, CAS programs, such as *Mathematica* and *Maple*, have "implicit plotting" capabilities that can graph equations such as (4). The graph of this equation, which is called the ***Folium of Descartes***, is shown in Figure 3.1.3*a*. Parts (*b*) and (*c*) of the figure show the graphs (in blue) of two functions that are defined implicitly by (4).



▲ **Figure 3.1.3**

IMPLICIT DIFFERENTIATION

In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly by the equation. To illustrate this, let us consider the simple equation

$$xy = 1 \tag{5}$$

One way to find dy/dx is to rewrite this equation as

$$y = \frac{1}{x} \tag{6}$$

from which it follows that

$$\frac{dy}{dx} = -\frac{1}{x^2} \tag{7}$$

Another way to obtain this derivative is to differentiate both sides of (5) *before* solving for y in terms of x, treating y as a (temporarily unspecified) differentiable function of x. With this approach we obtain

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1]$$
$$x\frac{d}{dx}[y] + y\frac{d}{dx}[x] = 0$$
$$x\frac{dy}{dx} + y = 0$$
$$\frac{dy}{dx} = -\frac{y}{x}$$

If we now substitute (6) into the last expression, we obtain

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

which agrees with Equation (7). This method of obtaining derivatives is called ***implicit differentiation***.

▶ **Example 2** Use implicit differentiation to find dy/dx if $5y^2 + \sin y = x^2$.

$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$
$$5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$$
$$5\left(2y\frac{dy}{dx}\right) + (\cos y)\frac{dy}{dx} = 2x$$

The chain rule was used here because y is a function of x.

$$10y\frac{dy}{dx} + (\cos y)\frac{dy}{dx} = 2x$$

René Descartes (1596–1650) Descartes, a French aristocrat, was the son of a government official. He graduated from the University of Poitiers with a law degree at age 20. After a brief probe into the pleasures of Paris he became a military engineer, first for the Dutch Prince of Nassau and then for the German Duke of Bavaria. It was during his service as a soldier that Descartes began to pursue mathematics seriously and develop his analytic geometry. After the wars, he returned to Paris where he stalked the city as an eccentric, wearing a sword in his belt and a plumed hat. He lived in leisure, seldom arose before 11 A.M., and dabbled in the study of human physiology, philosophy, glaciers, meteors, and rainbows. He eventually moved to Holland, where he published his *Discourse on the Method*, and finally to Sweden where he died while serving as tutor to Queen Christina. Descartes is regarded as a genius of the first magnitude. In addition to major contributions in mathematics and philosophy he is considered, along with William Harvey, to be a founder of modern physiology.

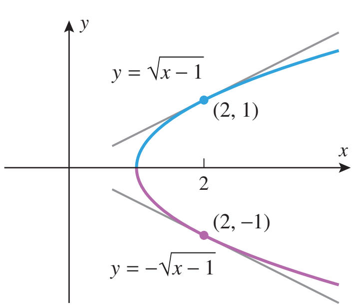
Solving for dy/dx we obtain
$$\frac{dy}{dx} = \frac{2x}{10y + \cos y} \tag{8}$$
Note that this formula involves both x and y. In order to obtain a formula for dy/dx that involves x alone, we would have to solve the original equation for y in terms of x and then substitute in (8). However, it is impossible to do this, so we are forced to leave the formula for dy/dx in terms of x and y. ◀

▶ Example 3 Use implicit differentiation to find d^2y/dx^2 if $4x^2 - 2y^2 = 9$.

Solution. Differentiating both sides of $4x^2 - 2y^2 = 9$ with respect to x yields
$$8x - 4y\frac{dy}{dx} = 0$$
from which we obtain
$$\frac{dy}{dx} = \frac{2x}{y} \tag{9}$$
Differentiating both sides of (9) yields
$$\frac{d^2y}{dx^2} = \frac{(y)(2) - (2x)(dy/dx)}{y^2} \tag{10}$$
Substituting (9) into (10) and simplifying using the original equation, we obtain
$$\frac{d^2y}{dx^2} = \frac{2y - 2x(2x/y)}{y^2} = \frac{2y^2 - 4x^2}{y^3} = -\frac{9}{y^3}$$ ◀

In Examples 2 and 3, the resulting formulas for dy/dx involved both x and y. Although it is usually more desirable to have the formula for dy/dx expressed in terms of x alone, having the formula in terms of x and y is not an impediment to finding slopes and equations of tangent lines provided the x- and y-coordinates of the point of tangency are known. This is illustrated in the following example.

▶ Example 4 Find the slopes of the tangent lines to the curve $y^2 - x + 1 = 0$ at the points $(2, -1)$ and $(2, 1)$.



▲ Figure 3.1.4

Solution. We could proceed by solving the equation for y in terms of x, and then evaluating the derivative of $y = \sqrt{x-1}$ at $(2, 1)$ and the derivative of $y = -\sqrt{x-1}$ at $(2, -1)$ (Figure 3.1.4). However, implicit differentiation is more efficient since it can be used for the slopes of *both* tangent lines. Differentiating implicitly yields
$$\frac{d}{dx}[y^2 - x + 1] = \frac{d}{dx}[0]$$
$$\frac{d}{dx}[y^2] - \frac{d}{dx}[x] + \frac{d}{dx}[1] = \frac{d}{dx}[0]$$
$$2y\frac{dy}{dx} - 1 = 0$$
$$\frac{dy}{dx} = \frac{1}{2y}$$
At $(2, -1)$ we have $y = -1$, and at $(2, 1)$ we have $y = 1$, so the slopes of the tangent lines to the curve at those points are
$$\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = -\frac{1}{2} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = \frac{1}{2}$$ ◀

▶ Example 5

(a) Use implicit differentiation to find dy/dx for the Folium of Descartes $x^3 + y^3 = 3xy$.
(b) Find an equation for the tangent line to the Folium of Descartes at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.
(c) At what point(s) in the first quadrant is the tangent line to the Folium of Descartes horizontal?

Solution (a). Differentiating implicitly yields

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$(y^2 - x)\frac{dy}{dx} = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x} \tag{11}$$

Formula (11) cannot be evaluated at (0, 0) and hence provides no information about the nature of the Folium of Descartes at the origin. Based on the graphs in Figure 3.1.3, what can you say about the differentiability of the implicitly defined functions graphed in blue in parts (*b*) and (*c*) of the figure?

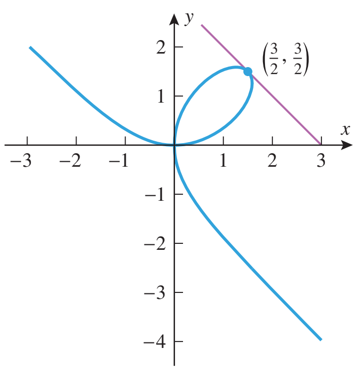
Solution (b). At the point $\left(\frac{3}{2}, \frac{3}{2}\right)$, we have $x = \frac{3}{2}$ and $y = \frac{3}{2}$, so from (11) the slope $m_{\tan}$ of the tangent line at this point is

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=3/2\\y=3/2}} = \frac{(3/2) - (3/2)^2}{(3/2)^2 - (3/2)} = -1$$

Thus, the equation of the tangent line at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ is

$$y - \tfrac{3}{2} = -1\left(x - \tfrac{3}{2}\right) \quad \text{or} \quad x + y = 3$$

which is consistent with Figure 3.1.5.



▲ **Figure 3.1.5**

Solution (c). The tangent line is horizontal at the points where $dy/dx = 0$, and from (11) this occurs only where $y - x^2 = 0$ or

$$y = x^2 \tag{12}$$

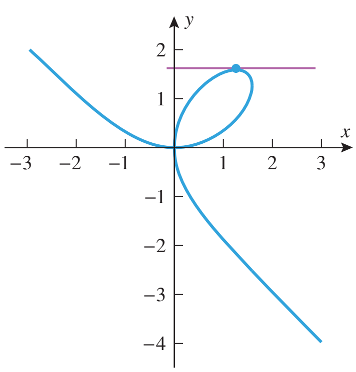
Substituting this expression for y in the equation $x^3 + y^3 = 3xy$ for the curve yields

$$x^3 + (x^2)^3 = 3x^3$$
$$x^6 - 2x^3 = 0$$
$$x^3(x^3 - 2) = 0$$

whose solutions are $x = 0$ and $x = 2^{1/3}$. From (12), the solutions $x = 0$ and $x = 2^{1/3}$ yield the points (0, 0) and $(2^{1/3}, 2^{2/3})$, respectively. Of these two, only $(2^{1/3}, 2^{2/3})$ is in the first quadrant. Substituting $x = 2^{1/3}$, $y = 2^{2/3}$ into (11) yields

$$\left.\frac{dy}{dx}\right|_{\substack{x=2^{1/3}\\y=2^{2/3}}} = \frac{0}{2^{4/3} - 2^{2/3}} = 0$$

We conclude that $(2^{1/3}, 2^{2/3}) \approx (1.26, 1.59)$ is the only point on the Folium of Descartes in the first quadrant at which the tangent line is horizontal (Figure 3.1.6). ◀



▲ **Figure 3.1.6**

DIFFERENTIABILITY OF FUNCTIONS DEFINED IMPLICITLY

When differentiating implicitly, it is assumed that y represents a differentiable function of x. If this is not so, then the resulting calculations may be nonsense. For example, if we differentiate the equation

$$x^2 + y^2 + 1 = 0 \tag{13}$$

we obtain

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}$$

However, this derivative is meaningless because there are no real values of x and y that satisfy (13) (why?); and hence (13) does not define any real functions implicitly.

The nonsensical conclusion of these computations conveys the importance of knowing whether an equation in x and y that is to be differentiated implicitly actually defines some differentiable function of x implicitly. Unfortunately, this can be a difficult problem, so we will leave the discussion of such matters for more advanced courses in analysis.

✔QUICK CHECK EXERCISES 3.1 (See page 192 for answers.)

1. The equation $xy + 2y = 1$ defines implicitly the function $y =$ ________.
2. Use implicit differentiation to find dy/dx for $x^2 - y^3 = xy$.
3. The slope of the tangent line to the graph of $x + y + xy = 3$ at $(1, 1)$ is ________.
4. Use implicit differentiation to find d^2y/dx^2 for $\sin y = x$.

EXERCISE SET 3.1 C CAS

1–2
(a) Find dy/dx by differentiating implicitly.
(b) Solve the equation for y as a function of x, and find dy/dx from that equation.
(c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone.

1. $x + xy - 2x^3 = 2$
2. $\sqrt{y} - \sin x = 2$

3–12 Find dy/dx by implicit differentiation.

3. $x^2 + y^2 = 100$
4. $x^3 + y^3 = 3xy^2$
5. $x^2y + 3xy^3 - x = 3$
6. $x^3y^2 - 5x^2y + x = 1$
7. $\dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}} = 1$
8. $x^2 = \dfrac{x + y}{x - y}$
9. $\sin(x^2y^2) = x$
10. $\cos(xy^2) = y$
11. $\tan^3(xy^2 + y) = x$
12. $\dfrac{xy^3}{1 + \sec y} = 1 + y^4$

13–18 Find d^2y/dx^2 by implicit differentiation.

13. $2x^2 - 3y^2 = 4$
14. $x^3 + y^3 = 1$
15. $x^3y^3 - 4 = 0$
16. $xy + y^2 = 2$
17. $y + \sin y = x$
18. $x \cos y = y$

19–20 Find the slope of the tangent line to the curve at the given points in two ways: first by solving for y in terms of x and differentiating and then by implicit differentiation.

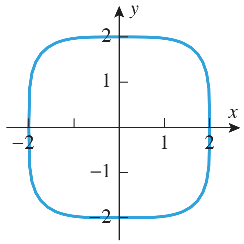
19. $x^2 + y^2 = 1$; $(1/2, \sqrt{3}/2)$, $(1/2, -\sqrt{3}/2)$
20. $y^2 - x + 1 = 0$; $(10, 3)$, $(10, -3)$

21–24 True–False Determine whether the statement is true or false. Explain your answer.

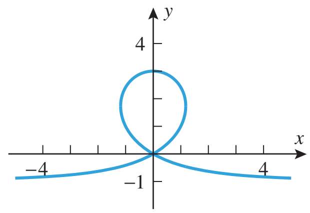
21. If an equation in x and y defines a function $y = f(x)$ implicitly, then the graph of the equation and the graph of f are identical.
22. The function
$$f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 < x \le 1 \\ -\sqrt{1 - x^2}, & -1 \le x \le 0 \end{cases}$$
is defined implicitly by the equation $x^2 + y^2 = 1$.
23. The function $|x|$ is not defined implicitly by the equation $(x + y)(x - y) = 0$.
24. If y is defined implicitly as a function of x by the equation $x^2 + y^2 = 1$, then $dy/dx = -x/y$.

25–28 Use implicit differentiation to find the slope of the tangent line to the curve at the specified point, and check that your answer is consistent with the accompanying graph on the next page.

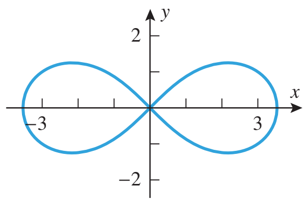
25. $x^4 + y^4 = 16$; $(1, \sqrt[4]{15})$ [***Lamé's special quartic***]
26. $y^3 + yx^2 + x^2 - 3y^2 = 0$; $(0, 3)$ [***trisectrix***]
27. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$; $(3, 1)$ [***lemniscate***]
28. $x^{2/3} + y^{2/3} = 4$; $(-1, 3\sqrt{3})$ [***four-cusped hypocycloid***]



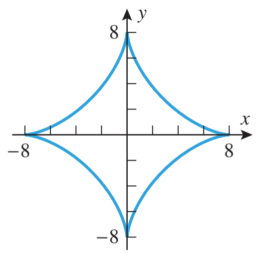
▲ Figure Ex-25



▲ Figure Ex-26



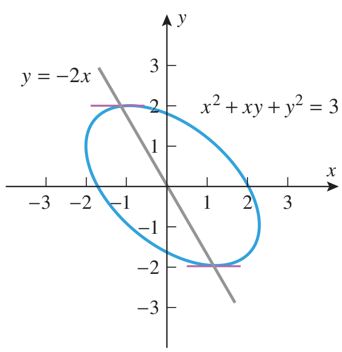
▲ Figure Ex-27



▲ Figure Ex-28

FOCUS ON CONCEPTS

29. In the accompanying figure, it appears that the ellipse $x^2 + xy + y^2 = 3$ has horizontal tangent lines at the points of intersection of the ellipse and the line $y = -2x$. Use implicit differentiation to explain why this is the case.



◀ Figure Ex-29

30. (a) A student claims that the ellipse $x^2 - xy + y^2 = 1$ has a horizontal tangent line at the point (1, 1). Without doing any computations, explain why the student's claim must be incorrect.
(b) Find all points on the ellipse $x^2 - xy + y^2 = 1$ at which the tangent line is horizontal.

C **31.** (a) Use the implicit plotting capability of a CAS to graph the equation $y^4 + y^2 = x(x - 1)$.
(b) Use implicit differentiation to help explain why the graph in part (a) has no horizontal tangent lines.
(c) Solve the equation $y^4 + y^2 = x(x - 1)$ for x in terms of y and explain why the graph in part (a) consists of two parabolas.

32. Use implicit differentiation to find all points on the graph of $y^4 + y^2 = x(x - 1)$ at which the tangent line is vertical.

33–34 These exercises deal with the rotated ellipse C whose equation is $x^2 - xy + y^2 = 4$. ■

33. Show that the line $y = x$ intersects C at two points P and Q and that the tangent lines to C at P and Q are parallel.

34. Prove that if $P(a, b)$ is a point on C, then so is $Q(-a, -b)$ and that the tangent lines to C through P and through Q are parallel.

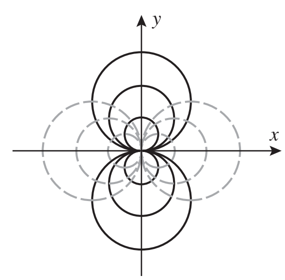
35. Find the values of a and b for the curve $x^2y + ay^2 = b$ if the point (1, 1) is on its graph and the tangent line at (1, 1) has the equation $4x + 3y = 7$.

36. At what point(s) is the tangent line to the curve $y^3 = 2x^2$ perpendicular to the line $x + 2y - 2 = 0$?

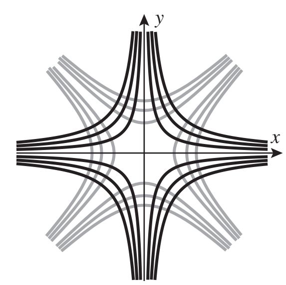
37–38 Two curves are said to be ***orthogonal*** if their tangent lines are perpendicular at each point of intersection, and two families of curves are said to be ***orthogonal trajectories*** of one another if each member of one family is orthogonal to each member of the other family. This terminology is used in these exercises. ■

37. The accompanying figure shows some typical members of the families of circles $x^2 + (y - c)^2 = c^2$ (black curves) and $(x - k)^2 + y^2 = k^2$ (gray curves). Show that these families are orthogonal trajectories of one another. [*Hint:* For the tangent lines to be perpendicular at a point of intersection, the slopes of those tangent lines must be negative reciprocals of one another.]

38. The accompanying figure shows some typical members of the families of hyperbolas $xy = c$ (black curves) and $x^2 - y^2 = k$ (gray curves), where $c \neq 0$ and $k \neq 0$. Use the hint in Exercise 37 to show that these families are orthogonal trajectories of one another.



▲ Figure Ex-37



▲ Figure Ex-38

C **39.** (a) Use the implicit plotting capability of a CAS to graph the curve C whose equation is $x^3 - 2xy + y^3 = 0$.
(b) Use the graph in part (a) to estimate the x-coordinates of a point in the first quadrant that is on C and at which the tangent line to C is parallel to the x-axis.
(c) Find the exact value of the x-coordinate in part (b).

C **40.** (a) Use the implicit plotting capability of a CAS to graph the curve C whose equation is $x^3 - 2xy + y^3 = 0$.
(b) Use the graph to guess the coordinates of a point in the first quadrant that is on C and at which the tangent line to C is parallel to the line $y = -x$. *(cont.)*

(c) Use implicit differentiation to verify your conjecture in part (b).

41. Prove that for every nonzero rational number r, the tangent line to the graph of $x^r + y^r = 2$ at the point $(1, 1)$ has slope -1.

42. Find equations for two lines through the origin that are tangent to the ellipse $2x^2 - 4x + y^2 + 1 = 0$.

43. Writing Write a paragraph that compares the concept of an *explicit* definition of a function with that of an *implicit* definition of a function.

44. Writing A student asks: "Suppose implicit differentiation yields an undefined expression at a point. Does this mean that dy/dx is undefined at that point?" Using the equation $x^2 - 2xy + y^2 = 0$ as a basis for your discussion, write a paragraph that answers the student's question.

✔ QUICK CHECK ANSWERS 3.1

1. $\dfrac{1}{x+2}$ **2.** $\dfrac{dy}{dx} = \dfrac{2x - y}{x + 3y^2}$ **3.** -1 **4.** $\dfrac{d^2y}{dx^2} = \sec^2 y \tan y$

3.2 DERIVATIVES OF LOGARITHMIC FUNCTIONS

In this section we will obtain derivative formulas for logarithmic functions, and we will explain why the natural logarithm function is preferred over logarithms with other bases in calculus.

