

If a dragster moves with varying velocity over a certain time interval, it is possible to find the distance it travels during that time interval using techniques of calculus.

In this chapter we will begin with an overview of the problem of finding areas-we will discuss what the term "area" means, and we will outline two approaches to defining and calculating areas. Following this overview, we will discuss the Fundamental Theorem of Calculus, which is the theorem that relates the problems of finding tangent lines and areas, and we will discuss techniques for calculating areas. We will then use the ideas in this chapter to define the average value of a function, to continue our study of rectilinear motion, and to examine some consequences of the chain rule in integral calculus. We conclude the chapter by studying functions defined by integrals, with a focus on the natural logarithm function.

### 5.1 AN OVERVIEW OF THE AREA PROBLEM

In this introductory section we will consider the problem of calculating areas of plane regions with curvilinear boundaries. All of the results in this section will be reexamined in more detail later in this chapter. Our purpose here is simply to introduce and motivate the fundamental concepts.

## THE AREA PROBLEM

Formulas for the areas of polygons, such as squares, rectangles, triangles, and trapezoids, were well known in many early civilizations. However, the problem of finding formulas for regions with curved boundaries (a circle being the simplest example) caused difficulties for early mathematicians.

The first real progress in dealing with the general area problem was made by the Greek mathematician Archimedes, who obtained areas of regions bounded by circular arcs, parabolas, spirals, and various other curves using an ingenious procedure that was later called the method of exhaustion. The method, when applied to a circle, consists of inscribing a succession of regular polygons in the circle and allowing the number of sides to increase indefinitely (Figure 5.1.1). As the number of sides increases, the polygons tend to "exhaust" the region inside the circle, and the areas of the polygons become better and better approximations of the exact area of the circle.

To see how this works numerically, let $A(n)$ denote the area of a regular $n$-sided polygon inscribed in a circle of radius 1 . Table 5.1 .1 shows the values of $A(n)$ for various choices of $n$. Note that for large values of $n$ the area $A(n)$ appears to be close to $\pi$ (square units),

Table 5.1.1

| $n$ | $A(n)$ |
| ---: | :---: |
| 100 | 3.13952597647 |
| 200 | 3.14107590781 |
| 300 | 3.14136298250 |
| 400 | 3.14146346236 |
| 500 | 3.14150997084 |
| 1000 | 3.14157198278 |
| 2000 | 3.14158748588 |
| 3000 | 3.14159035683 |
| 4000 | 3.14159136166 |
| 5000 | 3.14159182676 |
| 10,000 | 3.14159244688 |

as one would expect. This suggests that for a circle of radius 1 , the method of exhaustion is equivalent to an equation of the form

$$
\lim _{n \rightarrow \infty} A(n)=\pi
$$

Since Greek mathematicians were suspicious of the concept of "infinity," they avoided


- Figure 5.1.2


A Figure 5.1.3

Logically speaking, we cannot really talk about computing areas without a precise mathematical definition of the term "area." Later in this chapter we will give such a definition, but for now we will treat the concept intuitively. its use in mathematical arguments. As a result, computation of area using the method of exhaustion was a very cumbersome procedure. It remained for Newton and Leibniz to obtain a general method for finding areas that explicitly used the notion of a limit. We will discuss their method in the context of the following problem.
5.1.1 THE AREA PRObLEM Given a function $f$ that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of $f$ and the interval $[a, b]$ on the $x$-axis (Figure 5.1.2).

## THE RECTANGLE METHOD FOR FINDING AREAS

One approach to the area problem is to use Archimedes' method of exhaustion in the following way:

- Divide the interval $[a, b]$ into $n$ equal subintervals, and over each subinterval construct a rectangle that extends from the $x$-axis to any point on the curve $y=f(x)$ that is above the subinterval; the particular point does not matter-it can be above the center, above an endpoint, or above any other point in the subinterval. In Figure 5.1.3 it is above the center.
- For each $n$, the total area of the rectangles can be viewed as an approximation to the exact area under the curve over the interval $[a, b]$. Moreover, it is evident intuitively that as $n$ increases these approximations will get better and better and will approach the exact area as a limit (Figure 5.1.4). That is, if $A$ denotes the exact area under the curve and $A_{n}$ denotes the approximation to $A$ using $n$ rectangles, then

$$
A=\lim _{n \rightarrow+\infty} A_{n}
$$

We will call this the rectangle method for computing $A$.