DERIVATIVES OF LOGARITHMIC FUNCTIONS

We will establish that $f(x) = \ln x$ is differentiable for $x > 0$ by applying the derivative definition to $f(x)$. To evaluate the resulting limit, we will need the fact that $\ln x$ is continuous for $x > 0$ (Theorem 1.6.3), and we will need the limit

$$\lim_{v \to 0} (1 + v)^{1/v} = e \tag{1}$$

This limit can be obtained from limits (7) and (8) of Section 1.3 by making the substitution $v = 1/x$ and using the fact that $v \to 0^+$ as $x \to +\infty$ and $v \to 0^-$ as $x \to -\infty$. This produces two equal one-sided limits that together imply (1) (see Exercise 64 of Section 1.3).

$$\begin{aligned}
\frac{d}{dx}[\ln x] &= \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) && \text{The quotient property of logarithms in Theorem 0.5.2} \\
&= \lim_{h \to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right) \\
&= \lim_{v \to 0} \frac{1}{vx} \ln(1 + v) && \text{Let } v = h/x \text{ and note that } v \to 0 \text{ if and only if } h \to 0. \\
&= \frac{1}{x} \lim_{v \to 0} \frac{1}{v} \ln(1 + v) && x \text{ is fixed in this limit computation, so } 1/x \text{ can be moved through the limit sign.} \\
&= \frac{1}{x} \lim_{v \to 0} \ln(1 + v)^{1/v} && \text{The power property of logarithms in Theorem 0.5.2} \\
&= \frac{1}{x} \ln\left[\lim_{v \to 0} (1 + v)^{1/v}\right] && \ln x \text{ is continuous on } (0, +\infty) \text{ so we can move the limit through the function symbol.} \\
&= \frac{1}{x} \ln e \\
&= \frac{1}{x} && \text{Since } \ln e = 1
\end{aligned}$$

Thus,

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0 \tag{2}$$

A derivative formula for the general logarithmic function $\log_b x$ can be obtained from (2) by using Formula (6) of Section 0.5 to write

$$\frac{d}{dx}[\log_b x] = \frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{\ln b}\frac{d}{dx}[\ln x]$$

It follows from this that

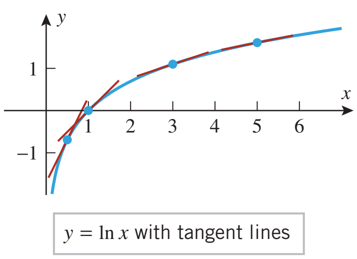
$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \quad x > 0 \tag{3}$$

Note that, among all possible bases, the base $b = e$ produces the simplest formula for the derivative of $\log_b x$. This is one of the reasons why the natural logarithm function is preferred over other logarithms in calculus.

▶ Example 1

(a) Figure 3.2.1 shows the graph of $y = \ln x$ and its tangent lines at the points $x = \frac{1}{2}$, 1, 3, and 5. Find the slopes of those tangent lines.

(b) Does the graph of $y = \ln x$ have any horizontal tangent lines? Use the derivative of $\ln x$ to justify your answer.



▲ **Figure 3.2.1**

Solution (a). From (2), the slopes of the tangent lines at the points $x = \frac{1}{2}$, 1, 3, and 5 are $1/x = 2$, 1, $\frac{1}{3}$, and $\frac{1}{5}$, respectively, which is consistent with Figure 3.2.1.

Solution (b). It does not appear from the graph of $y = \ln x$ that there are any horizontal tangent lines. This is confirmed by the fact that $dy/dx = 1/x$ is not equal to zero for any real value of x. ◀

If u is a differentiable function of x, and if $u(x) > 0$, then applying the chain rule to (2) and (3) produces the following generalized derivative formulas:

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[\log_b u] = \frac{1}{u \ln b} \cdot \frac{du}{dx} \tag{4–5}$$

▶ Example 2

Find $\frac{d}{dx}[\ln(x^2 + 1)]$.

Solution. Using (4) with $u = x^2 + 1$ we obtain

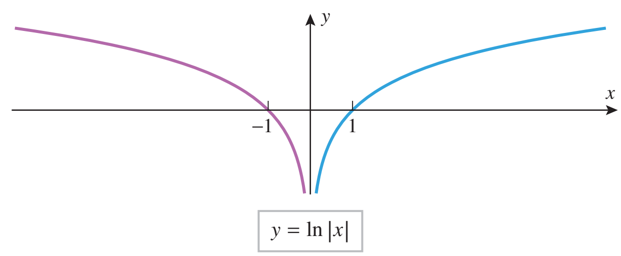
$$\frac{d}{dx}[\ln(x^2 + 1)] = \frac{1}{x^2 + 1} \cdot \frac{d}{dx}[x^2 + 1] = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1} \quad ◀$$

When possible, the properties of logarithms in Theorem 0.5.2 should be used to convert products, quotients, and exponents into sums, differences, and constant multiples *before* differentiating a function involving logarithms.

▶ Example 3

$$\begin{aligned}
\frac{d}{dx}\left[\ln\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right)\right] &= \frac{d}{dx}\left[2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1+x)\right] \\
&= \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)} \\
&= \frac{2}{x} + \cot x - \frac{1}{2+2x} \quad ◀
\end{aligned}$$

Figure 3.2.2 shows the graph of $f(x) = \ln|x|$. This function is important because it "extends" the domain of the natural logarithm function in the sense that the values of $\ln|x|$ and $\ln x$ are the same for $x > 0$, but $\ln|x|$ is defined for all nonzero values of x, and $\ln x$ is only defined for positive values of x.



▶ **Figure 3.2.2**

The derivative of $\ln|x|$ for $x \neq 0$ can be obtained by considering the cases $x > 0$ and $x < 0$ separately:

Case $x > 0$. In this case $|x| = x$, so

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln x] = \frac{1}{x}$$

Case $x < 0$. In this case $|x| = -x$, so it follows from (4) that

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln(-x)] = \frac{1}{(-x)} \cdot \frac{d}{dx}[-x] = \frac{1}{x}$$

Since the same formula results in both cases, we have shown that

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{if } x \neq 0 \tag{6}$$

▶ **Example 4** From (6) and the chain rule,

$$\frac{d}{dx}[\ln|\sin x|] = \frac{1}{\sin x} \cdot \frac{d}{dx}[\sin x] = \frac{\cos x}{\sin x} = \cot x \blacktriangleleft$$

LOGARITHMIC DIFFERENTIATION

We now consider a technique called ***logarithmic differentiation*** that is useful for differentiating functions that are composed of products, quotients, and powers.

▶ **Example 5** The derivative of

$$y = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4} \tag{7}$$

is messy to calculate directly. However, if we first take the natural logarithm of both sides and then use its properties, we can write

$$\ln y = 2\ln x + \tfrac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

Differentiating both sides with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{7/3}{7x-14} - \frac{8x}{1+x^2}$$

Thus, on solving for dy/dx and using (7) we obtain

$$\frac{dy}{dx} = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\left[\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right] \blacktriangleleft$$

REMARK Since $\ln y$ is only defined for $y > 0$, the computations in Example 5 are only valid for $x > 2$ (verify). However, because the derivative of $\ln y$ is the same as the derivative of $\ln|y|$, and because $\ln|y|$ is defined for $y < 0$ as well as $y > 0$, it follows that the formula obtained for dy/dx is valid for $x < 2$ as well as $x > 2$. In general, whenever a derivative dy/dx is obtained by logarithmic differentiation, the resulting derivative formula will be valid for all values of x for which $y \neq 0$. It may be valid at those points as well, but it is not guaranteed.

DERIVATIVES OF REAL POWERS OF *x*

We know from Theorem 2.3.2 and Exercise 82 in Section 2.3 that the differentiation formula

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{8}$$

holds for constant integer values of r. We will now use logarithmic differentiation to show that this formula holds if r is *any* real number (rational or irrational). In our computations we will assume that x^r is a differentiable function and that the familiar laws of exponents hold for real exponents.

In the next section we will discuss differentiating functions that have exponents which are not constant.

Let $y = x^r$, where r is a real number. The derivative dy/dx can be obtained by logarithmic differentiation as follows:

$$\ln|y| = \ln|x^r| = r\ln|x|$$

$$\frac{d}{dx}[\ln|y|] = \frac{d}{dx}[r\ln|x|]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x}$$

$$\frac{dy}{dx} = \frac{r}{x}y = \frac{r}{x}x^r = rx^{r-1}$$

✔ QUICK CHECK EXERCISES 3.2 *(See page 196 for answers.)*

1. The equation of the tangent line to the graph of $y = \ln x$ at $x = e^2$ is ________.
2. Find dy/dx.
 (a) $y = \ln 3x$ (b) $y = \ln\sqrt{x}$
 (c) $y = \log(1/|x|)$
3. Use logarithmic differentiation to find the derivative of
$$f(x) = \frac{\sqrt{x+1}}{\sqrt[3]{x-1}}$$
4. $\lim_{h\to 0}\frac{\ln(1+h)}{h} =$ ________

EXERCISE SET 3.2

1–26 Find dy/dx.

1. $y = \ln 5x$
2. $y = \ln\frac{x}{3}$
3. $y = \ln|1+x|$
4. $y = \ln(2+\sqrt{x})$
5. $y = \ln|x^2-1|$
6. $y = \ln|x^3-7x^2-3|$
7. $y = \ln\left(\frac{x}{1+x^2}\right)$
8. $y = \ln\left|\frac{1+x}{1-x}\right|$
9. $y = \ln x^2$
10. $y = (\ln x)^3$
11. $y = \sqrt{\ln x}$
12. $y = \ln\sqrt{x}$
13. $y = x\ln x$
14. $y = x^3\ln x$
15. $y = x^2\log_2(3-2x)$
16. $y = x[\log_2(x^2-2x)]^3$
17. $y = \frac{x^2}{1+\log x}$
18. $y = \frac{\log x}{1+\log x}$

19. $y=\ln(\ln x)$

20. $y=\ln(\ln(\ln x))$

21. $y=\ln(\tan x)$

22. $y=\ln(\cos x)$

23. $y=\cos(\ln x)$

24. $y=\sin^2(\ln x)$

25. $y=\log(\sin^2 x)$

26. $y=\log(1-\sin^2 x)$

27–30 Use the method of Example 3 to help perform the indicated differentiation. ■

27. $\dfrac{d}{dx}[\ln((x-1)^3(x^2+1)^4)]$

28. $\dfrac{d}{dx}[\ln((\cos^2 x)\sqrt{1+x^4})]$

29. $\dfrac{d}{dx}\left[\ln\dfrac{\cos x}{\sqrt{4-3x^2}}\right]$

30. $\dfrac{d}{dx}\left[\ln\sqrt{\dfrac{x-1}{x+1}}\right]$

31–34 True–False Determine whether the statement is true or false. Explain your answer. ■

31. The slope of the tangent line to the graph of $y=\ln x$ at $x=a$ approaches infinity as $a\to 0^+$.

32. If $\lim_{x\to+\infty} f'(x)=0$, then the graph of $y=f(x)$ has a horizontal asymptote.

33. The derivative of $\ln|x|$ is an odd function.

34. We have

$$\frac{d}{dx}((\ln x)^2)=\frac{d}{dx}(2(\ln x))=\frac{2}{x}$$

35–38 Find dy/dx using logarithmic differentiation. ■

35. $y=x\sqrt[3]{1+x^2}$

36. $y=\sqrt[5]{\dfrac{x-1}{x+1}}$

37. $y=\dfrac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}$

38. $y=\dfrac{\sin x\cos x\tan^3 x}{\sqrt{x}}$

39. Find

(a) $\dfrac{d}{dx}[\log_x e]$ (b) $\dfrac{d}{dx}[\log_x 2]$.

40. Find

(a) $\dfrac{d}{dx}[\log_{(1/x)} e]$ (b) $\dfrac{d}{dx}[\log_{(\ln x)} e]$.

41–44 Find the equation of the tangent line to the graph of $y=f(x)$ at $x=x_0$. ■

41. $f(x)=\ln x;\ x_0=e^{-1}$

42. $f(x)=\log x;\ x_0=10$

43. $f(x)=\ln(-x);\ x_0=-e$

44. $f(x)=\ln|x|;\ x_0=-2$

FOCUS ON CONCEPTS

45. (a) Find the equation of a line through the origin that is tangent to the graph of $y=\ln x$.

(b) Explain why the y-intercept of a tangent line to the curve $y=\ln x$ must be 1 unit less than the y-coordinate of the point of tangency.

46. Use logarithmic differentiation to verify the product and quotient rules. Explain what properties of $\ln x$ are important for this verification.

47. Find a formula for the area $A(w)$ of the triangle bounded by the tangent line to the graph of $y=\ln x$ at $P(w,\ln w)$, the horizontal line through P, and the y-axis.

48. Find a formula for the area $A(w)$ of the triangle bounded by the tangent line to the graph of $y=\ln x^2$ at $P(w,\ln w^2)$, the horizontal line through P, and the y-axis.

49. Verify that $y=\ln(x+e)$ satisfies $dy/dx=e^{-y}$, with $y=1$ when $x=0$.

50. Verify that $y=-\ln(e^2-x)$ satisfies $dy/dx=e^y$, with $y=-2$ when $x=0$.

51. Find a function f such that $y=f(x)$ satisfies $dy/dx=e^{-y}$, with $y=0$ when $x=0$.

52. Find a function f such that $y=f(x)$ satisfies $dy/dx=e^y$, with $y=-\ln 2$ when $x=0$.

53–55 Find the limit by interpreting the expression as an appropriate derivative. ■

53. (a) $\displaystyle\lim_{x\to 0}\frac{\ln(1+3x)}{x}$ (b) $\displaystyle\lim_{x\to 0}\frac{\ln(1-5x)}{x}$

54. (a) $\displaystyle\lim_{\Delta x\to 0}\frac{\ln(e^2+\Delta x)-2}{\Delta x}$ (b) $\displaystyle\lim_{w\to 1}\frac{\ln w}{w-1}$

55. (a) $\displaystyle\lim_{x\to 0}\frac{\ln(\cos x)}{x}$ (b) $\displaystyle\lim_{h\to 0}\frac{(1+h)^{\sqrt{2}}-1}{h}$

56. Modify the derivation of Equation (2) to give another proof of Equation (3).

57. Writing Review the derivation of the formula

$$\frac{d}{dx}[\ln x]=\frac{1}{x}$$

and then write a paragraph that discusses all the ingredients (theorems, limit properties, etc.) that are needed for this derivation.

58. Writing Write a paragraph that explains how logarithmic differentiation can replace a difficult differentiation computation with a simpler computation.

✔ QUICK CHECK ANSWERS 3.2

1. $y=\dfrac{x}{e^2}+1$ **2.** (a) $\dfrac{dy}{dx}=\dfrac{1}{x}$ (b) $\dfrac{dy}{dx}=\dfrac{1}{2x}$ (c) $\dfrac{dy}{dx}=-\dfrac{1}{x\ln 10}$ **3.** $\dfrac{\sqrt{x+1}}{\sqrt[3]{x-1}}\left[\dfrac{1}{2(x+1)}-\dfrac{1}{3(x-1)}\right]$ **4.** 1

3.3 DERIVATIVES OF EXPONENTIAL AND INVERSE TRIGONOMETRIC FUNCTIONS

See Section 0.4 for a review of one-to-one functions and inverse functions.

In this section we will show how the derivative of a one-to-one function can be used to obtain the derivative of its inverse function. This will provide the tools we need to obtain derivative formulas for exponential functions from the derivative formulas for logarithmic functions and to obtain derivative formulas for inverse trigonometric functions from the derivative formulas for trigonometric functions.

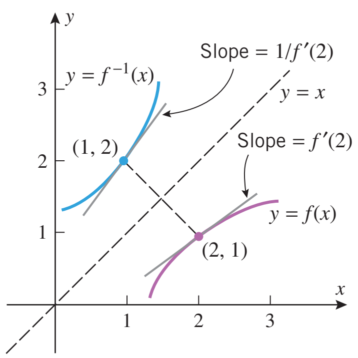
Our first goal in this section is to obtain a formula relating the derivative of the inverse function f^{-1} to the derivative of the function f.

Example 1 Suppose that f is a one-to-one differentiable function such that $f(2) = 1$ and $f'(2) = \frac{3}{4}$. Then the tangent line to $y = f(x)$ at the point (2, 1) has equation

$$y - 1 = \tfrac{3}{4}(x - 2)$$

The tangent line to $y = f^{-1}(x)$ at the point (1, 2) is the reflection about the line $y = x$ of the tangent line to $y = f(x)$ at the point (2, 1) (Figure 3.3.1), and its equation can be obtained by interchanging x and y:

$$x - 1 = \tfrac{3}{4}(y - 2) \quad \text{or} \quad y - 2 = \tfrac{4}{3}(x - 1)$$



▲ **Figure 3.3.1**

Notice that the slope of the tangent line to $y = f^{-1}(x)$ at $x = 1$ is the reciprocal of the slope of the tangent line to $y = f(x)$ at $x = 2$. That is,

$$(f^{-1})'(1) = \frac{1}{f'(2)} = \frac{4}{3} \quad \blacktriangleleft \tag{1}$$

Since $2 = f^{-1}(1)$ for the function f in Example 1, it follows that $f'(2) = f'(f^{-1}(1))$. Thus, Formula (1) can also be expressed as

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

In general, if f is a differentiable and one-to-one function, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \tag{2}$$

provided $f'(f^{-1}(x)) \neq 0$.

Formula (2) can be confirmed using implicit differentiation. The equation $y = f^{-1}(x)$ is equivalent to $x = f(y)$. Differentiating with respect to x we obtain

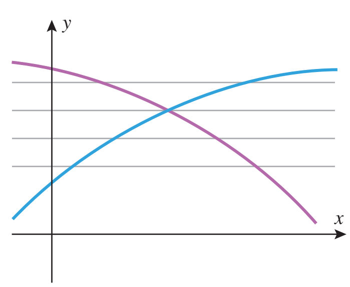
$$1 = \frac{d}{dx}[x] = \frac{d}{dx}[f(y)] = f'(y) \cdot \frac{dy}{dx}$$

so that

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Also from $x = f(y)$ we have $dx/dy = f'(y)$, which gives the following alternative version of Formula (2):

$$\frac{dy}{dx} = \frac{1}{dx/dy} \tag{3}$$



▲ **Figure 3.3.2** The graph of an increasing function (blue) or a decreasing function (purple) is cut at most once by any horizontal line.

INCREASING OR DECREASING FUNCTIONS ARE ONE-TO-ONE

If the graph of a function f is always increasing or always decreasing over the domain of f, then a horizontal line will cut the graph of f in at most one point (Figure 3.3.2), so f

must have an inverse function (see Section 0.4). We will prove in the next chapter that f is increasing on any interval on which $f'(x) > 0$ (since the graph has positive slope) and that f is decreasing on any interval on which $f'(x) < 0$ (since the graph has negative slope). These intuitive observations, together with Formula (2), suggest the following theorem, which we state without formal proof.

3.3.1 THEOREM *Suppose that the domain of a function f is an open interval on which $f'(x) > 0$ or on which $f'(x) < 0$. Then f is one-to-one, $f^{-1}(x)$ is differentiable at all values of x in the range of f, and the derivative of $f^{-1}(x)$ is given by Formula (2).*

▶ **Example 2** Consider the function $f(x) = x^5 + x + 1$.

(a) Show that f is one-to-one on the interval $(-\infty, +\infty)$.
(b) Find a formula for the derivative of f^{-1}.
(c) Compute $(f^{-1})'(1)$.

In general, once it is established that f^{-1} is differentiable, one has the option of calculating the derivative of f^{-1} using Formula (2) or (3), or by differentiating implicitly, as in Example 2.

Solution (a). Since

$$f'(x) = 5x^4 + 1 > 0$$

for all real values of x, it follows from Theorem 3.3.1 that f is one-to-one on the interval $(-\infty, +\infty)$.

Solution (b). Let $y = f^{-1}(x)$. Differentiating $x = f(y) = y^5 + y + 1$ implicitly with respect to x yields

$$\frac{d}{dx}[x] = \frac{d}{dx}[y^5 + y + 1]$$

$$1 = (5y^4 + 1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{5y^4 + 1} \tag{4}$$

We cannot solve $x = y^5 + y + 1$ for y in terms of x, so we leave the expression for dy/dx in Equation (4) in terms of y.

Solution (c). From Equation (4),

$$(f^{-1})'(1) = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{1}{5y^4 + 1}\right|_{x=1}$$

Thus, we need to know the value of $y = f^{-1}(x)$ at $x = 1$, which we can obtain by solving the equation $f(y) = 1$ for y. This equation is $y^5 + y + 1 = 1$, which, by inspection, is satisfied by $y = 0$. Thus,

$$(f^{-1})'(1) = \left.\frac{1}{5y^4 + 1}\right|_{y=0} = 1 \blacktriangleleft$$

DERIVATIVES OF EXPONENTIAL FUNCTIONS

Our next objective is to show that the general exponential function b^x $(b > 0, b \neq 1)$ is differentiable everywhere and to find its derivative. To do this, we will use the fact that

b^x is the inverse of the function $f(x) = \log_b x$. We will assume that $b > 1$. With this assumption we have $\ln b > 0$, so

$$f'(x) = \frac{d}{dx}[\log_b x] = \frac{1}{x \ln b} > 0 \quad \text{for all } x \text{ in the interval } (0, +\infty)$$

It now follows from Theorem 3.3.1 that $f^{-1}(x) = b^x$ is differentiable for all x in the range of $f(x) = \log_b x$. But we know from Table 0.5.3 that the range of $\log_b x$ is $(-\infty, +\infty)$, so we have established that b^x is differentiable everywhere.

To obtain a derivative formula for b^x we rewrite $y = b^x$ as

$$x = \log_b y$$

and differentiate implicitly using Formula (5) of Section 3.2 to obtain

$$1 = \frac{1}{y \ln b} \cdot \frac{dy}{dx}$$

Solving for dy/dx and replacing y by b^x we have

$$\frac{dy}{dx} = y \ln b = b^x \ln b$$

Thus, we have shown that

$$\frac{d}{dx}[b^x] = b^x \ln b \qquad (5)$$

How does the derivation of Formula (5) change if $0 < b < 1$?

In the special case where $b = e$ we have $\ln e = 1$, so that (5) becomes

$$\frac{d}{dx}[e^x] = e^x \qquad (6)$$

In Section 0.5 we stated that $b = e$ is the only base for which the slope of the tangent line to the curve $y = b^x$ at any point P on the curve is the y-coordinate at P (see page 54). Verify this statement.

Moreover, if u is a differentiable function of x, then it follows from (5) and (6) that

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx} \qquad (7\text{–}8)$$

It is important to distinguish between differentiating an exponential function b^x (variable exponent and constant base) and a power function x^b (variable base and constant exponent). For example, compare the derivative

$$\frac{d}{dx}[x^2] = 2x$$

to the derivative of 2^x in Example 3.

▶ **Example 3** The following computations use Formulas (7) and (8).

$$\frac{d}{dx}[2^x] = 2^x \ln 2$$

$$\frac{d}{dx}[e^{-2x}] = e^{-2x} \cdot \frac{d}{dx}[-2x] = -2e^{-2x}$$

$$\frac{d}{dx}[e^{x^3}] = e^{x^3} \cdot \frac{d}{dx}[x^3] = 3x^2 e^{x^3}$$

$$\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}[\cos x] = -(\sin x) e^{\cos x}$$ ◀

Functions of the form $f(x) = u^v$ in which u and v are *nonconstant* functions of x are neither exponential functions nor power functions. Functions of this form can be differentiated using logarithmic differentiation.

▶ **Example 4** Use logarithmic differentiation to find $\frac{d}{dx}[(x^2+1)^{\sin x}]$.

Solution. Setting $y = (x^2+1)^{\sin x}$ we have

$$\ln y = \ln[(x^2+1)^{\sin x}] = (\sin x)\ln(x^2+1)$$

Differentiating both sides with respect to x yields

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}[(\sin x)\ln(x^2+1)]\\ &= (\sin x)\frac{1}{x^2+1}(2x) + (\cos x)\ln(x^2+1)\end{aligned}$$

Thus,

$$\begin{aligned}\frac{dy}{dx} &= y\left[\frac{2x\sin x}{x^2+1} + (\cos x)\ln(x^2+1)\right]\\ &= (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + (\cos x)\ln(x^2+1)\right]\end{aligned}$$ ◀

DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS

To obtain formulas for the derivatives of the inverse trigonometric functions, we will need to use some of the identities given in Formulas (11) to (17) of Section 0.4. Rather than memorize those identities, we recommend that you review the "triangle technique" that we used to obtain them.

To begin, consider the function $\sin^{-1} x$. If we let $f(x) = \sin x$ $(-\pi/2 \le x \le \pi/2)$, then it follows from Formula (2) that $f^{-1}(x) = \sin^{-1} x$ will be differentiable at any point x where $\cos(\sin^{-1} x) \neq 0$. This is equivalent to the condition

$$\sin^{-1} x \neq -\frac{\pi}{2} \quad \text{and} \quad \sin^{-1} x \neq \frac{\pi}{2}$$

so it follows that $\sin^{-1} x$ is differentiable on the interval $(-1, 1)$.

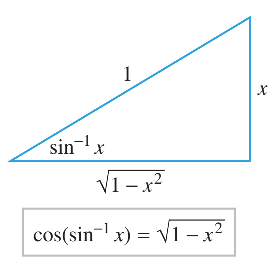
A derivative formula for $\sin^{-1} x$ on $(-1, 1)$ can be obtained by using Formula (2) or (3) or by differentiating implicitly. We will use the latter method. Rewriting the equation $y = \sin^{-1} x$ as $x = \sin y$ and differentiating implicitly with respect to x, we obtain

Observe that $\sin^{-1} x$ is only differentiable on the interval $(-1, 1)$, even though its domain is $[-1, 1]$. This is because the graph of $y = \sin x$ has horizontal tangent lines at the points $(\pi/2, 1)$ and $(-\pi/2, -1)$, so the graph of $y = \sin^{-1} x$ has vertical tangent lines at $x = \pm 1$.

$$\begin{aligned}\frac{d}{dx}[x] &= \frac{d}{dx}[\sin y]\\ 1 &= \cos y \cdot \frac{dy}{dx}\\ \frac{dy}{dx} &= \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}\end{aligned}$$

At this point we have succeeded in obtaining the derivative; however, this derivative formula can be simplified using the identity indicated in Figure 3.3.3. This yields

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$



▲ **Figure 3.3.3**

Thus, we have shown that

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1)$$

More generally, if u is a differentiable function of x, then the chain rule produces the following generalized version of this formula:

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (-1 < u < 1)$$

The method used to derive this formula can be used to obtain generalized derivative formulas for the remaining inverse trigonometric functions. The following is a complete list of these formulas, each of which is valid on the natural domain of the function that multiplies du/dx.

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}[\cos^{-1} u] = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (9\text{–}10)$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx} \qquad (11\text{–}12)$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad \frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad (13\text{–}14)$$

The appearance of $|u|$ in (13) and (14) will be explained in Exercise 58.

Example 5 Find dy/dx if

(a) $y = \sin^{-1}(x^3)$ (b) $y = \sec^{-1}(e^x)$

Solution (a). From (9)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}(3x^2) = \frac{3x^2}{\sqrt{1-x^6}}$$

Solution (b). From (13)

$$\frac{dy}{dx} = \frac{1}{e^x\sqrt{(e^x)^2-1}}(e^x) = \frac{1}{\sqrt{e^{2x}-1}}$$ ◀

✔ QUICK CHECK EXERCISES 3.3 *(See page 203 for answers.)*

1. Suppose that a one-to-one function f has tangent line $y = 5x + 3$ at the point $(1, 8)$. Evaluate $(f^{-1})'(8)$.

2. In each case, from the given derivative, determine whether the function f is invertible.
(a) $f'(x) = x^2 + 1$ (b) $f'(x) = x^2 - 1$
(c) $f'(x) = \sin x$ (d) $f'(x) = \frac{\pi}{2} + \tan^{-1} x$

3. Evaluate the derivative.
(a) $\frac{d}{dx}[e^x]$ (b) $\frac{d}{dx}[7^x]$
(c) $\frac{d}{dx}[\cos(e^x+1)]$ (d) $\frac{d}{dx}[e^{3x-2}]$

4. Let $f(x) = e^{x^3+x}$. Use $f'(x)$ to verify that f is one-to-one.

EXERCISE SET 3.3 Graphing Utility

FOCUS ON CONCEPTS

1. Let $f(x) = x^5 + x^3 + x$.
(a) Show that f is one-to-one and confirm that $f(1) = 3$.
(b) Find $(f^{-1})'(3)$.

2. Let $f(x) = x^3 + 2e^x$.
(a) Show that f is one-to-one and confirm that $f(0) = 2$.
(b) Find $(f^{-1})'(2)$.

3–4 Find $(f^{-1})'(x)$ using Formula (2), and check your answer by differentiating f^{-1} directly. ■

3. $f(x) = 2/(x+3)$ **4.** $f(x) = \ln(2x+1)$

5–6 Determine whether the function f is one-to-one by examining the sign of $f'(x)$. ■

5. (a) $f(x) = x^2 + 8x + 1$
(b) $f(x) = 2x^5 + x^3 + 3x + 2$
(c) $f(x) = 2x + \sin x$
(d) $f(x) = \left(\frac{1}{2}\right)^x$

6. (a) $f(x) = x^3 + 3x^2 - 8$
(b) $f(x) = x^5 + 8x^3 + 2x - 1$
(c) $f(x) = \frac{x}{x+1}$
(d) $f(x) = \log_b x,\quad 0 < b < 1$

7–10 Find the derivative of f^{-1} by using Formula (3), and check your result by differentiating implicitly. ■

7. $f(x) = 5x^3 + x - 7$
8. $f(x) = 1/x^2, \quad x > 0$
9. $f(x) = 2x^5 + x^3 + 1$
10. $f(x) = 5x - \sin 2x, \quad -\frac{\pi}{4} < x < \frac{\pi}{4}$

FOCUS ON CONCEPTS

11. Figure 0.4.8 is a "proof by picture" that the reflection of a point $P(a, b)$ about the line $y = x$ is the point $Q(b, a)$. Establish this result rigorously by completing each part.
    (a) Prove that if P is not on the line $y = x$, then P and Q are distinct, and the line $\overleftrightarrow{PQ}$ is perpendicular to the line $y = x$.
    (b) Prove that if P is not on the line $y = x$, the midpoint of segment PQ is on the line $y = x$.
    (c) Carefully explain what it means geometrically to reflect P about the line $y = x$.
    (d) Use the results of parts (a)–(c) to prove that Q is the reflection of P about the line $y = x$.
12. Prove that the reflection about the line $y = x$ of a line with slope m, $m \neq 0$, is a line with slope $1/m$. [*Hint:* Apply the result of the previous exercise to a pair of points on the line of slope m and to a corresponding pair of points on the reflection of this line about the line $y = x$.]
13. Suppose that f and g are increasing functions. Determine which of the functions $f(x) + g(x)$, $f(x)g(x)$, and $f(g(x))$ must also be increasing.
14. Suppose that f and g are one-to-one functions. Determine which of the functions $f(x) + g(x)$, $f(x)g(x)$, and $f(g(x))$ must also be one-to-one.

15–26 Find dy/dx. ■

15. $y = e^{7x}$
16. $y = e^{-5x^2}$
17. $y = x^3e^x$
18. $y = e^{1/x}$
19. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
20. $y = \sin(e^x)$
21. $y = e^{x \tan x}$
22. $y = \dfrac{e^x}{\ln x}$
23. $y = e^{(x - e^{3x})}$
24. $y = \exp(\sqrt{1 + 5x^3})$
25. $y = \ln(1 - xe^{-x})$
26. $y = \ln(\cos e^x)$

27–30 Find $f'(x)$ by Formula (7) and then by logarithmic differentiation. ■

27. $f(x) = 2^x$
28. $f(x) = 3^{-x}$
29. $f(x) = \pi^{\sin x}$
30. $f(x) = \pi^{x \tan x}$

31–35 Find dy/dx using the method of logarithmic differentiation. ■

31. $y = (x^3 - 2x)^{\ln x}$
32. $y = x^{\sin x}$
33. $y = (\ln x)^{\tan x}$
34. $y = (x^2 + 3)^{\ln x}$
35. $y = (\ln x)^{\ln x}$
36. (a) Explain why Formula (5) cannot be used to find $(d/dx)[x^x]$.
    (b) Find this derivative by logarithmic differentiation.

37–52 Find dy/dx. ■

37. $y = \sin^{-1}(3x)$
38. $y = \cos^{-1}\left(\dfrac{x+1}{2}\right)$
39. $y = \sin^{-1}(1/x)$
40. $y = \cos^{-1}(\cos x)$
41. $y = \tan^{-1}(x^3)$
42. $y = \sec^{-1}(x^5)$
43. $y = (\tan x)^{-1}$
44. $y = \dfrac{1}{\tan^{-1} x}$
45. $y = e^x \sec^{-1} x$
46. $y = \ln(\cos^{-1} x)$
47. $y = \sin^{-1} x + \cos^{-1} x$
48. $y = x^2(\sin^{-1} x)^3$
49. $y = \sec^{-1} x + \csc^{-1} x$
50. $y = \csc^{-1}(e^x)$
51. $y = \cot^{-1}(\sqrt{x})$
52. $y = \sqrt{\cot^{-1} x}$

53–56 True–False Determine whether the statement is true or false. Explain your answer. ■

53. If a function $y = f(x)$ satisfies $dy/dx = y$, then $y = e^x$.
54. If $y = f(x)$ is a function such that dy/dx is a rational function, then $f(x)$ is also a rational function.
55. $\dfrac{d}{dx}(\log_b |x|) = \dfrac{1}{x \ln b}$
56. We can conclude from the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ that $\sin^{-1} x + \cos^{-1} x$ is constant.
57. (a) Use Formula (2) to prove that
$$\frac{d}{dx}[\cot^{-1} x]\bigg|_{x=0} = -1$$
    (b) Use part (a) above, part (a) of Exercise 48 in Section 0.4, and the chain rule to show that
$$\frac{d}{dx}[\cot^{-1} x] = -\frac{1}{1 + x^2}$$
    for $-\infty < x < +\infty$.
    (c) Conclude from part (b) that
$$\frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1 + u^2}\frac{du}{dx}$$
    for $-\infty < u < +\infty$.
58. (a) Use part (c) of Exercise 48 in Section 0.4 and the chain rule to show that
$$\frac{d}{dx}[\csc^{-1} x] = -\frac{1}{|x|\sqrt{x^2 - 1}}$$
    for $1 < |x|$.
    (b) Conclude from part (a) that
$$\frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}$$
    for $1 < |u|$.

(cont.)

(c) Use Equation (11) in Section 0.4 and parts (b) and (c) of Exercise 48 in that section to show that if $|x| \geq 1$ then, $\sec^{-1} x + \csc^{-1} x = \pi/2$. Conclude from part (a) that

$$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2 - 1}}$$

(d) Conclude from part (c) that

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}$$

59–60 Find dy/dx by implicit differentiation. ■

59. $x^3 + x\tan^{-1} y = e^y$

60. $\sin^{-1}(xy) = \cos^{-1}(x - y)$

61. (a) Show that $f(x) = x^3 - 3x^2 + 2x$ is not one-to-one on $(-\infty, +\infty)$.
(b) Find the largest value of k such that f is one-to-one on the interval $(-k, k)$.

62. (a) Show that the function $f(x) = x^4 - 2x^3$ is not one-to-one on $(-\infty, +\infty)$.
(b) Find the smallest value of k such that f is one-to-one on the interval $[k, +\infty)$.

63. Let $f(x) = x^4 + x^3 + 1$, $0 \leq x \leq 2$.
(a) Show that f is one-to-one.
(b) Let $g(x) = f^{-1}(x)$ and define $F(x) = f(2g(x))$. Find an equation for the tangent line to $y = F(x)$ at $x = 3$.

64. Let $f(x) = \dfrac{\exp(4 - x^2)}{x}$, $x > 0$.
(a) Show that f is one-to-one.
(b) Let $g(x) = f^{-1}(x)$ and define $F(x) = f([g(x)]^2)$. Find $F'\left(\frac{1}{2}\right)$.

65. Show that for any constants A and k, the function $y = Ae^{kt}$ satisfies the equation $dy/dt = ky$.

66. Show that for any constants A and B, the function

$$y = Ae^{2x} + Be^{-4x}$$

satisfies the equation

$$y'' + 2y' - 8y = 0$$

67. Show that
(a) $y = xe^{-x}$ satisfies the equation $xy' = (1 - x)y$
(b) $y = xe^{-x^2/2}$ satisfies the equation $xy' = (1 - x^2)y$.

68. Show that the rate of change of $y = 100e^{-0.2x}$ with respect to x is proportional to y.

69. Show that

$$y = \frac{60}{5 + 7e^{-t}} \quad \text{satisfies} \quad \frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

for some constants r and K, and determine the values of these constants.

70. Suppose that the population of oxygen-dependent bacteria in a pond is modeled by the equation

$$P(t) = \frac{60}{5 + 7e^{-t}}$$

where $P(t)$ is the population (in billions) t days after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t \to +\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

71–76 Find the limit by interpreting the expression as an appropriate derivative. ■

71. $\displaystyle\lim_{x \to 0} \frac{e^{3x} - 1}{x}$

72. $\displaystyle\lim_{x \to 0} \frac{\exp(x^2) - 1}{x}$

73. $\displaystyle\lim_{h \to 0} \frac{10^h - 1}{h}$

74. $\displaystyle\lim_{h \to 0} \frac{\tan^{-1}(1 + h) - \pi/4}{h}$

75. $\displaystyle\lim_{\Delta x \to 0} \frac{9\left[\sin^{-1}\left(\frac{\sqrt{3}}{2} + \Delta x\right)\right]^2 - \pi^2}{\Delta x}$

76. $\displaystyle\lim_{w \to 2} \frac{3\sec^{-1} w - \pi}{w - 2}$

77. Writing Let G denote the graph of an invertible function f and consider G as a fixed set of points in the plane. Suppose we relabel the coordinate axes so that the x-axis becomes the y-axis and vice versa. Carefully explain why now the same set of points G becomes the graph of f^{-1} (with the coordinate axes in a nonstandard position). Use this result to explain Formula (2).

78. Writing Suppose that f has an inverse function. Carefully explain the connection between Formula (2) and implicit differentiation of the equation $x = f(y)$.

✔ QUICK CHECK ANSWERS 3.3

1. $\frac{1}{5}$ **2.** (a) yes (b) no (c) no (d) yes **3.** (a) e^x (b) $7^x \ln 7$ (c) $-e^x \sin(e^x + 1)$ (d) $3e^{3x-2}$
4. $f'(x) = e^{x^3+x} \cdot (3x^2 + 1) > 0$ for all x

3.4 RELATED RATES

In this section we will study related rates problems. In such problems one tries to find the rate at which some quantity is changing by relating the quantity to other quantities whose rates of change are known.

DIFFERENTIATING EQUATIONS TO RELATE RATES

Figure 3.4.1 shows a liquid draining through a conical filter. As the liquid drains, its volume V, height h, and radius r are functions of the elapsed time t, and at each instant these variables are related by the equation

$$V = \frac{\pi}{3}r^2h$$

If we were interested in finding the rate of change of the volume V with respect to the time t, we could begin by differentiating both sides of this equation with respect to t to obtain

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Thus, to find dV/dt at a specific time t from this equation we would need to have values for r, h, dh/dt, and dr/dt at that time. This is called a ***related rates problem*** because the goal is to find an unknown rate of change by *relating* it to other variables whose values and whose rates of change at time t are known or can be found in some way. Let us begin with a simple example.