$\Delta$ Figure 5.1.4

To illustrate this idea, we will use the rectangle method to approximate the area under the curve $y=x^{2}$ over the interval $[0,1]$ (Figure 5.1 .5 ). We will begin by dividing the interval $[0,1]$ into $n$ equal subintervals, from which it follows that each subinterval has length $1 / n$; the endpoints of the subintervals occur at

$$
0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, 1
$$



Archimedes (287 b.C.-212 b.c.) Greek mathematician and scientist. Born in Syracuse, Sicily, Archimedes was the son of the astronomer Pheidias and possibly related to Heiron II, king of Syracuse. Most of the facts about his life come from the Roman biographer, Plutarch, who inserted a few tantalizing pages about him in the massive biography of the Roman soldier, Marcellus. In the words of one writer, "the account of Archimedes is slipped like a tissue-thin shaving of ham in a bull-choking sandwich."

Archimedes ranks with Newton and Gauss as one of the three greatest mathematicians who ever lived, and he is certainly the greatest mathematician of antiquity. His mathematical work is so modern in spirit and technique that it is barely distinguishable from that of a seventeenth-century mathematician, yet it was all done without benefit of algebra or a convenient number system. Among his mathematical achievements, Archimedes developed a general method (exhaustion) for finding areas and volumes, and he used the method to find areas bounded by parabolas and spirals and to find volumes of cylinders, paraboloids, and segments of spheres. He gave a procedure for approximating $\pi$ and bounded its value between $3 \frac{10}{71}$ and $3 \frac{1}{7}$. In spite of the limitations of the Greek numbering system, he devised methods for finding square roots and invented a method based on the Greek myriad $(10,000)$ for representing numbers as large as 1 followed by 80 million billion zeros.

Of all his mathematical work, Archimedes was most proud of his discovery of a method for finding the volume of a sphere-he showed that the volume of a sphere is two-thirds the volume of the smallest cylinder that can contain it. At his request, the figure of a sphere and cylinder was engraved on his tombstone.

In addition to mathematics, Archimedes worked extensively in mechanics and hydrostatics. Nearly every schoolchild knows Archimedes as the absent-minded scientist who, on realizing that a floating object displaces its weight of liquid, leaped from his bath and ran naked through the streets of Syracuse shouting, "Eureka, Eureka!"-(meaning, "I have found it!"). Archimedes actually cre-
ated the discipline of hydrostatics and used it to find equilibrium positions for various floating bodies. He laid down the fundamental postulates of mechanics, discovered the laws of levers, and calculated centers of gravity for various flat surfaces and solids. In the excitement of discovering the mathematical laws of the lever, he is said to have declared, "Give me a place to stand and I will move the earth."

Although Archimedes was apparently more interested in pure mathematics than its applications, he was an engineering genius. During the second Punic war, when Syracuse was attacked by the Roman fleet under the command of Marcellus, it was reported by Plutarch that Archimedes' military inventions held the fleet at bay for three years. He invented super catapults that showered the Romans with rocks weighing a quarter ton or more, and fearsome mechanical devices with iron "beaks and claws" that reached over the city walls, grasped the ships, and spun them against the rocks. After the first repulse, Marcellus called Archimedes a "geometrical Briareus (a hundred-armed mythological monster) who uses our ships like cups to ladle water from the sea."

Eventually the Roman army was victorious and contrary to Marcellus' specific orders the 75 -year-old Archimedes was killed by a Roman soldier. According to one report of the incident, the soldier cast a shadow across the sand in which Archimedes was working on a mathematical problem. When the annoyed Archimedes yelled, "Don't disturb my circles," the soldier flew into a rage and cut the old man down.

Although there is no known likeness or statue of this great man, nine works of Archimedes have survived to the present day. Especially important is his treatise, The Method of Mechanical Theorems, which was part of a palimpsest found in Constantinople in 1906. In this treatise Archimedes explains how he made some of his discoveries, using reasoning that anticipated ideas of the integral calculus. Thought to be lost, the Archimedes palimpsest later resurfaced in 1998, when it was purchased by an anonymous private collector for two million dollars.


A Figure 5.1.5

$$
\begin{aligned}
& \text { Width }=\frac{1}{n} \\
& \begin{array}{l}
\text { Subdivision of }[0,1] \text { into } n \\
\text { subintervals of equal length }
\end{array}
\end{aligned}
$$

$\Delta$ Figure 5.1.6

## TECHNOLOGY MASTERY

Use a calculating utility to compute the value of $A_{10}$ in Table 5.1.2. Some calculating utilities have special commands for computing sums such as that in (1) for any specified value of $n$. If your utility has this feature, use it to compute $A_{100}$ as well.