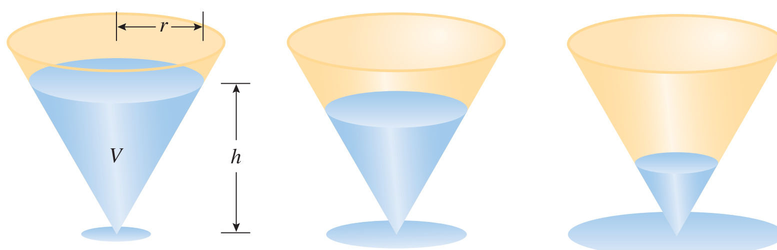


Figure 3.4.1

Example 1 Suppose that x and y are differentiable functions of t and are related by the equation $y = x^3$. Find dy/dt at time $t = 1$ if $x = 2$ and $dx/dt = 4$ at time $t = 1$.

Solution. Using the chain rule to differentiate both sides of the equation $y = x^3$ with respect to t yields

$$\frac{dy}{dt} = \frac{d}{dt}[x^3] = 3x^2\frac{dx}{dt}$$

Thus, the value of dy/dt at time $t = 1$ is

$$\left.\frac{dy}{dt}\right|_{t=1} = 3(2)^2\left.\frac{dx}{dt}\right|_{t=1} = 12 \cdot 4 = 48$$ ◀

Oil spill from a ruptured tanker.

▶ **Example 2** Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

Solution. Let

$$\begin{aligned} t &= \text{number of seconds elapsed from the time of the spill} \\ r &= \text{radius of the spill in feet after } t \text{ seconds} \\ A &= \text{area of the spill in square feet after } t \text{ seconds} \end{aligned}$$

(Figure 3.4.2). We know the rate at which the radius is increasing, and we want to find the rate at which the area is increasing at the instant when $r = 60$; that is, we want to find

$$\left.\frac{dA}{dt}\right|_{r=60} \quad \text{given that} \quad \frac{dr}{dt} = 2 \text{ ft/s}$$

This suggests that we look for an equation relating A and r that we can differentiate with respect to t to produce a relationship between dA/dt and dr/dt. But A is the area of a circle of radius r, so

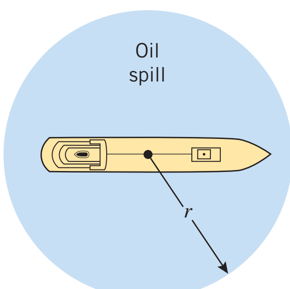
$$A = \pi r^2 \tag{1}$$

Differentiating both sides of (1) with respect to t yields

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \tag{2}$$

Thus, when $r = 60$ the area of the spill is increasing at the rate of

$$\left.\frac{dA}{dt}\right|_{r=60} = 2\pi(60)(2) = 240\pi \text{ ft}^2/\text{s} \approx 754 \text{ ft}^2/\text{s} \blacktriangleleft$$



▲ **Figure 3.4.2**

With some minor variations, the method used in Example 2 can be used to solve a variety of related rates problems. We can break the method down into five steps.

A Strategy for Solving Related Rates Problems

Step 1. Assign letters to all quantities that vary with time and any others that seem relevant to the problem. Give a definition for each letter.

Step 2. Identify the rates of change that are known and the rate of change that is to be found. Interpret each rate as a derivative.

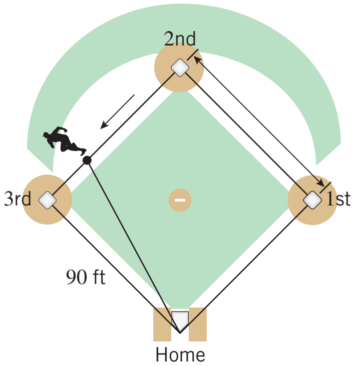
Step 3. Find an equation that relates the variables whose rates of change were identified in Step 2. To do this, it will often be helpful to draw an appropriately labeled figure that illustrates the relationship.

Step 4. Differentiate both sides of the equation obtained in Step 3 with respect to time to produce a relationship between the known rates of change and the unknown rate of change.

Step 5. *After* completing Step 4, substitute all known values for the rates of change and the variables, and then solve for the unknown rate of change.

WARNING

We have italicized the word "After" in Step 5 because it is a common error to substitute numerical values before performing the differentiation. For instance, in Example 2 had we substituted the known value of $r = 60$ in (1) before differentiating, we would have obtained $dA/dt = 0$, which is obviously incorrect.



▲ **Figure 3.4.3**

The quantity

$$\left.\frac{dx}{dt}\right|_{x=20}$$

is negative because x is decreasing with respect to t.

▶ **Example 3** A baseball diamond is a square whose sides are 90 ft long (Figure 3.4.3). Suppose that a player running from second base to third base has a speed of 30 ft/s at the instant when he is 20 ft from third base. At what rate is the player's distance from home plate changing at that instant?

Solution. We are given a constant speed with which the player is approaching third base, and we want to find the rate of change of the distance between the player and home plate at a particular instant. Thus, let

t = number of seconds since the player left second base
x = distance in feet from the player to third base
y = distance in feet from the player to home plate

(Figure 3.4.4). Thus, we want to find

$$\left.\frac{dy}{dt}\right|_{x=20} \quad \text{given that} \quad \left.\frac{dx}{dt}\right|_{x=20} = -30 \text{ ft/s}$$

As suggested by Figure 3.4.4, an equation relating the variables x and y can be obtained using the Theorem of Pythagoras:

$$x^2 + 90^2 = y^2 \tag{3}$$

Differentiating both sides of this equation with respect to t yields

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

from which we obtain

$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} \tag{4}$$

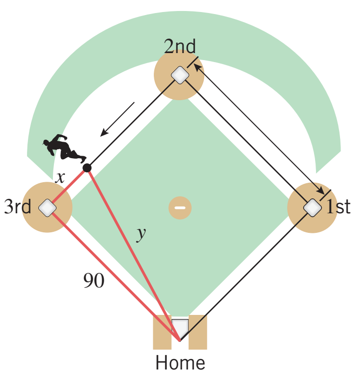
When $x = 20$, it follows from (3) that

$$y = \sqrt{20^2 + 90^2} = \sqrt{8500} = 10\sqrt{85}$$

so that (4) yields

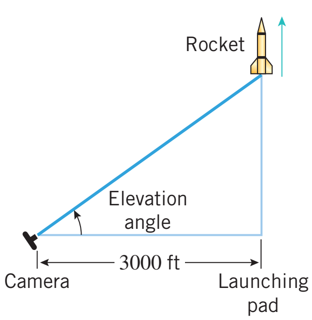
$$\left.\frac{dy}{dt}\right|_{x=20} = \frac{20}{10\sqrt{85}}(-30) = -\frac{60}{\sqrt{85}} \approx -6.51 \text{ ft/s}$$

The negative sign in the answer tells us that y is decreasing, which makes sense physically from Figure 3.4.4. ◀



▲ **Figure 3.4.4**

▶ **Example 4** In Figure 3.4.5 we have shown a camera mounted at a point 3000 ft from the base of a rocket launching pad. If the rocket is rising vertically at 880 ft/s when it is 4000 ft above the launching pad, how fast must the camera elevation angle change at that instant to keep the camera aimed at the rocket?

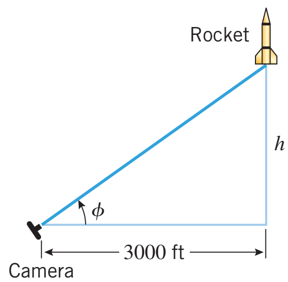


▲ **Figure 3.4.5**

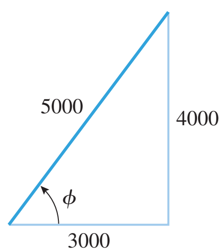
Solution. Let

t = number of seconds elapsed from the time of launch
ϕ = camera elevation angle in radians after t seconds
h = height of the rocket in feet after t seconds

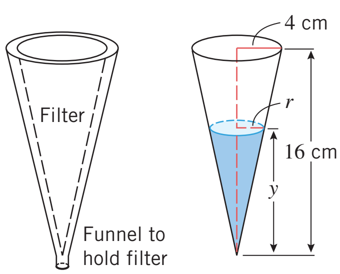
(Figure 3.4.6). At each instant the rate at which the camera elevation angle must change



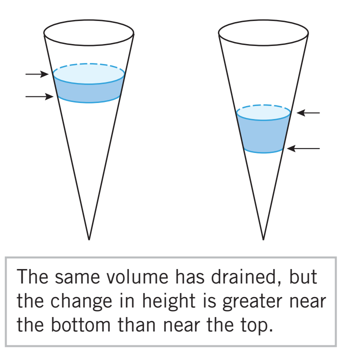
▲ **Figure 3.4.6**



▲ **Figure 3.4.7**



▲ **Figure 3.4.8**



▲ **Figure 3.4.9**

is $d\phi/dt$, and the rate at which the rocket is rising is dh/dt. We want to find

$$\left.\frac{d\phi}{dt}\right|_{h=4000} \quad \text{given that} \quad \left.\frac{dh}{dt}\right|_{h=4000} = 880 \text{ ft/s}$$

From Figure 3.4.6 we see that

$$\tan\phi = \frac{h}{3000} \tag{5}$$

Differentiating both sides of (5) with respect to t yields

$$(\sec^2\phi)\frac{d\phi}{dt} = \frac{1}{3000}\frac{dh}{dt} \tag{6}$$

When $h = 4000$, it follows that

$$(\sec\phi)\big|_{h=4000} = \frac{5000}{3000} = \frac{5}{3}$$

(see Figure 3.4.7), so that from (6)

$$\left(\frac{5}{3}\right)^2 \left.\frac{d\phi}{dt}\right|_{h=4000} = \frac{1}{3000}\cdot 880 = \frac{22}{75}$$

$$\left.\frac{d\phi}{dt}\right|_{h=4000} = \frac{22}{75}\cdot\frac{9}{25} = \frac{66}{625} \approx 0.11 \text{ rad/s} \approx 6.05 \text{ deg/s}$$ ◀

▶ **Example 5** Suppose that liquid is to be cleared of sediment by allowing it to drain through a conical filter that is 16 cm high and has a radius of 4 cm at the top (Figure 3.4.8). Suppose also that the liquid is forced out of the cone at a constant rate of 2 cm^3/min.

(a) Do you think that the depth of the liquid will decrease at a constant rate? Give a verbal argument that justifies your conclusion.

(b) Find a formula that expresses the rate at which the depth of the liquid is changing in terms of the depth, and use that formula to determine whether your conclusion in part (a) is correct.

(c) At what rate is the depth of the liquid changing at the instant when the liquid in the cone is 8 cm deep?

Solution (a). For the volume of liquid to decrease by a *fixed amount*, it requires a greater decrease in depth when the cone is close to empty than when it is almost full (Figure 3.4.9). This suggests that for the volume to decrease at a constant rate, the depth must decrease at an increasing rate.

Solution (b). Let

t = time elapsed from the initial observation (min)
V = volume of liquid in the cone at time t (cm^3)
y = depth of the liquid in the cone at time t (cm)
r = radius of the liquid surface at time t (cm)

(Figure 3.4.8). At each instant the rate at which the volume of liquid is changing is dV/dt, and the rate at which the depth is changing is dy/dt. We want to express dy/dt in terms of y given that dV/dt has a constant value of $dV/dt = -2$. (We must use a minus sign here because V *decreases* as t increases.)

From the formula for the volume of a cone, the volume V, the radius r, and the depth y are related by

$$V = \tfrac{1}{3}\pi r^2 y \tag{7}$$

If we differentiate both sides of (7) with respect to t, the right side will involve the quantity dr/dt. Since we have no direct information about dr/dt, it is desirable to eliminate r from (7) before differentiating. This can be done using similar triangles. From Figure 3.4.8 we see that

$$\frac{r}{y} = \frac{4}{16} \quad \text{or} \quad r = \frac{1}{4}y$$

Substituting this expression in (7) gives

$$V = \frac{\pi}{48}y^3 \tag{8}$$

Differentiating both sides of (8) with respect to t we obtain

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3y^2\frac{dy}{dt}\right)$$

or

$$\frac{dy}{dt} = \frac{16}{\pi y^2}\frac{dV}{dt} = \frac{16}{\pi y^2}(-2) = -\frac{32}{\pi y^2} \tag{9}$$

which expresses dy/dt in terms of y. The minus sign tells us that y is decreasing with time, and

$$\left|\frac{dy}{dt}\right| = \frac{32}{\pi y^2}$$

tells us how fast y is decreasing. From this formula we see that $|dy/dt|$ increases as y decreases, which confirms our conjecture in part (a) that the depth of the liquid decreases more quickly as the liquid drains through the filter.

Solution (c). The rate at which the depth is changing when the depth is 8 cm can be obtained from (9) with $y = 8$:

$$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{32}{\pi(8^2)} = -\frac{1}{2\pi} \approx -0.16 \text{ cm/min}$$ ◀

✔QUICK CHECK EXERCISES 3.4 (See page 211 for answers.)

1. If $A = x^2$ and $\dfrac{dx}{dt} = 3$, find $\left.\dfrac{dA}{dt}\right|_{x=10}$.

2. If $A = x^2$ and $\dfrac{dA}{dt} = 3$, find $\left.\dfrac{dx}{dt}\right|_{x=10}$.

3. A 10-foot ladder stands on a horizontal floor and leans against a vertical wall. Use x to denote the distance along the floor from the wall to the foot of the ladder, and use y to denote the distance along the wall from the floor to the top of the ladder. If the foot of the ladder is dragged away from the wall, find an equation that relates rates of change of x and y with respect to time.

4. Suppose that a block of ice in the shape of a right circular cylinder melts so that it retains its cylindrical shape. Find an equation that relates the rates of change of the volume (V), height (h), and radius (r) of the block of ice.

EXERCISE SET 3.4

1–4 Both x and y denote functions of t that are related by the given equation. Use this equation and the given derivative information to find the specified derivative. ■

1. Equation: $y = 3x + 5$.
(a) Given that $dx/dt = 2$, find dy/dt when $x = 1$.
(b) Given that $dy/dt = -1$, find dx/dt when $x = 0$.

2. Equation: $x + 4y = 3$.
(a) Given that $dx/dt = 1$, find dy/dt when $x = 2$.
(b) Given that $dy/dt = 4$, find dx/dt when $x = 3$.

3. Equation: $4x^2 + 9y^2 = 1$.
(a) Given that $dx/dt = 3$, find dy/dt when $(x, y) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$.

(cont.)

(b) Given that $dy/dt = 8$, find dx/dt when $(x, y) = \left(\frac{1}{3}, -\frac{\sqrt{5}}{9}\right)$.

4. Equation: $x^2 + y^2 = 2x + 4y$.
(a) Given that $dx/dt = -5$, find dy/dt when $(x, y) = (3, 1)$.
(b) Given that $dy/dt = 6$, find dx/dt when $(x, y) = (1 + \sqrt{2}, 2 + \sqrt{3})$.

FOCUS ON CONCEPTS

5. Let A be the area of a square whose sides have length x, and assume that x varies with the time t.
(a) Draw a picture of the square with the labels A and x placed appropriately.
(b) Write an equation that relates A and x.
(c) Use the equation in part (b) to find an equation that relates dA/dt and dx/dt.
(d) At a certain instant the sides are 3 ft long and increasing at a rate of 2 ft/min. How fast is the area increasing at that instant?

6. In parts (a)–(d), let A be the area of a circle of radius r, and assume that r increases with the time t.
(a) Draw a picture of the circle with the labels A and r placed appropriately.
(b) Write an equation that relates A and r.
(c) Use the equation in part (b) to find an equation that relates dA/dt and dr/dt.
(d) At a certain instant the radius is 5 cm and increasing at the rate of 2 cm/s. How fast is the area increasing at that instant?

7. Let V be the volume of a cylinder having height h and radius r, and assume that h and r vary with time.
(a) How are dV/dt, dh/dt, and dr/dt related?
(b) At a certain instant, the height is 6 in and increasing at 1 in/s, while the radius is 10 in and decreasing at 1 in/s. How fast is the volume changing at that instant? Is the volume increasing or decreasing at that instant?

8. Let l be the length of a diagonal of a rectangle whose sides have lengths x and y, and assume that x and y vary with time.
(a) How are dl/dt, dx/dt, and dy/dt related?
(b) If x increases at a constant rate of $\frac{1}{2}$ ft/s and y decreases at a constant rate of $\frac{1}{4}$ ft/s, how fast is the size of the diagonal changing when $x = 3$ ft and $y = 4$ ft? Is the diagonal increasing or decreasing at that instant?

9. Let θ (in radians) be an acute angle in a right triangle, and let x and y, respectively, be the lengths of the sides adjacent to and opposite θ. Suppose also that x and y vary with time.
(a) How are $d\theta/dt$, dx/dt, and dy/dt related?
(b) At a certain instant, $x = 2$ units and is increasing at 1 unit/s, while $y = 2$ units and is decreasing at $\frac{1}{4}$ unit/s. How fast is θ changing at that instant? Is θ increasing or decreasing at that instant?

10. Suppose that $z = x^3y^2$, where both x and y are changing with time. At a certain instant when $x = 1$ and $y = 2$, x is decreasing at the rate of 2 units/s, and y is increasing at the rate of 3 units/s. How fast is z changing at this instant? Is z increasing or decreasing?

11. The minute hand of a certain clock is 4 in long. Starting from the moment when the hand is pointing straight up, how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

12. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/s. How rapidly is the area enclosed by the ripple increasing at the end of 10 s?

13. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/h. How fast is the radius of the spill increasing when the area is 9 mi^2?

14. A spherical balloon is inflated so that its volume is increasing at the rate of 3 ft^3/min. How fast is the diameter of the balloon increasing when the radius is 1 ft?

15. A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 9 cm?

16. A 17 ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 5 ft/s, how fast will the top of the ladder be moving down the wall when it is 8 ft above the ground?

17. A 13 ft ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/s, how fast will the foot be moving away from the wall when the top is 5 ft above the ground?

18. A 10 ft plank is leaning against a wall. If at a certain instant the bottom of the plank is 2 ft from the wall and is being pushed toward the wall at the rate of 6 in/s, how fast is the acute angle that the plank makes with the ground increasing?

19. A softball diamond is a square whose sides are 60 ft long. Suppose that a player running from first to second base has a speed of 25 ft/s at the instant when she is 10 ft from second base. At what rate is the player's distance from home plate changing at that instant?

20. A rocket, rising vertically, is tracked by a radar station that is on the ground 5 mi from the launchpad. How fast is the rocket rising when it is 4 mi high and its distance from the radar station is increasing at a rate of 2000 mi/h?

21. For the camera and rocket shown in Figure 3.4.5, at what rate is the camera-to-rocket distance changing when the rocket is 4000 ft up and rising vertically at 880 ft/s?

22. For the camera and rocket shown in Figure 3.4.5, at what rate is the rocket rising when the elevation angle is $\pi/4$ radians and increasing at a rate of 0.2 rad/s?

23. A satellite is in an elliptical orbit around the Earth. Its distance r (in miles) from the center of the Earth is given by
$$r = \frac{4995}{1 + 0.12\cos\theta}$$
where θ is the angle measured from the point on the orbit nearest the Earth's surface (see the accompanying figure).
(a) Find the altitude of the satellite at ***perigee*** (the point nearest the surface of the Earth) and at ***apogee*** (the point farthest from the surface of the Earth). Use 3960 mi as the radius of the Earth.
(b) At the instant when θ is 120°, the angle θ is increasing at the rate of 2.7°/min. Find the altitude of the satellite and the rate at which the altitude is changing at this instant. Express the rate in units of mi/min.

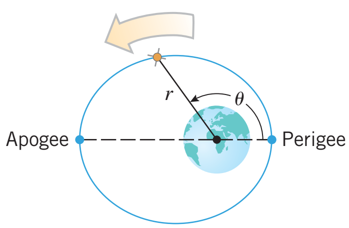


Figure Ex-23

24. An aircraft is flying horizontally at a constant height of 4000 ft above a fixed observation point (see the accompanying figure). At a certain instant the angle of elevation θ is 30° and decreasing, and the speed of the aircraft is 300 mi/h.
(a) How fast is θ decreasing at this instant? Express the result in units of deg/s.
(b) How fast is the distance between the aircraft and the observation point changing at this instant? Express the result in units of ft/s. Use 1 mi = 5280 ft.

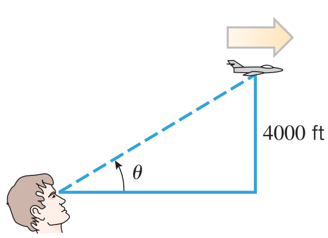


Figure Ex-24

25. A conical water tank with vertex down has a radius of 10 ft at the top and is 24 ft high. If water flows into the tank at a rate of 20 ft^3/min, how fast is the depth of the water increasing when the water is 16 ft deep?

26. Grain pouring from a chute at the rate of 8 ft^3/min forms a conical pile whose height is always twice its radius. How fast is the height of the pile increasing at the instant when the pile is 6 ft high?

27. Sand pouring from a chute forms a conical pile whose height is always equal to the diameter. If the height increases at a constant rate of 5 ft/min, at what rate is sand pouring from the chute when the pile is 10 ft high?

28. Wheat is poured through a chute at the rate of 10 ft^3/min and falls in a conical pile whose bottom radius is always half the altitude. How fast will the circumference of the base be increasing when the pile is 8 ft high?

29. An aircraft is climbing at a 30° angle to the horizontal. How fast is the aircraft gaining altitude if its speed is 500 mi/h?

30. A boat is pulled into a dock by means of a rope attached to a pulley on the dock (see the accompanying figure). The rope is attached to the bow of the boat at a point 10 ft below the pulley. If the rope is pulled through the pulley at a rate of 20 ft/min, at what rate will the boat be approaching the dock when 125 ft of rope is out?

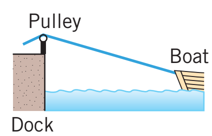


Figure Ex-30

31. For the boat in Exercise 30, how fast must the rope be pulled if we want the boat to approach the dock at a rate of 12 ft/min at the instant when 125 ft of rope is out?

32. A man 6 ft tall is walking at the rate of 3 ft/s toward a streetlight 18 ft high (see the accompanying figure).
(a) At what rate is his shadow length changing?
(b) How fast is the tip of his shadow moving?

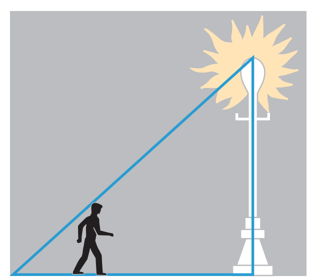


Figure Ex-32

33. A beacon that makes one revolution every 10 s is located on a ship anchored 4 kilometers from a straight shoreline. How fast is the beam moving along the shoreline when it makes an angle of 45° with the shore?

34. An aircraft is flying at a constant altitude with a constant speed of 600 mi/h. An antiaircraft missile is fired on a straight line perpendicular to the flight path of the aircraft so that it will hit the aircraft at a point P (see the accompanying figure). At the instant the aircraft is 2 mi from the impact point P the missile is 4 mi from P and flying at 1200 mi/h. At that instant, how rapidly is the distance between missile and aircraft decreasing?

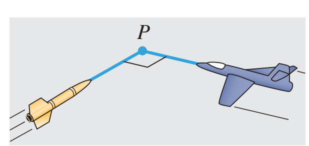


Figure Ex-34

35. Solve Exercise 34 under the assumption that the angle between the flight paths is 120° instead of the assumption that the paths are perpendicular. [*Hint:* Use the law of cosines.]

36. A police helicopter is flying due north at 100 mi/h and at a constant altitude of $\frac{1}{2}$ mi. Below, a car is traveling west on a highway at 75 mi/h. At the moment the helicopter crosses over the highway the car is 2 mi east of the helicopter.
(a) How fast is the distance between the car and helicopter changing at the moment the helicopter crosses the highway?
(b) Is the distance between the car and helicopter increasing or decreasing at that moment?

37. A particle is moving along the curve whose equation is
$$\frac{xy^3}{1+y^2} = \frac{8}{5}$$
Assume that the x-coordinate is increasing at the rate of 6 units/s when the particle is at the point $(1, 2)$.
(a) At what rate is the y-coordinate of the point changing at that instant?
(b) Is the particle rising or falling at that instant?

38. A point P is moving along the curve whose equation is $y = \sqrt{x^3 + 17}$. When P is at $(2, 5)$, y is increasing at the rate of 2 units/s. How fast is x changing?

39. A point P is moving along the line whose equation is $y = 2x$. How fast is the distance between P and the point $(3, 0)$ changing at the instant when P is at $(3, 6)$ if x is decreasing at the rate of 2 units/s at that instant?

40. A point P is moving along the curve whose equation is $y = \sqrt{x}$. Suppose that x is increasing at the rate of 4 units/s when $x = 3$.
(a) How fast is the distance between P and the point $(2, 0)$ changing at this instant?
(b) How fast is the angle of inclination of the line segment from P to $(2, 0)$ changing at this instant?

41. A particle is moving along the curve $y = x/(x^2 + 1)$. Find all values of x at which the rate of change of x with respect to time is three times that of y. [Assume that dx/dt is never zero.]

42. A particle is moving along the curve $16x^2 + 9y^2 = 144$. Find all points (x, y) at which the rates of change of x and y with respect to time are equal. [Assume that dx/dt and dy/dt are never both zero at the same point.]

43. The ***thin lens equation*** in physics is
$$\frac{1}{s} + \frac{1}{S} = \frac{1}{f}$$
where s is the object distance from the lens, S is the image distance from the lens, and f is the focal length of the lens. Suppose that a certain lens has a focal length of 6 cm and that an object is moving toward the lens at the rate of 2 cm/s. How fast is the image distance changing at the instant when the object is 10 cm from the lens? Is the image moving away from the lens or toward the lens?

44. Water is stored in a cone-shaped reservoir (vertex down). Assuming the water evaporates at a rate proportional to the surface area exposed to the air, show that the depth of the water will decrease at a constant rate that does not depend on the dimensions of the reservoir.

45. A meteor enters the Earth's atmosphere and burns up at a rate that, at each instant, is proportional to its surface area. Assuming that the meteor is always spherical, show that the radius decreases at a constant rate.

46. On a certain clock the minute hand is 4 in long and the hour hand is 3 in long. How fast is the distance between the tips of the hands changing at 9 o'clock?

47. Coffee is poured at a uniform rate of 20 cm^3/s into a cup whose inside is shaped like a truncated cone (see the accompanying figure). If the upper and lower radii of the cup are 4 cm and 2 cm and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up? [*Hint:* Extend the cup downward to form a cone.]

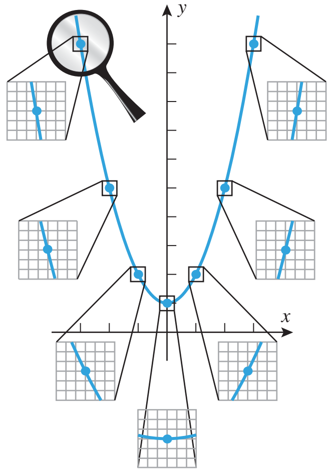
◀ **Figure Ex-47**

✔ QUICK CHECK ANSWERS 3.4

1. 60 **2.** $\frac{3}{20}$ **3.** $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$ **4.** $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$

3.5 LOCAL LINEAR APPROXIMATION; DIFFERENTIALS

In this section we will show how derivatives can be used to approximate nonlinear functions by linear functions. Also, up to now we have been interpreting dy/dx as a single entity representing the derivative. In this section we will define the quantities dx and dy themselves, thereby allowing us to interpret dy/dx as an actual ratio.



▲ **Figure 3.5.1**

Recall from Section 2.2 that if a function f is differentiable at x_0, then a sufficiently magnified portion of the graph of f centered at the point $P(x_0, f(x_0))$ takes on the appearance of a straight line segment. Figure 3.5.1 illustrates this at several points on the graph of $y = x^2 + 1$. For this reason, a function that is differentiable at x_0 is sometimes said to be ***locally linear*** at x_0.

The line that best approximates the graph of f in the vicinity of $P(x_0, f(x_0))$ is the tangent line to the graph of f at x_0, given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

[see Formula (3) of Section 2.2]. Thus, for values of x near x_0 we can approximate values of $f(x)$ by

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{1}$$

This is called the ***local linear approximation*** of f at x_0. This formula can also be expressed in terms of the increment $\Delta x = x - x_0$ as

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x \tag{2}$$

▶ Example 1

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$, and compare your approximation to the result produced directly by a calculating utility.

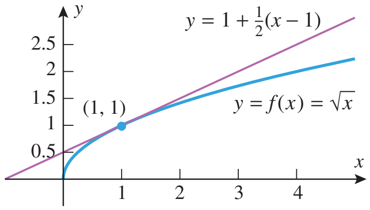
Solution (a). Since $f'(x) = 1/(2\sqrt{x})$, it follows from (1) that the local linear approximation of $\sqrt{x}$ at a point x_0 is

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0)$$

Thus, the local linear approximation at $x_0 = 1$ is

$$\sqrt{x} \approx 1 + \tfrac{1}{2}(x - 1) \tag{3}$$

The graphs of $y = \sqrt{x}$ and the local linear approximation $y = 1 + \frac{1}{2}(x - 1)$ are shown in Figure 3.5.2.



▲ **Figure 3.5.2**

Solution (b). Applying (3) with $x = 1.1$ yields

$$\sqrt{1.1} \approx 1 + \tfrac{1}{2}(1.1 - 1) = 1.05$$

Since the tangent line $y = 1 + \frac{1}{2}(x - 1)$ in Figure 3.5.2 lies above the graph of $f(x) = \sqrt{x}$, we would expect this approximation to be slightly too large. This expectation is confirmed by the calculator approximation $\sqrt{1.1} \approx 1.04881$. ◀

Examples 1 and 2 illustrate important ideas and are not meant to suggest that you should use local linear approximations for computations that your calculating utility can perform. The main application of local linear approximation is in modeling problems where it is useful to replace complicated functions by simpler ones.

Example 2

(a) Find the local linear approximation of $f(x) = \sin x$ at $x_0 = 0$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sin 2°$, and compare your approximation to the result produced directly by your calculating device.

Solution (a). Since $f'(x) = \cos x$, it follows from (1) that the local linear approximation of $\sin x$ at a point x_0 is

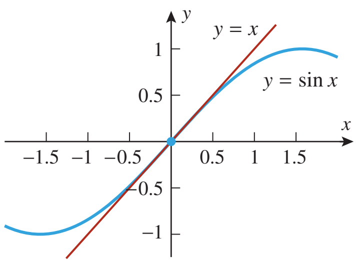
$$\sin x \approx \sin x_0 + (\cos x_0)(x - x_0)$$

Thus, the local linear approximation at $x_0 = 0$ is

$$\sin x \approx \sin 0 + (\cos 0)(x - 0)$$

which simplifies to

$$\sin x \approx x \tag{4}$$



▲ **Figure 3.5.3**

Solution (b). The variable x in (4) is in radian measure, so we must first convert 2° to radians before we can apply this approximation. Since

$$2° = 2\left(\frac{\pi}{180}\right) = \frac{\pi}{90} \approx 0.0349066 \text{ radian}$$

it follows from (4) that $\sin 2° \approx 0.0349066$. Comparing the two graphs in Figure 3.5.3, we would expect this approximation to be slightly larger than the exact value. The calculator approximation $\sin 2° \approx 0.0348995$ shows that this is indeed the case. ◀

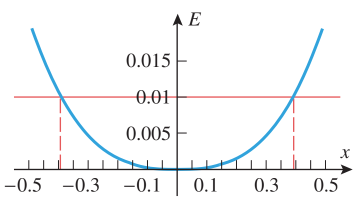
ERROR IN LOCAL LINEAR APPROXIMATIONS

As a general rule, the accuracy of the local linear approximation to $f(x)$ at x_0 will deteriorate as x gets progressively farther from x_0. To illustrate this for the approximation $\sin x \approx x$ in Example 2, let us graph the function

$$E(x) = |\sin x - x|$$

which is the absolute value of the error in the approximation (Figure 3.5.4).

In Figure 3.5.4, the graph shows how the absolute error in the local linear approximation of $\sin x$ increases as x moves progressively farther from 0 in either the positive or negative direction. The graph also tells us that for values of x between the two vertical lines, the absolute error does not exceed 0.01. Thus, for example, we could use the local linear approximation $\sin x \approx x$ for all values of x in the interval $-0.35 < x < 0.35$ (radians) with confidence that the approximation is within ± 0.01 of the exact value.



▲ **Figure 3.5.4**

DIFFERENTIALS

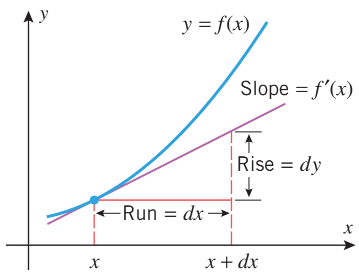
Newton and Leibniz each used a different notation when they published their discoveries of calculus, thereby creating a notational divide between Britain and the European continent that lasted for more than 50 years. The ***Leibniz notation*** dy/dx eventually prevailed because it suggests correct formulas in a natural way, the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

being a good example.

Up to now we have interpreted dy/dx as a single entity representing the derivative of y with respect to x; the symbols "dy" and "dx," which are called ***differentials***, have had no meanings attached to them. Our next goal is to define these symbols in such a way that dy/dx can be treated as an actual ratio. To do this, assume that f is differentiable at a point x, *define* dx to be an independent variable that can have any real value, and *define* dy by the formula

$$dy = f'(x)\,dx \tag{5}$$



▲ **Figure 3.5.5**

If $dx \neq 0$, then we can divide both sides of (5) by dx to obtain

$$\frac{dy}{dx} = f'(x) \tag{6}$$

Thus, we have achieved our goal of defining dy and dx so their ratio is $f'(x)$. Formula (5) is said to express (6) in ***differential form***.

To interpret (5) geometrically, note that $f'(x)$ is the slope of the tangent line to the graph of f at x. The differentials dy and dx can be viewed as a corresponding rise and run of this tangent line (Figure 3.5.5).

▶ Example 3 Express the derivative with respect to x of $y = x^2$ in differential form, and discuss the relationship between dy and dx at $x = 1$.

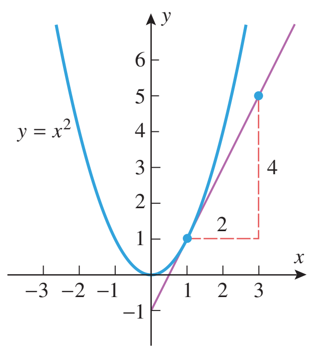
Solution. The derivative of y with respect to x is $dy/dx = 2x$, which can be expressed in differential form as

$$dy = 2x\,dx$$

When $x = 1$ this becomes

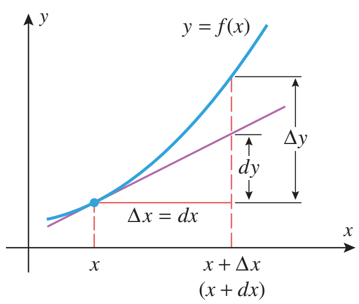
$$dy = 2\,dx$$

This tells us that if we travel along the tangent line to the curve $y = x^2$ at $x = 1$, then a change of dx units in x produces a change of $2\,dx$ units in y. Thus, for example, a run of $dx = 2$ units produces a rise of $dy = 4$ units along the tangent line (Figure 3.5.6). ◀



▲ **Figure 3.5.6**

It is important to understand the distinction between the increment Δy and the differential dy. To see the difference, let us assign the independent variables dx and Δx the same value, so $dx = \Delta x$. Then Δy represents the change in y that occurs when we start at x and travel *along the curve* $y = f(x)$ until we have moved Δx $(= dx)$ units in the x-direction, while dy represents the change in y that occurs if we start at x and travel *along the tangent line* until we have moved dx $(= \Delta x)$ units in the x-direction (Figure 3.5.7).



▲ **Figure 3.5.7**

▶ Example 4 Let $y = \sqrt{x}$. Find dy and Δy at $x = 4$ with $dx = \Delta x = 3$. Then make a sketch of $y = \sqrt{x}$, showing dy and Δy in the picture.

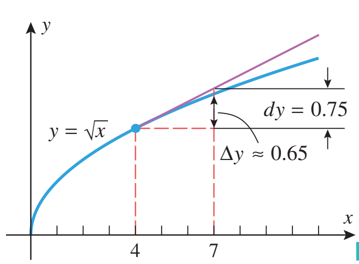
Solution. With $f(x) = \sqrt{x}$ we obtain

$$\Delta y = f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{7} - \sqrt{4} \approx 0.65$$

If $y = \sqrt{x}$, then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad dy = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4} = 0.75$$

Figure 3.5.8 shows the curve $y = \sqrt{x}$ together with dy and Δy. ◀



▲ **Figure 3.5.8**

LOCAL LINEAR APPROXIMATION FROM THE DIFFERENTIAL POINT OF VIEW

Although Δy and dy are generally different, the differential dy will nonetheless be a good approximation of Δy provided $dx = \Delta x$ is close to 0. To see this, recall from Section 2.2 that

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

It follows that if Δx is close to 0, then we will have $f'(x) \approx \Delta y/\Delta x$ or, equivalently,

$$\Delta y \approx f'(x)\Delta x$$

If we agree to let $dx = \Delta x$, then we can rewrite this as

$$\Delta y \approx f'(x)\,dx = dy \tag{7}$$

In words, this states that for values of dx near zero the differential dy closely approximates the increment Δy (Figure 3.5.7). But this is to be expected since the graph of the tangent line at x is the local linear approximation of the graph of f.

ERROR PROPAGATION

Real-world measurements inevitably have small errors.

In real-world applications, small errors in measured quantities will invariably occur. These measurement errors are of importance in scientific research—all scientific measurements come with measurement errors included. For example, your height might be measured as 170 ± 0.5 cm, meaning that your exact height lies somewhere between 169.5 and 170.5 cm. Researchers often must use these inexactly measured quantities to compute other quantities, thereby *propagating* the errors from the measured quantities to the computed quantities. This phenomenon is called ***error propagation***. Researchers must be able to estimate errors in the computed quantities. Our goal is to show how to estimate these errors using local linear approximation and differentials. For this purpose, suppose

x_0 is the exact value of the quantity being measured
$y_0 = f(x_0)$ is the exact value of the quantity being computed
x is the measured value of x_0
$y = f(x)$ is the computed value of y

We define

$dx\ (= \Delta x) = x - x_0$ to be the ***measurement error*** of x
$\Delta y = f(x) - f(x_0)$ to be the ***propagated error*** of y

It follows from (7) with x_0 replacing x that the propagated error Δy can be approximated by

$$\Delta y \approx dy = f'(x_0)\,dx \tag{8}$$

Unfortunately, there is a practical difficulty in applying this formula since the value of x_0 is unknown. (Keep in mind that only the measured value x is known to the researcher.) This being the case, it is standard practice in research to use the measured value x in place of x_0 in (8) and use the approximation

$$\Delta y \approx dy = f'(x)\,dx \tag{9}$$

for the propagated error.