Aigure 5.1.7
(Figure 5.1.6). We want to construct a rectangle over each of these subintervals whose height is the value of the function $f(x)=x^{2}$ at some point in the subinterval. To be specific, let us use the right endpoints, in which case the heights of our rectangles will be

$$
\left(\frac{1}{n}\right)^{2},\left(\frac{2}{n}\right)^{2},\left(\frac{3}{n}\right)^{2}, \ldots, 1^{2}
$$

and since each rectangle has a base of width $1 / n$, the total area $A_{n}$ of the $n$ rectangles will be

$$
\begin{equation*}
A_{n}=\left[\left(\frac{1}{n}\right)^{2}+\left(\frac{2}{n}\right)^{2}+\left(\frac{3}{n}\right)^{2}+\cdots+1^{2}\right]\left(\frac{1}{n}\right) \tag{1}
\end{equation*}
$$

For example, if $n=4$, then the total area of the four approximating rectangles would be

$$
A_{4}=\left[\left(\frac{1}{4}\right)^{2}+\left(\frac{2}{4}\right)^{2}+\left(\frac{3}{4}\right)^{2}+1^{2}\right]\left(\frac{1}{4}\right)=\frac{15}{32}=0.46875
$$

Table 5.1.2 shows the result of evaluating (1) on a computer for some increasingly large values of $n$. These computations suggest that the exact area is close to $\frac{1}{3}$. Later in this chapter we will prove that this area is exactly $\frac{1}{3}$ by showing that

$$
\lim _{n \rightarrow \infty} A_{n}=\frac{1}{3}
$$

Table 5.1.2

| $n$ | 4 | 10 | 100 | 1000 | 10,000 | 100,000 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $A_{n}$ | 0.468750 | 0.385000 | 0.338350 | 0.333834 | 0.333383 | 0.333338 |

## THE ANTIDERIVATIVE METHOD FOR FINDING AREAS

Although the rectangle method is appealing intuitively, the limits that result can only be evaluated in certain cases. For this reason, progress on the area problem remained at a rudimentary level until the latter part of the seventeenth century when Isaac Newton and Gottfried Leibniz independently discovered a fundamental relationship between areas and derivatives. Briefly stated, they showed that if $f$ is a nonnegative continuous function on the interval $[a, b]$, and if $A(x)$ denotes the area under the graph of $f$ over the interval $[a, x]$, where $x$ is any point in the interval $[a, b]$ (Figure 5.1.7), then

$$
\begin{equation*}
A^{\prime}(x)=f(x) \tag{2}
\end{equation*}
$$

The following example confirms Formula (2) in some cases where a formula for $A(x)$ can be found using elementary geometry.

- Example 1 For each of the functions $f$, find the area $A(x)$ between the graph of $f$ and the interval $[a, x]=[-1, x]$, and find the derivative $A^{\prime}(x)$ of this area function.
(a) $f(x)=2$
(b) $f(x)=x+1$
(c) $f(x)=2 x+3$

Solution (a). From Figure 5.1.8a we see that

$$
A(x)=2(x-(-1))=2(x+1)=2 x+2
$$

is the area of a rectangle of height 2 and base $x+1$. For this area function,

$$
A^{\prime}(x)=2=f(x)
$$


(a)

(b)

(c)
$\triangle$ Figure 5.1.8

How does the solution to Example 2 change if the interval $[0,1]$ is replaced by the interval $[-1,1]$ ?

Solution (b). From Figure $5.1 .8 b$ we see that

$$
A(x)=\frac{1}{2}(x+1)(x+1)=\frac{x^{2}}{2}+x+\frac{1}{2}
$$

is the area of an isosceles right triangle with base and height equal to $x+1$. For this area function,

$$
A^{\prime}(x)=x+1=f(x)
$$

Solution (c). Recall that the formula for the area of a trapezoid is $A=\frac{1}{2}\left(b+b^{\prime}\right) h$, where $b$ and $b^{\prime}$ denote the lengths of the parallel sides of the trapezoid, and the altitude $h$ denotes the distance between the parallel sides. From Figure 5.1.8c we see that

$$
A(x)=\frac{1}{2}((2 x+3)+1)(x-(-1))=x^{2}+3 x+2
$$

is the area of a trapezoid with parallel sides of lengths 1 and $2 x+3$ and with altitude $x-(-1)=x+1$. For this area function,

$$
A^{\prime}(x)=2 x+3=f(x)
$$

Formula (2) is important because it relates the area function $A$ and the region-bounding function $f$. Although a formula for $A(x)$ may be difficult to obtain directly, its derivative, $f(x)$, is given. If a formula for $A(x)$ can be recovered from the given formula for $A^{\prime}(x)$, then the area under the graph of $f$ over the interval $[a, b]$ can be obtained by computing A(b).