Note that measurement error is positive if the measured value is greater than the exact value and is negative if it is less than the exact value. The sign of the propagated error conveys similar information.

Explain why an error estimate of at most $\pm\frac{1}{32}$ inch is reasonable for a ruler that is calibrated in sixteenths of an inch.

▶ **Example 5** Suppose that the side of a square is measured with a ruler to be 10 inches with a measurement error of at most $\pm\frac{1}{32}$ in. Estimate the error in the computed area of the square.

Solution. Let x denote the exact length of a side and y the exact area so that $y = x^2$. It follows from (9) with $f(x) = x^2$ that if dx is the measurement error, then the propagated error Δy can be approximated as

$$\Delta y \approx dy = 2x\,dx$$

Substituting the measured value $x = 10$ into this equation yields

$$dy = 20\,dx \tag{10}$$

But to say that the measurement error is at most $\pm\frac{1}{32}$ means that

$$-\frac{1}{32} \le dx \le \frac{1}{32}$$

Multiplying these inequalities through by 20 and applying (10) yields

$$20\left(-\tfrac{1}{32}\right) \le dy \le 20\left(\tfrac{1}{32}\right) \quad \text{or equivalently} \quad -\tfrac{5}{8} \le dy \le \tfrac{5}{8}$$

Thus, the propagated error in the area is estimated to be within $\pm\frac{5}{8}$ in^2. ◀

If the true value of a quantity is q and a measurement or calculation produces an error Δq, then $\Delta q/q$ is called the ***relative error*** in the measurement or calculation; when expressed as a percentage, $\Delta q/q$ is called the ***percentage error***. As a practical matter, the true value q is usually unknown, so that the measured or calculated value of q is used instead; and the relative error is approximated by dq/q.

Example 6 The radius of a sphere is measured with a percentage error within $\pm 0.04\%$. Estimate the percentage error in the calculated volume of the sphere.

> Formula (11) tells us that, as a rule of thumb, the percentage error in the computed volume of a sphere is approximately 3 times the percentage error in the measured value of its radius. As a rule of thumb, how is the percentage error in the computed area of a square related to the percentage error in the measured value of a side?

Solution. The volume V of a sphere is $V = \frac{4}{3}\pi r^3$, so

$$\frac{dV}{dr} = 4\pi r^2$$

from which it follows that $dV = 4\pi r^2\,dr$. Thus, the relative error in V is approximately

$$\frac{dV}{V} = \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r} \tag{11}$$

We are given that the relative error in the measured value of r is $\pm 0.04\%$, which means that

$$-0.0004 \le \frac{dr}{r} \le 0.0004$$

Multiplying these inequalities through by 3 and applying (11) yields

$$3(-0.0004) \le \frac{dV}{V} \le 3(0.0004) \quad \text{or equivalently} \quad -0.0012 \le \frac{dV}{V} \le 0.0012$$

Thus, we estimate the percentage error in the calculated value of V to be within $\pm 0.12\%$. ◀

MORE NOTATION; DIFFERENTIAL FORMULAS

The symbol df is another common notation for the differential of a function $y = f(x)$. For example, if $f(x) = \sin x$, then we can write $df = \cos x\,dx$. We can also view the symbol "d" as an *operator* that acts on a function to produce the corresponding differential. For example, $d[x^2] = 2x\,dx$, $d[\sin x] = \cos x\,dx$, and so on. All of the general rules of differentiation then have corresponding differential versions:

DERIVATIVE FORMULA	DIFFERENTIAL FORMULA
$\frac{d}{dx}[c] = 0$	$d[c] = 0$
$\frac{d}{dx}[cf] = c\frac{df}{dx}$	$d[cf] = c\,df$
$\frac{d}{dx}[f+g] = \frac{df}{dx} + \frac{dg}{dx}$	$d[f+g] = df + dg$
$\frac{d}{dx}[fg] = f\frac{dg}{dx} + g\frac{df}{dx}$	$d[fg] = f\,dg + g\,df$
$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$	$d\left[\frac{f}{g}\right] = \frac{g\,df - f\,dg}{g^2}$

For example,

$$\begin{aligned} d[x^2\sin x] &= (x^2\cos x + 2x\sin x)\,dx \\ &= x^2(\cos x\,dx) + (2x\,dx)\sin x \\ &= x^2 d[\sin x] + (\sin x)\,d[x^2] \end{aligned}$$

illustrates the differential version of the product rule.

✔ QUICK CHECK EXERCISES 3.5 (See page 219 for answers.)

1. The local linear approximation of f at x_0 uses the ________ line to the graph of $y = f(x)$ at $x = x_0$ to approximate values of ________ for values of x near ________.

2. Find an equation for the local linear approximation to $y = 5 - x^2$ at $x_0 = 2$.

3. Let $y = 5 - x^2$. Find dy and Δy at $x = 2$ with $dx = \Delta x = 0.1$.

4. The intensity of light from a light source is a function $I = f(x)$ of the distance x from the light source. Suppose that a small gemstone is measured to be 10 m from a light source, $f(10) = 0.2\ \text{W/m}^2$, and $f'(10) = -0.04\ \text{W/m}^3$. If the distance $x = 10$ m was obtained with a measurement error within ± 0.05 m, estimate the percentage error in the calculated intensity of the light on the gemstone.

EXERCISE SET 3.5 Graphing Utility

1. (a) Use Formula (1) to obtain the local linear approximation of x^3 at $x_0 = 1$.
   (b) Use Formula (2) to rewrite the approximation obtained in part (a) in terms of Δx.
   (c) Use the result obtained in part (a) to approximate $(1.02)^3$, and confirm that the formula obtained in part (b) produces the same result.

2. (a) Use Formula (1) to obtain the local linear approximation of $1/x$ at $x_0 = 2$.
   (b) Use Formula (2) to rewrite the approximation obtained in part (a) in terms of Δx.
   (c) Use the result obtained in part (a) to approximate $1/2.05$, and confirm that the formula obtained in part (b) produces the same result.

FOCUS ON CONCEPTS

3. (a) Find the local linear approximation of the function $f(x) = \sqrt{1+x}$ at $x_0 = 0$, and use it to approximate $\sqrt{0.9}$ and $\sqrt{1.1}$.
   (b) Graph f and its tangent line at x_0 together, and use the graphs to illustrate the relationship between the exact values and the approximations of $\sqrt{0.9}$ and $\sqrt{1.1}$.

4. A student claims that whenever a local linear approximation is used to approximate the square root of a number, the approximation is too large.
   (a) Write a few sentences that make the student's claim precise, and justify this claim geometrically.
   (b) Verify the student's claim algebraically using approximation (1).

5–10 Confirm that the stated formula is the local linear approximation at $x_0 = 0$. ■

5. $(1+x)^{15} \approx 1 + 15x$

6. $\dfrac{1}{\sqrt{1-x}} \approx 1 + \dfrac{1}{2}x$

7. $\tan x \approx x$

8. $\dfrac{1}{1+x} \approx 1 - x$

9. $e^x \approx 1 + x$

10. $\ln(1+x) \approx x$

11–16 Confirm that the stated formula is the local linear approximation of f at $x_0 = 1$, where $\Delta x = x - 1$. ■

11. $f(x) = x^4;\ (1+\Delta x)^4 \approx 1 + 4\Delta x$

12. $f(x) = \sqrt{x};\ \sqrt{1+\Delta x} \approx 1 + \frac{1}{2}\Delta x$

13. $f(x) = \dfrac{1}{2+x};\ \dfrac{1}{3+\Delta x} \approx \dfrac{1}{3} - \dfrac{1}{9}\Delta x$

14. $f(x) = (4+x)^3;\ (5+\Delta x)^3 \approx 125 + 75\Delta x$

15. $\tan^{-1} x;\ \tan^{-1}(1+\Delta x) \approx \dfrac{\pi}{4} + \dfrac{1}{2}\Delta x$

16. $\sin^{-1}\left(\dfrac{x}{2}\right);\ \sin^{-1}\left(\dfrac{1}{2} + \dfrac{1}{2}\Delta x\right) \approx \dfrac{\pi}{6} + \dfrac{1}{\sqrt{3}}\Delta x$

17–20 Confirm that the formula is the local linear approximation at $x_0 = 0$, and use a graphing utility to estimate an interval of x-values on which the error is at most ± 0.1. ■

17. $\sqrt{x+3} \approx \sqrt{3} + \dfrac{1}{2\sqrt{3}}x$

18. $\dfrac{1}{\sqrt{9-x}} \approx \dfrac{1}{3} + \dfrac{1}{54}x$

19. $\tan 2x \approx 2x$

20. $\dfrac{1}{(1+2x)^5} \approx 1 - 10x$

21. (a) Use the local linear approximation of $\sin x$ at $x_0 = 0$ obtained in Example 2 to approximate $\sin 1^\circ$, and compare the approximation to the result produced directly by your calculating device.
    (b) How would you choose x_0 to approximate $\sin 44^\circ$?
    (c) Approximate $\sin 44^\circ$; compare the approximation to the result produced directly by your calculating device.

22. (a) Use the local linear approximation of $\tan x$ at $x_0 = 0$ to approximate $\tan 2^\circ$, and compare the approximation to the result produced directly by your calculating device.
    (b) How would you choose x_0 to approximate $\tan 61^\circ$?
    (c) Approximate $\tan 61^\circ$; compare the approximation to the result produced directly by your calculating device.

23–31 Use an appropriate local linear approximation to estimate the value of the given quantity. ■

23. $(3.02)^4$

24. $(1.97)^3$

25. $\sqrt{65}$

26. $\sqrt{24}$ **27.** $\sqrt{80.9}$ **28.** $\sqrt{36.03}$

29. $\sin 0.1$ **30.** $\tan 0.2$ **31.** $\cos 31°$

32. $\ln(1.01)$ **33.** $\tan^{-1}(0.99)$

FOCUS ON CONCEPTS

34. The approximation $(1+x)^k \approx 1+kx$ is commonly used by engineers for quick calculations.
(a) Derive this result, and use it to make a rough estimate of $(1.001)^{37}$.
(b) Compare your estimate to that produced directly by your calculating device.
(c) If k is a positive integer, how is the approximation $(1+x)^k \approx 1+kx$ related to the expansion of $(1+x)^k$ using the binomial theorem?

35. Use the approximation $(1+x)^k \approx 1+kx$, along with some mental arithmetic to show that $\sqrt[3]{8.24} \approx 2.02$ and $4.08^{3/2} \approx 8.24$.

36. Referring to the accompanying figure, suppose that the angle of elevation of the top of the building, as measured from a point 500 ft from its base, is found to be $\theta = 6°$. Use an appropriate local linear approximation, along with some mental arithmetic to show that the building is about 52 ft high.

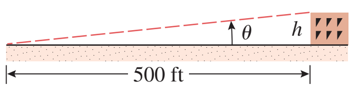


Figure Ex-36

37. (a) Let $y = x^2$. Find dy and Δy at $x = 2$ with $dx = \Delta x = 1$.
(b) Sketch the graph of $y = x^2$, showing dy and Δy in the picture.

38. (a) Let $y = x^3$. Find dy and Δy at $x = 1$ with $dx = \Delta x = 1$.
(b) Sketch the graph of $y = x^3$, showing dy and Δy in the picture.

39–42 Find formulas for dy and Δy.

39. $y = x^3$ **40.** $y = 8x - 4$

41. $y = x^2 - 2x + 1$ **42.** $y = \sin x$

43–46 Find the differential dy.

43. (a) $y = 4x^3 - 7x^2$ (b) $y = x\cos x$

44. (a) $y = 1/x$ (b) $y = 5\tan x$

45. (a) $y = x\sqrt{1-x}$ (b) $y = (1+x)^{-17}$

46. (a) $y = \dfrac{1}{x^3 - 1}$ (b) $y = \dfrac{1-x^3}{2-x}$

47–50 True–False Determine whether the statement is true or false. Explain your answer.

47. A differential dy is defined to be a very small change in y.

48. The error in approximation (2) is the same as the error in approximation (7).

49. A local linear approximation to a function can never be identically equal to the function.

50. A local linear approximation to a nonconstant function can never be constant.

51–54 Use the differential dy to approximate Δy when x changes as indicated.

51. $y = \sqrt{3x-2}$; from $x = 2$ to $x = 2.03$

52. $y = \sqrt{x^2+8}$; from $x = 1$ to $x = 0.97$

53. $y = \dfrac{x}{x^2+1}$; from $x = 2$ to $x = 1.96$

54. $y = x\sqrt{8x+1}$; from $x = 3$ to $x = 3.05$

55. The side of a square is measured to be 10 ft, with a possible error of ± 0.1 ft.
(a) Use differentials to estimate the error in the calculated area.
(b) Estimate the percentage errors in the side and the area.

56. The side of a cube is measured to be 25 cm, with a possible error of ± 1 cm.
(a) Use differentials to estimate the error in the calculated volume.
(b) Estimate the percentage errors in the side and volume.

57. The hypotenuse of a right triangle is known to be 10 in exactly, and one of the acute angles is measured to be 30°, with a possible error of $\pm 1°$.
(a) Use differentials to estimate the errors in the sides opposite and adjacent to the measured angle.
(b) Estimate the percentage errors in the sides.

58. One side of a right triangle is known to be 25 cm exactly. The angle opposite to this side is measured to be 60°, with a possible error of $\pm 0.5°$.
(a) Use differentials to estimate the errors in the adjacent side and the hypotenuse.
(b) Estimate the percentage errors in the adjacent side and hypotenuse.

59. The electrical resistance R of a certain wire is given by $R = k/r^2$, where k is a constant and r is the radius of the wire. Assuming that the radius r has a possible error of $\pm 5\%$, use differentials to estimate the percentage error in R. (Assume k is exact.)

60. A 12-foot ladder leaning against a wall makes an angle θ with the floor. If the top of the ladder is h feet up the wall, express h in terms of θ and then use dh to estimate the change in h if θ changes from 60° to 59°.

61. The area of a right triangle with a hypotenuse of H is calculated using the formula $A = \frac{1}{4}H^2 \sin 2\theta$, where θ is one of the acute angles. Use differentials to approximate the error in calculating A if $H = 4$ cm (exactly) and θ is measured to be 30°, with a possible error of $\pm 15'$.

62. The side of a square is measured with a possible percentage error of $\pm 1\%$. Use differentials to estimate the percentage error in the area.

63. The side of a cube is measured with a possible percentage error of $\pm 2\%$. Use differentials to estimate the percentage error in the volume.

64. The volume of a sphere is to be computed from a measured value of its radius. Estimate the maximum permissible percentage error in the measurement if the percentage error in the volume must be kept within $\pm 3\%$. ($V = \frac{4}{3}\pi r^3$ is the volume of a sphere of radius r.)

65. The area of a circle is to be computed from a measured value of its diameter. Estimate the maximum permissible percentage error in the measurement if the percentage error in the area must be kept within $\pm 1\%$.

66. A steel cube with 1-inch sides is coated with 0.01 inch of copper. Use differentials to estimate the volume of copper in the coating. [*Hint:* Let ΔV be the change in the volume of the cube.]

67. A metal rod 15 cm long and 5 cm in diameter is to be covered (except for the ends) with insulation that is 0.1 cm thick. Use differentials to estimate the volume of insulation. [*Hint:* Let ΔV be the change in volume of the rod.]

68. The time required for one complete oscillation of a pendulum is called its ***period***. If L is the length of the pendulum and the oscillation is small, then the period is given by $P = 2\pi\sqrt{L/g}$, where g is the constant acceleration due to gravity. Use differentials to show that the percentage error in P is approximately half the percentage error in L.

69. If the temperature T of a metal rod of length L is changed by an amount ΔT, then the length will change by the amount $\Delta L = \alpha L \Delta T$, where α is called the ***coefficient of linear expansion***. For moderate changes in temperature α is taken as constant.

(a) Suppose that a rod 40 cm long at 20°C is found to be 40.006 cm long when the temperature is raised to 30°C. Find α.

(b) If an aluminum pole is 180 cm long at 15°C, how long is the pole if the temperature is raised to 40°C? [Take $\alpha = 2.3 \times 10^{-5}/°\text{C}$.]

70. If the temperature T of a solid or liquid of volume V is changed by an amount ΔT, then the volume will change by the amount $\Delta V = \beta V \Delta T$, where β is called the ***coefficient of volume expansion***. For moderate changes in temperature β is taken as constant. Suppose that a tank truck loads 4000 gallons of ethyl alcohol at a temperature of 35°C and delivers its load sometime later at a temperature of 15°C. Using $\beta = 7.5 \times 10^{-4}/°\text{C}$ for ethyl alcohol, find the number of gallons delivered.

71. **Writing** Explain why the local linear approximation of a function value is equivalent to the use of a differential to approximate a change in the function.

72. **Writing** The local linear approximation

$$\sin x \approx x$$

is known as the *small angle approximation* and has both practical and theoretical applications. Do some research on some of these applications, and write a short report on the results of your investigations.

✔ QUICK CHECK ANSWERS 3.5

1. tangent; $f(x)$; x_0 **2.** $y = 1 + (-4)(x - 2)$ or $y = -4x + 9$ **3.** $dy = -0.4$, $\Delta y = -0.41$ **4.** within $\pm 1\%$

3.6 L'HÔPITAL'S RULE; INDETERMINATE FORMS

In this section we will discuss a general method for using derivatives to find limits. This method will enable us to establish limits with certainty that earlier in the text we were only able to conjecture using numerical or graphical evidence. The method that we will discuss in this section is an extremely powerful tool that is used internally by many computer programs to calculate limits of various types.

INDETERMINATE FORMS OF TYPE 0/0

Recall that a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)} \tag{1}$$

in which $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ is called an ***indeterminate form of type 0/0***. Some examples encountered earlier in the text are

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2, \quad \lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

The first limit was obtained algebraically by factoring the numerator and canceling the common factor of $x - 1$, and the second two limits were obtained using geometric methods. However, there are many indeterminate forms for which neither algebraic nor geometric methods will produce the limit, so we need to develop a more general method.

To motivate such a method, suppose that (1) is an indeterminate form of type $0/0$ in which f' and g' are continuous at $x = a$ and $g'(a) \neq 0$. Since f and g can be closely approximated by their local linear approximations near a, it is reasonable to expect that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} \tag{2}$$

Since we are assuming that f' and g' are continuous at $x = a$, we have

$$\lim_{x \to a} f'(x) = f'(a) \quad \text{and} \quad \lim_{x \to a} g'(x) = g'(a)$$

and since the differentiability of f and g at $x = a$ implies the continuity of f and g at $x = a$, we have

$$f(a) = \lim_{x \to a} f(x) = 0 \quad \text{and} \quad g(a) = \lim_{x \to a} g(x) = 0$$

Thus, we can rewrite (2) as

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \lim_{x \to a} \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{3}$$

This result, called ***L'Hôpital's rule***, converts the given indeterminate form into a limit involving derivatives that is often easier to evaluate.

Although we motivated (3) by assuming that f and g have continuous derivatives at $x = a$ and that $g'(a) \neq 0$, the result is true under less stringent conditions and is also valid for one-sided limits and limits at $+\infty$ and $-\infty$. The proof of the following precise statement of L'Hôpital's rule is omitted.

3.6.1 THEOREM (*L'Hôpital's Rule for Form* 0/0) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

WARNING

Note that in L'Hôpital's rule the numerator and denominator are differentiated individually. This is *not* the same as differentiating $f(x)/g(x)$.

In the examples that follow we will apply L'Hôpital's rule using the following three-step process:

Applying L'Hôpital's Rule

Step 1. Check that the limit of $f(x)/g(x)$ is an indeterminate form of type $0/0$.

Step 2. Differentiate f and g separately.

Step 3. Find the limit of $f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

▶ **Example 1** Find the limit

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

using L'Hôpital's rule, and check the result by factoring.

Solution. The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{\dfrac{d}{dx}[x^2 - 4]}{\dfrac{d}{dx}[x - 2]} = \lim_{x \to 2} \frac{2x}{1} = 4$$

The limit in Example 1 can be interpreted as the limit form of a certain derivative. Use that derivative to evaluate the limit.

This agrees with the computation

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2}(x + 2) = 4 \blacktriangleleft$$

▶ **Example 2** In each part confirm that the limit is an indeterminate form of type 0/0, and evaluate it using L'Hôpital's rule.

(a) $\lim_{x \to 0} \dfrac{\sin 2x}{x}$ (b) $\lim_{x \to \pi/2} \dfrac{1 - \sin x}{\cos x}$ (c) $\lim_{x \to 0} \dfrac{e^x - 1}{x^3}$

(d) $\lim_{x \to 0^-} \dfrac{\tan x}{x^2}$ (e) $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$ (f) $\lim_{x \to +\infty} \dfrac{x^{-4/3}}{\sin(1/x)}$

Solution (a). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{\dfrac{d}{dx}[\sin 2x]}{\dfrac{d}{dx}[x]} = \lim_{x \to 0} \frac{2 \cos 2x}{1} = 2$$

Observe that this result agrees with that obtained by substitution in Example 4(b) of Section 1.6.

WARNING

Applying L'Hôpital's rule to limits that are not indeterminate forms can produce incorrect results. For example, the computation

$$\lim_{x \to 0} \frac{x + 6}{x + 2} = \lim_{x \to 0} \frac{\dfrac{d}{dx}[x + 6]}{\dfrac{d}{dx}[x + 2]} = \lim_{x \to 0} \frac{1}{1} = 1$$

is ***not valid***, since the limit is not an indeterminate form. The correct result is

$$\lim_{x \to 0} \frac{x + 6}{x + 2} = \frac{0 + 6}{0 + 2} = 3$$

Solution (b). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \to \pi/2} \frac{\dfrac{d}{dx}[1 - \sin x]}{\dfrac{d}{dx}[\cos x]} = \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0$$

Guillaume François Antoine de L'Hôpital (1661–1704) French mathematician. L'Hôpital, born to parents of the French high nobility, held the title of Marquis de Sainte-Mesme Comte d'Autrement. He showed mathematical talent quite early and at age 15 solved a difficult problem about cycloids posed by Pascal. As a young man he served briefly as a cavalry officer, but resigned because of nearsightedness. In his own time he gained fame as the author of the first textbook ever published on differential calculus, *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (1696). L'Hôpital's rule appeared for the first time in that book. Actually, L'Hôpital's rule and most of the material in the calculus text were due to John Bernoulli, who was L'Hôpital's teacher. L'Hôpital dropped his plans for a book on integral calculus when Leibniz informed him that he intended to write such a text. L'Hôpital was apparently generous and personable, and his many contacts with major mathematicians provided the vehicle for disseminating major discoveries in calculus throughout Europe.

Solution (c). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{\frac{d}{dx}[e^x-1]}{\frac{d}{dx}[x^3]}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$

Solution (d). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^-}\frac{\tan x}{x^2}=\lim_{x\to 0^-}\frac{\sec^2 x}{2x}=-\infty$$

Solution (e). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}$$

Since the new limit is another indeterminate form of type 0/0, we apply L'Hôpital's rule again:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

Solution (f). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}=\lim_{x\to +\infty}\frac{-\frac{4}{3}x^{-7/3}}{(-1/x^2)\cos(1/x)}=\lim_{x\to +\infty}\frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)}=\frac{0}{1}=0 \quad \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE ∞/∞

When we want to indicate that the limit (or a one-sided limit) of a function is $+\infty$ or $-\infty$ without being specific about the sign, we will say that the limit is ∞. For example,

$$\lim_{x\to a^+} f(x)=\infty \quad \text{means} \quad \lim_{x\to a^+} f(x)=+\infty \quad \text{or} \quad \lim_{x\to a^+} f(x)=-\infty$$

$$\lim_{x\to +\infty} f(x)=\infty \quad \text{means} \quad \lim_{x\to +\infty} f(x)=+\infty \quad \text{or} \quad \lim_{x\to +\infty} f(x)=-\infty$$

$$\lim_{x\to a} f(x)=\infty \quad \text{means} \quad \lim_{x\to a^+} f(x)=\pm\infty \quad \text{and} \quad \lim_{x\to a^-} f(x)=\pm\infty$$

The limit of a ratio, $f(x)/g(x)$, in which the numerator has limit ∞ and the denominator has limit ∞ is called an ***indeterminate form of type*** ∞/∞. The following version of L'Hôpital's rule, which we state without proof, can often be used to evaluate limits of this type.

3.6.2 THEOREM (*L'Hôpital's Rule for Form* ∞/∞) *Suppose that f and g are differentiable functions on an open interval containing $x=a$, except possibly at $x=a$, and that*

$$\lim_{x\to a} f(x)=\infty \quad \textit{and} \quad \lim_{x\to a} g(x)=\infty$$

If $\lim_{x\to a}[f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to -\infty$, or as $x\to +\infty$.

▶ **Example 3** In each part confirm that the limit is an indeterminate form of type ∞/∞ and apply L'Hôpital's rule.

$$\text{(a)}\quad \lim_{x\to+\infty}\frac{x}{e^x} \qquad \text{(b)}\quad \lim_{x\to0^+}\frac{\ln x}{\csc x}$$

Solution (a). The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to+\infty}\frac{x}{e^x} = \lim_{x\to+\infty}\frac{1}{e^x} = 0$$

Solution (b). The numerator has a limit of $-\infty$ and the denominator has a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to0^+}\frac{\ln x}{\csc x} = \lim_{x\to0^+}\frac{1/x}{-\csc x\cot x} \tag{4}$$

This last limit is again an indeterminate form of type ∞/∞. Moreover, any additional applications of L'Hôpital's rule will yield powers of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms. We must try something else. The last limit in (4) can be rewritten as

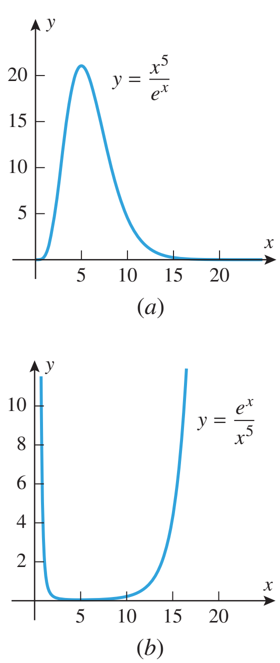
$$\lim_{x\to0^+}\left(-\frac{\sin x}{x}\tan x\right) = -\lim_{x\to0^+}\frac{\sin x}{x}\cdot\lim_{x\to0^+}\tan x = -(1)(0) = 0$$

Thus,

$$\lim_{x\to0^+}\frac{\ln x}{\csc x} = 0 \; ◀$$

ANALYZING THE GROWTH OF EXPONENTIAL FUNCTIONS USING L'HÔPITAL'S RULE

If n is any positive integer, then $x^n \to +\infty$ as $x \to +\infty$. Such integer powers of x are sometimes used as "measuring sticks" to describe how rapidly other functions grow. For example, we know that $e^x \to +\infty$ as $x \to +\infty$ and that the growth of e^x is very rapid (Table 0.5.5); however, the growth of x^n is also rapid when n is a high power, so it is reasonable to ask whether high powers of x grow more or less rapidly than e^x. One way to investigate this is to examine the behavior of the ratio x^n/e^x as $x \to +\infty$. For example, Figure 3.6.1a shows the graph of $y = x^5/e^x$. This graph suggests that $x^5/e^x \to 0$ as $x \to +\infty$, and this implies that the growth of the function e^x is sufficiently rapid that its values eventually overtake those of x^5 and force the ratio toward zero. Stated informally, "e^x eventually grows more rapidly than x^5." The same conclusion could have been reached by putting e^x on top and examining the behavior of e^x/x^5 as $x \to +\infty$ (Figure 3.6.1b). In this case the values of e^x eventually overtake those of x^5 and force the ratio toward $+\infty$. More generally, we can use L'Hôpital's rule to show that e^x *eventually grows more rapidly than any positive integer power of* x, that is,



▲ Figure 3.6.1

$$\lim_{x\to+\infty}\frac{x^n}{e^x} = 0 \quad \text{and} \quad \lim_{x\to+\infty}\frac{e^x}{x^n} = +\infty \tag{5–6}$$

Both limits are indeterminate forms of type ∞/∞ that can be evaluated using L'Hôpital's rule. For example, to establish (5), we will need to apply L'Hôpital's rule n times. For this purpose, observe that successive differentiations of x^n reduce the exponent by 1 each time, thus producing a constant for the nth derivative. For example, the successive derivatives of x^3 are $3x^2$, $6x$, and 6. In general, the nth derivative of x^n is $n(n-1)(n-2)\cdots 1 = n!$ (verify).* Thus, applying L'Hôpital's rule n times to (5) yields

$$\lim_{x\to+\infty}\frac{x^n}{e^x} = \lim_{x\to+\infty}\frac{n!}{e^x} = 0$$

Limit (6) can be established similarly.

INDETERMINATE FORMS OF TYPE $0 \cdot \infty$

Thus far we have discussed indeterminate forms of type $0/0$ and ∞/∞. However, these are not the only possibilities; in general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)},\quad f(x)\cdot g(x),\quad f(x)^{g(x)},\quad f(x)-g(x),\quad f(x)+g(x)$$

is called an *indeterminate form* if the limits of $f(x)$ and $g(x)$ individually exert conflicting influences on the limit of the entire expression. For example, the limit

$$\lim_{x\to 0^+} x\ln x$$

is an ***indeterminate form of type*** $\mathbf{0\cdot\infty}$ because the limit of the first factor is 0, the limit of the second factor is $-\infty$, and these two limits exert conflicting influences on the product. On the other hand, the limit

$$\lim_{x\to+\infty}[\sqrt{x}(1-x^2)]$$

is not an indeterminate form because the first factor has a limit of $+\infty$, the second factor has a limit of $-\infty$, and these influences work together to produce a limit of $-\infty$ for the product.

Indeterminate forms of type $0\cdot\infty$ can sometimes be evaluated by rewriting the product as a ratio, and then applying L'Hôpital's rule for indeterminate forms of type $0/0$ or ∞/∞.

WARNING

It is tempting to argue that an indeterminate form of type $0\cdot\infty$ has value 0 since "zero times anything is zero." However, this is fallacious since $0\cdot\infty$ is not a product of numbers, but rather a statement about limits. For example, here are two indeterminate forms of type $0\cdot\infty$ whose limits are *not* zero:

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x}\right) = \lim_{x\to 0} 1 = 1$$

$$\lim_{x\to 0^+}\left(\sqrt{x}\cdot\frac{1}{x}\right) = \lim_{x\to 0^+}\left(\frac{1}{\sqrt{x}}\right) = +\infty$$

▶ Example 4 Evaluate

$$\text{(a)}\ \lim_{x\to 0^+} x\ln x \qquad \text{(b)}\ \lim_{x\to\pi/4}(1-\tan x)\sec 2x$$

Solution (a). The factor x has a limit of 0 and the factor $\ln x$ has a limit of $-\infty$, so the stated problem is an indeterminate form of type $0\cdot\infty$. There are two possible approaches: we can rewrite the limit as

$$\lim_{x\to 0^+}\frac{\ln x}{1/x} \quad\text{or}\quad \lim_{x\to 0^+}\frac{x}{1/\ln x}$$

the first being an indeterminate form of type ∞/∞ and the second an indeterminate form of type $0/0$. However, the first form is the preferred initial choice because the derivative of $1/x$ is less complicated than the derivative of $1/\ln x$. That choice yields

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

Solution (b). The stated problem is an indeterminate form of type $0\cdot\infty$. We will convert it to an indeterminate form of type $0/0$:

$$\lim_{x\to\pi/4}(1-\tan x)\sec 2x = \lim_{x\to\pi/4}\frac{1-\tan x}{1/\sec 2x} = \lim_{x\to\pi/4}\frac{1-\tan x}{\cos 2x}$$

$$= \lim_{x\to\pi/4}\frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1 \ ◀$$

*Recall that for $n \geq 1$ the expression $n!$, read ***n-factorial***, denotes the product of the first n positive integers.

INDETERMINATE FORMS OF TYPE $\infty - \infty$

A limit problem that leads to one of the expressions

$$(+\infty) - (+\infty), \quad (-\infty) - (-\infty),$$
$$(+\infty) + (-\infty), \quad (-\infty) + (+\infty)$$

is called an ***indeterminate form of type*** $\boldsymbol{\infty - \infty}$. Such limits are indeterminate because the two terms exert conflicting influences on the expression: one pushes it in the positive direction and the other pushes it in the negative direction. However, limit problems that lead to one of the expressions

$$(+\infty) + (+\infty), \quad (+\infty) - (-\infty),$$
$$(-\infty) + (-\infty), \quad (-\infty) - (+\infty)$$

are not indeterminate, since the two terms work together (those on the top produce a limit of $+\infty$ and those on the bottom produce a limit of $-\infty$).

Indeterminate forms of type $\infty - \infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $0/0$ or ∞/∞.

▶ Example 5 Evaluate $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Solution. Both terms have a limit of $+\infty$, so the stated problem is an indeterminate form of type $\infty - \infty$. Combining the two terms yields

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+}\frac{\sin x - x}{x\sin x}$$

which is an indeterminate form of type $0/0$. Applying L'Hôpital's rule twice yields

$$\lim_{x\to 0^+}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0^+}\frac{\cos x - 1}{\sin x + x\cos x}$$
$$= \lim_{x\to 0^+}\frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0 \quad ◀$$

INDETERMINATE FORMS OF TYPE 0^0, ∞^0, 1^∞

Limits of the form

$$\lim f(x)^{g(x)}$$

can give rise to ***indeterminate forms of the types*** $\boldsymbol{0^0}$***,*** $\boldsymbol{\infty^0}$***, and*** $\boldsymbol{1^\infty}$. (The interpretations of these symbols should be clear.) For example, the limit

$$\lim_{x\to 0^+}(1+x)^{1/x}$$

whose value we know to be e [see Formula (1) of Section 3.2] is an indeterminate form of type 1^∞. It is indeterminate because the expressions $1 + x$ and $1/x$ exert two conflicting influences: the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$.

Indeterminate forms of types 0^0, ∞^0, and 1^∞ can sometimes be evaluated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then computing the limit of $\ln y$. Since

$$\ln y = \ln[f(x)^{g(x)}] = g(x)\cdot\ln[f(x)]$$

the limit of $\ln y$ will be an indeterminate form of type $0\cdot\infty$ (verify), which can be evaluated by methods we have already studied. Once the limit of $\ln y$ is known, it is a straightforward matter to determine the limit of $y = f(x)^{g(x)}$, as we will illustrate in the next example.

Example 6 Find $\lim_{x\to 0}(1+\sin x)^{1/x}$.

Solution. As discussed above, we begin by introducing a dependent variable

$$y=(1+\sin x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y=\ln(1+\sin x)^{1/x}=\frac{1}{x}\ln(1+\sin x)=\frac{\ln(1+\sin x)}{x}$$

Thus,

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(1+\sin x)}{x}$$

which is an indeterminate form of type $0/0$, so by L'Hôpital's rule

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(1+\sin x)}{x}=\lim_{x\to 0}\frac{(\cos x)/(1+\sin x)}{1}=1$$

Since we have shown that $\ln y\to 1$ as $x\to 0$, the continuity of the exponential function implies that $e^{\ln y}\to e^1$ as $x\to 0$, and this implies that $y\to e$ as $x\to 0$. Thus,

$$\lim_{x\to 0}(1+\sin x)^{1/x}=e$$ ◀

✔ QUICK CHECK EXERCISES 3.6 (See page 228 for answers.)

1. In each part, does L'Hôpital's rule apply to the given limit?
   (a) $\lim_{x\to 1}\frac{2x-2}{x^3+x-2}$ (b) $\lim_{x\to 0}\frac{\cos x}{x}$
   (c) $\lim_{x\to 0}\frac{e^{2x}-1}{\tan x}$

2. Evaluate each of the limits in Quick Check Exercise 1.

3. Using L'Hôpital's rule, $\lim_{x\to+\infty}\frac{e^x}{500x^2}=$ ________.

EXERCISE SET 3.6 Graphing Utility C CAS

1–2 Evaluate the given limit without using L'Hôpital's rule, and then check that your answer is correct using L'Hôpital's rule.

1. (a) $\lim_{x\to 2}\frac{x^2-4}{x^2+2x-8}$ (b) $\lim_{x\to+\infty}\frac{2x-5}{3x+7}$

2. (a) $\lim_{x\to 0}\frac{\sin x}{\tan x}$ (b) $\lim_{x\to 1}\frac{x^2-1}{x^3-1}$

3–6 True–False Determine whether the statement is true or false. Explain your answer.

3. L'Hôpital's rule does not apply to $\lim_{x\to-\infty}\frac{\ln x}{x}$.

4. For any polynomial $p(x)$, $\lim_{x\to+\infty}\frac{p(x)}{e^x}=0$.

5. If n is chosen sufficiently large, then $\lim_{x\to+\infty}\frac{(\ln x)^n}{x}=+\infty$.

6. $\lim_{x\to 0^+}(\sin x)^{1/x}=0$

7–45 Find the limits.

7. $\lim_{x\to 0}\frac{e^x-1}{\sin x}$

8. $\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}$

9. $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$

10. $\lim_{t\to 0}\frac{te^t}{1-e^t}$

11. $\lim_{x\to\pi^+}\frac{\sin x}{x-\pi}$

12. $\lim_{x\to 0^+}\frac{\sin x}{x^2}$

13. $\lim_{x\to+\infty}\frac{\ln x}{x}$

14. $\lim_{x\to+\infty}\frac{e^{3x}}{x^2}$

15. $\lim_{x\to 0^+}\frac{\cot x}{\ln x}$

16. $\lim_{x\to 0^+}\frac{1-\ln x}{e^{1/x}}$

17. $\lim_{x\to+\infty}\frac{x^{100}}{e^x}$

18. $\lim_{x\to 0^+}\frac{\ln(\sin x)}{\ln(\tan x)}$

19. $\lim_{x\to 0}\frac{\sin^{-1}2x}{x}$

20. $\lim_{x\to 0}\frac{x-\tan^{-1}x}{x^3}$

21. $\lim_{x\to+\infty}xe^{-x}$

22. $\lim_{x\to\pi^-}(x-\pi)\tan\frac{1}{2}x$

23. $\lim_{x\to+\infty}x\sin\frac{\pi}{x}$

24. $\lim_{x\to 0^+}\tan x\ln x$

25. $\lim_{x\to\pi/2^-}\sec 3x\cos 5x$

26. $\lim_{x\to\pi}(x-\pi)\cot x$

27. $\lim_{x\to+\infty}(1-3/x)^x$

28. $\lim_{x\to 0}(1+2x)^{-3/x}$

29. $\lim_{x\to 0}(e^x + x)^{1/x}$

30. $\lim_{x\to +\infty}(1 + a/x)^{bx}$

31. $\lim_{x\to 1}(2 - x)^{\tan[(\pi/2)x]}$

32. $\lim_{x\to +\infty}[\cos(2/x)]^{x^2}$

33. $\lim_{x\to 0}(\csc x - 1/x)$

34. $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{\cos 3x}{x^2}\right)$

35. $\lim_{x\to +\infty}(\sqrt{x^2 + x} - x)$

36. $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

37. $\lim_{x\to +\infty}[x - \ln(x^2 + 1)]$

38. $\lim_{x\to +\infty}[\ln x - \ln(1 + x)]$

39. $\lim_{x\to 0^+} x^{\sin x}$

40. $\lim_{x\to 0^+}(e^{2x} - 1)^x$

41. $\lim_{x\to 0^+}\left[-\frac{1}{\ln x}\right]^x$

42. $\lim_{x\to +\infty} x^{1/x}$

43. $\lim_{x\to +\infty}(\ln x)^{1/x}$

44. $\lim_{x\to 0^+}(-\ln x)^x$

45. $\lim_{x\to \pi/2^-}(\tan x)^{(\pi/2)-x}$

46. Show that for any positive integer n

(a) $\lim_{x\to +\infty}\frac{\ln x}{x^n} = 0$ (b) $\lim_{x\to +\infty}\frac{x^n}{\ln x} = +\infty.$

FOCUS ON CONCEPTS

47. (a) Find the error in the following calculation:

$$\lim_{x\to 1}\frac{x^3 - x^2 + x - 1}{x^3 - x^2} = \lim_{x\to 1}\frac{3x^2 - 2x + 1}{3x^2 - 2x} = \lim_{x\to 1}\frac{6x - 2}{6x - 2} = 1$$

(b) Find the correct limit.

48. (a) Find the error in the following calculation:

$$\lim_{x\to 2}\frac{e^{3x^2-12x+12}}{x^4 - 16} = \lim_{x\to 2}\frac{(6x - 12)e^{3x^2-12x+12}}{4x^3} = 0$$

(b) Find the correct limit.

49–52 Make a conjecture about the limit by graphing the function involved with a graphing utility; then check your conjecture using L'Hôpital's rule. ■

49. $\lim_{x\to +\infty}\frac{\ln(\ln x)}{\sqrt{x}}$

50. $\lim_{x\to 0^+} x^x$

51. $\lim_{x\to 0^+}(\sin x)^{3/\ln x}$

52. $\lim_{x\to (\pi/2)^-}\frac{4\tan x}{1 + \sec x}$

53–56 Make a conjecture about the equations of horizontal asymptotes, if any, by graphing the equation with a graphing utility; then check your answer using L'Hôpital's rule. ■

53. $y = \ln x - e^x$

54. $y = x - \ln(1 + 2e^x)$

55. $y = (\ln x)^{1/x}$

56. $y = \left(\frac{x + 1}{x + 2}\right)^x$

57. Limits of the type

$$0/\infty,\quad \infty/0,\quad 0^\infty,\quad \infty\cdot\infty,\quad +\infty + (+\infty),$$
$$+\infty - (-\infty),\quad -\infty + (-\infty),\quad -\infty - (+\infty)$$

are *not* indeterminate forms. Find the following limits by inspection.

(a) $\lim_{x\to 0^+}\frac{x}{\ln x}$ (b) $\lim_{x\to +\infty}\frac{x^3}{e^{-x}}$

(c) $\lim_{x\to (\pi/2)^-}(\cos x)^{\tan x}$ (d) $\lim_{x\to 0^+}(\ln x)\cot x$

(e) $\lim_{x\to 0^+}\left(\frac{1}{x} - \ln x\right)$ (f) $\lim_{x\to -\infty}(x + x^3)$

58. There is a myth that circulates among beginning calculus students which states that all indeterminate forms of types 0^0, ∞^0, and 1^∞ have value 1 because "anything to the zero power is 1" and "1 to any power is 1." The fallacy is that 0^0, ∞^0, and 1^∞ are not powers of numbers, but rather descriptions of limits. The following examples, which were suggested by Prof. Jack Staib of Drexel University, show that such indeterminate forms can have any positive real value:

(a) $\lim_{x\to 0^+}[x^{(\ln a)/(1+\ln x)}] = a$ (form 0^0)

(b) $\lim_{x\to +\infty}[x^{(\ln a)/(1+\ln x)}] = a$ (form ∞^0)

(c) $\lim_{x\to 0}[(x + 1)^{(\ln a)/x}] = a$ (form 1^∞).

Verify these results.

59–62 Verify that L'Hôpital's rule is of no help in finding the limit; then find the limit, if it exists, by some other method. ■

59. $\lim_{x\to +\infty}\frac{x + \sin 2x}{x}$

60. $\lim_{x\to +\infty}\frac{2x - \sin x}{3x + \sin x}$

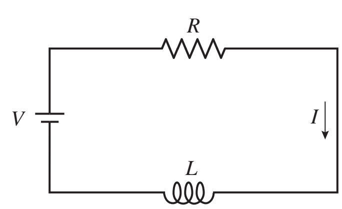
61. $\lim_{x\to +\infty}\frac{x(2 + \sin 2x)}{x + 1}$

62. $\lim_{x\to +\infty}\frac{x(2 + \sin x)}{x^2 + 1}$

63. The accompanying schematic diagram represents an electrical circuit consisting of an electromotive force that produces a voltage V, a resistor with resistance R, and an inductor with inductance L. It is shown in electrical circuit theory that if the voltage is first applied at time $t = 0$, then the current I flowing through the circuit at time t is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L})$$

What is the effect on the current at a fixed time t if the resistance approaches 0 (i.e., $R \to 0^+$)?



◀ **Figure Ex-63**

64. (a) Show that $\lim_{x\to \pi/2}(\pi/2 - x)\tan x = 1$.

(b) Show that

$$\lim_{x\to \pi/2}\left(\frac{1}{\pi/2 - x} - \tan x\right) = 0$$

(c) It follows from part (b) that the approximation

$$\tan x \approx \frac{1}{\pi/2 - x}$$

should be good for values of x near $\pi/2$. Use a calculator to find $\tan x$ and $1/(\pi/2 - x)$ for $x = 1.57$; compare the results.

C **65.** (a) Use a CAS to show that if k is a positive constant, then

$$\lim_{x\to+\infty} x(k^{1/x} - 1) = \ln k$$

(b) Confirm this result using L'Hôpital's rule. [*Hint:* Express the limit in terms of $t = 1/x$.]

(c) If n is a positive integer, then it follows from part (a) with $x = n$ that the approximation

$$n(\sqrt[n]{k} - 1) \approx \ln k$$

should be good when n is large. Use this result and the square root key on a calculator to approximate the values of $\ln 0.3$ and $\ln 2$ with $n = 1024$, then compare the values obtained with values of the logarithms generated directly from the calculator. [*Hint:* The nth roots for which n is a power of 2 can be obtained as successive square roots.]

66. Find all values of k and l such that

$$\lim_{x\to 0} \frac{k + \cos lx}{x^2} = -4$$

FOCUS ON CONCEPTS

67. Let $f(x) = x^2 \sin(1/x)$.

(a) Are the limits $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ indeterminate forms?

(b) Use a graphing utility to generate the graph of f, and use the graph to make conjectures about the limits in part (a).

(c) Use the Squeezing Theorem (1.6.4) to confirm that your conjectures in part (b) are correct.

68. (a) Explain why L'Hôpital's rule does not apply to the problem

$$\lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x}$$

(b) Find the limit.

69. Find $\displaystyle\lim_{x\to 0^+} \frac{x \sin(1/x)}{\sin x}$ if it exists.

70. Suppose that functions f and g are differentiable at $x = a$ and that $f(a) = g(a) = 0$. If $g'(a) \neq 0$, show that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

without using L'Hôpital's rule. [*Hint:* Divide the numerator and denominator of $f(x)/g(x)$ by $x - a$ and use the definitions for $f'(a)$ and $g'(a)$.]

71. Writing Were we to use L'Hôpital's rule to evaluate either

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{or} \quad \lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x$$

we could be accused of circular reasoning. Explain why.

72. Writing Exercise 58 shows that the indeterminate forms 0^0 and ∞^0 can assume any positive real value. However, it is often the case that these indeterminate forms have value 1. Read the article "Indeterminate Forms of Exponential Type" by John Baxley and Elmer Hayashi in the June–July 1978 issue of *The American Mathematical Monthly*, and write a short report on why this is the case.

✔ QUICK CHECK ANSWERS 3.6

1. (a) yes (b) no (c) yes **2.** (a) $\frac{1}{2}$ (b) does not exist (c) 2 **3.** $+\infty$

CHAPTER 3 REVIEW EXERCISES

Graphing Utility

1–2 (a) Find dy/dx by differentiating implicitly. (b) Solve the equation for y as a function of x, and find dy/dx from that equation. (c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone. ■

1. $x^3 + xy - 2x = 1$ **2.** $xy = x - y$

3–6 Find dy/dx by implicit differentiation. ■

3. $\dfrac{1}{y} + \dfrac{1}{x} = 1$ **4.** $x^3 - y^3 = 6xy$

5. $\sec(xy) = y$ **6.** $x^2 = \dfrac{\cot y}{1 + \csc y}$

7–8 Find d^2y/dx^2 by implicit differentiation. ■

7. $3x^2 - 4y^2 = 7$ **8.** $2xy - y^2 = 3$

9. Use implicit differentiation to find the slope of the tangent line to the curve $y = x\tan(\pi y/2)$, $x > 0$, $y > 0$ (***the quadratrix of Hippias***) at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

10. At what point(s) is the tangent line to the curve $y^2 = 2x^3$ perpendicular to the line $4x - 3y + 1 = 0$?

11. Prove that if P and Q are two distinct points on the rotated ellipse $x^2 + xy + y^2 = 4$ such that P, Q, and the origin are collinear, then the tangent lines to the ellipse at P and Q are parallel.

12. Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is parallel to the x-axis.

13. Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is parallel to the y-axis.

14. Use implicit differentiation to show that the equation of the tangent line to the curve $y^2 = kx$ at (x_0, y_0) is

$$y_0 y = \tfrac{1}{2}k(x + x_0)$$

15–16 Find dy/dx by first using algebraic properties of the natural logarithm function. ■

15. $y = \ln\left(\dfrac{(x+1)(x+2)^2}{(x+3)^3(x+4)^4}\right)$ **16.** $y = \ln\left(\dfrac{\sqrt{x}\sqrt[3]{x+1}}{\sin x \sec x}\right)$

17–34 Find dy/dx. ■

17. $y = \ln 2x$ **18.** $y = (\ln x)^2$

19. $y = \sqrt[3]{\ln x + 1}$ **20.** $y = \ln(\sqrt[3]{x+1})$

21. $y = \log(\ln x)$ **22.** $y = \dfrac{1 + \log x}{1 - \log x}$

23. $y = \ln(x^{3/2}\sqrt{1+x^4})$ **24.** $y = \ln\left(\dfrac{\sqrt{x}\cos x}{1+x^2}\right)$

25. $y = e^{\ln(x^2+1)}$ **26.** $y = \ln\left(\dfrac{1 + e^x + e^{2x}}{1 - e^{3x}}\right)$

27. $y = 2xe^{\sqrt{x}}$ **28.** $y = \dfrac{a}{1 + be^{-x}}$

29. $y = \dfrac{1}{\pi}\tan^{-1} 2x$ **30.** $y = 2^{\sin^{-1} x}$

31. $y = x^{(e^x)}$ **32.** $y = (1+x)^{1/x}$

33. $y = \sec^{-1}(2x + 1)$ **34.** $y = \sqrt{\cos^{-1} x^2}$

35–36 Find dy/dx using logarithmic differentiation. ■

35. $y = \dfrac{x^3}{\sqrt{x^2+1}}$ **36.** $y = \sqrt[3]{\dfrac{x^2-1}{x^2+1}}$

37. (a) Make a conjecture about the shape of the graph of $y = \frac{1}{2}x - \ln x$, and draw a rough sketch.
(b) Check your conjecture by graphing the equation over the interval $0 < x < 5$ with a graphing utility.
(c) Show that the slopes of the tangent lines to the curve at $x = 1$ and $x = e$ have opposite signs.
(d) What does part (c) imply about the existence of a horizontal tangent line to the curve? Explain.
(e) Find the exact x-coordinates of all horizontal tangent lines to the curve.

38. Recall from Section 0.5 that the loudness β of a sound in decibels (dB) is given by $\beta = 10\log(I/I_0)$, where I is the intensity of the sound in watts per square meter (W/m^2) and I_0 is a constant that is approximately the intensity of a sound at the threshold of human hearing. Find the rate of change of β with respect to I at the point where
(a) $I/I_0 = 10$ (b) $I/I_0 = 100$ (c) $I/I_0 = 1000$.

39. A particle is moving along the curve $y = x\ln x$. Find all values of x at which the rate of change of y with respect to time is three times that of x. [Assume that dx/dt is never zero.]

40. Find the equation of the tangent line to the graph of $y = \ln(5 - x^2)$ at $x = 2$.

41. Find the value of b so that the line $y = x$ is tangent to the graph of $y = \log_b x$. Confirm your result by graphing both $y = x$ and $y = \log_b x$ in the same coordinate system.

42. In each part, find the value of k for which the graphs of $y = f(x)$ and $y = \ln x$ share a common tangent line at their point of intersection. Confirm your result by graphing $y = f(x)$ and $y = \ln x$ in the same coordinate system.
(a) $f(x) = \sqrt{x} + k$ (b) $f(x) = k\sqrt{x}$

43. If f and g are inverse functions and f is differentiable on its domain, must g be differentiable on its domain? Give a reasonable informal argument to support your answer.

44. In each part, find $(f^{-1})'(x)$ using Formula (2) of Section 3.3, and check your answer by differentiating f^{-1} directly.
(a) $f(x) = 3/(x+1)$ (b) $f(x) = \sqrt{e^x}$

45. Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

46. Show that the rate of change of $y = 5000e^{1.07x}$ is proportional to y.

47. Show that the rate of change of $y = 3^{2x}5^{7x}$ is proportional to y.

48. The equilibrium constant k of a balanced chemical reaction changes with the absolute temperature T according to the law

$$k = k_0 \exp\left(-\frac{q(T - T_0)}{2T_0 T}\right)$$

where k_0, q, and T_0 are constants. Find the rate of change of k with respect to T.

49. Show that the function $y = e^{ax}\sin bx$ satisfies

$$y'' - 2ay' + (a^2 + b^2)y = 0$$

for any real constants a and b.

50. Show that the function $y = \tan^{-1} x$ satisfies

$$y'' = -2\sin y \cos^3 y$$

51. Suppose that the population of deer on an island is modeled by the equation

$$P(t) = \frac{95}{5 - 4e^{-t/4}}$$

where $P(t)$ is the number of deer t weeks after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t\to+\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

52. In each part, find each limit by interpreting the expression as an appropriate derivative.
(a) $\displaystyle\lim_{h\to 0} \frac{(1+h)^\pi - 1}{h}$ (b) $\displaystyle\lim_{x\to e} \frac{1 - \ln x}{(x - e)\ln x}$

53. Suppose that $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$. In each of the four possible cases, state whether $\lim[f(x) - g(x)]$ is an indeterminate form, and give a reasonable informal argument to support your answer.

54. (a) Under what conditions will a limit of the form

$$\lim_{x\to a} [f(x)/g(x)]$$

be an indeterminate form?

(b) If $\lim_{x\to a} g(x) = 0$, must $\lim_{x\to a}[f(x)/g(x)]$ be an indeterminate form? Give some examples to support your answer.

55–58 Evaluate the given limit. ■

55. $\lim_{x\to+\infty} (e^x - x^2)$

56. $\lim_{x\to 1} \sqrt{\dfrac{\ln x}{x^4 - 1}}$

57. $\lim_{x\to 0} \dfrac{x^2 e^x}{\sin^2 3x}$

58. $\lim_{x\to 0} \dfrac{a^x - 1}{x}, \quad a > 0$

59. An oil slick on a lake is surrounded by a floating circular containment boom. As the boom is pulled in, the circular containment area shrinks. If the boom is pulled in at the rate of 5 m/min, at what rate is the containment area shrinking when the containment area has a diameter of 100 m?

60. The hypotenuse of a right triangle is growing at a constant rate of a centimeters per second and one leg is decreasing at a constant rate of b centimeters per second. How fast is the acute angle between the hypotenuse and the other leg changing at the instant when both legs are 1 cm?

61. In each part, use the given information to find Δx, Δy, and dy.

(a) $y = 1/(x-1)$; x decreases from 2 to 1.5.

(b) $y = \tan x$; x increases from $-\pi/4$ to 0.

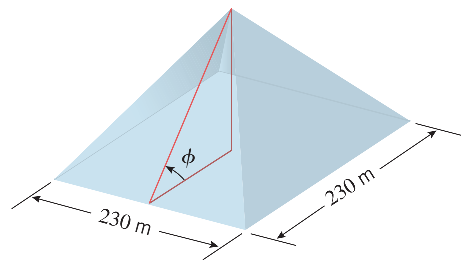
(c) $y = \sqrt{25 - x^2}$; x increases from 0 to 3.

62. Use an appropriate local linear approximation to estimate the value of $\cot 46°$, and compare your answer to the value obtained with a calculating device.

63. The base of the Great Pyramid at Giza is a square that is 230 m on each side.

(a) As illustrated in the accompanying figure, suppose that an archaeologist standing at the center of a side measures the angle of elevation of the apex to be $\phi = 51°$ with an error of $\pm 0.5°$. What can the archaeologist reasonably say about the height of the pyramid?

(b) Use differentials to estimate the allowable error in the elevation angle that will ensure that the error in calculating the height is at most ± 5 m.



▲ **Figure Ex-63**

CHAPTER 3 MAKING CONNECTIONS

In these exercises we explore an application of exponential functions to radioactive decay, and we consider another approach to computing the derivative of the natural exponential function.

1. Consider a simple model of radioactive decay. We assume that given any quantity of a radioactive element, the fraction of the quantity that decays over a period of time will be a constant that depends on only the particular element and the length of the time period. We choose a time parameter $-\infty < t < +\infty$ and let $A = A(t)$ denote the amount of the element remaining at time t. We also choose units of measure such that the initial amount of the element is $A(0) = 1$, and we let $b = A(1)$ denote the amount at time $t = 1$. Prove that the function $A(t)$ has the following properties.

(a) $A(-t) = \dfrac{1}{A(t)}$ [*Hint:* For $t > 0$, you can interpret $A(t)$ as the fraction of any given amount that remains after a time period of length t.]

(b) $A(s + t) = A(s) \cdot A(t)$ [*Hint:* First consider positive s and t. For the other cases use the property in part (a).]

(c) If n is any nonzero integer, then

$$A\left(\frac{1}{n}\right) = (A(1))^{1/n} = b^{1/n}$$

(d) If m and n are integers with $n \neq 0$, then

$$A\left(\frac{m}{n}\right) = (A(1))^{m/n} = b^{m/n}$$

(e) Assuming that $A(t)$ is a continuous function of t, then $A(t) = b^t$. [*Hint:* Prove that if two continuous functions agree on the set of rational numbers, then they are equal.]

(f) If we replace the assumption that $A(0) = 1$ by the condition $A(0) = A_0$, prove that $A = A_0 b^t$.

2. Refer to Figure 1.3.4.

(a) Make the substitution $h = 1/x$ and conclude that

$$(1+h)^{1/h} < e < (1-h)^{-1/h} \quad \text{for } h > 0$$

and

$$(1-h)^{-1/h} < e < (1+h)^{1/h} \quad \text{for } h < 0$$

(b) Use the inequalities in part (a) and the Squeezing Theorem to prove that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

(c) Explain why the limit in part (b) confirms Figure 0.5.4.

(d) Use the limit in part (b) to prove that

$$\frac{d}{dx}(e^x) = e^x$$