The process of finding a function from its derivative is called antidifferentiation, and a procedure for finding areas via antidifferentiation is called the antiderivative method. To illustrate this method, let us revisit the problem of finding the area in Figure 5.1.5.

- Example 2 Use the antiderivative method to find the area under the graph of $y=x^{2}$ over the interval $[0,1]$.

Solution. Let $x$ be any point in the interval $[0,1]$, and let $A(x)$ denote the area under the graph of $f(x)=x^{2}$ over the interval $[0, x]$. It follows from (2) that

$$
\begin{equation*}
A^{\prime}(x)=x^{2} \tag{3}
\end{equation*}
$$

To find $A(x)$ we must look for a function whose derivative is $x^{2}$. By guessing, we see that one such function is $\frac{1}{3} x^{3}$, so by Theorem 4.8.3

$$
\begin{equation*}
A(x)=\frac{1}{3} x^{3}+C \tag{4}
\end{equation*}
$$

for some real constant $C$. We can determine the specific value for $C$ by considering the case where $x=0$. In this case (4) implies that

$$
\begin{equation*}
A(0)=C \tag{5}
\end{equation*}
$$

But if $x=0$, then the interval $[0, x]$ reduces to a single point. If we agree that the area above a single point should be taken as zero, then $A(0)=0$ and (5) implies that $C=0$. Thus, it follows from (4) that

$$
A(x)=\frac{1}{3} x^{3}
$$

is the area function we are seeking. This implies that the area under the graph of $y=x^{2}$ over the interval $[0,1]$ is

$$
A(1)=\frac{1}{3}\left(1^{3}\right)=\frac{1}{3}
$$

This is consistent with the result that we previously obtained numerically.

As Example 2 illustrates, antidifferentiation is a process in which one tries to "undo" a differentiation. One of the objectives in this chapter is to develop efficient antidifferentiation procedures.

## the rectangle method and the antiderivative method compared

The rectangle method and the antiderivative method provide two very different approaches to the area problem, each of which is important. The antiderivative method is usually the more efficient way to compute areas, but it is the rectangle method that is used to formally define the notion of area, thereby allowing us to prove mathematical results about areas. The underlying idea of the rectangle approach is also important because it can be adapted readily to such diverse problems as finding the volume of a solid, the length of a curve, the mass of an object, and the work done in pumping water out of a tank, to name a few.

## QUICK CHECK EXERCISES 5.1 (See page 322 for answers.)

1. Let $R$ denote the region below the graph of $f(x)=\sqrt{1-x^{2}}$ and above the interval $[-1,1]$.
(a) Use a geometric argument to find the area of $R$.
(b) What estimate results if the area of $R$ is approximated by the total area within the rectangles of the accompanying figure?


4Figure Ex-1
2. Suppose that when the area $A$ between the graph of a function $y=f(x)$ and an interval $[a, b]$ is approximated by the areas of $n$ rectangles, the total area of the rectangles is $A_{n}=2+(2 / n), n=1,2, \ldots$ Then, $A=$ $\qquad$ —.
3. The area under the graph of $y=x^{2}$ over the interval $[0,3]$ is $\qquad$
4. Find a formula for the area $A(x)$ between the graph of the function $f(x)=x$ and the interval $[0, x]$, and verify that $A^{\prime}(x)=f(x)$.
5. The area under the graph of $y=f(x)$ over the interval $[0, x]$ is $A(x)=x+e^{x}-1$. It follows that $f(x)=$ $\qquad$

## EXERCISE SET 5.1

1-12 Estimate the area between the graph of the function $f$ and the interval $[a, b]$. Use an approximation scheme with $n$ rectangles similar to our treatment of $f(x)=x^{2}$ in this section. If your calculating utility will perform automatic summations, estimate the specified area using $n=10,50$, and 100 rectangles. Otherwise, estimate this area using $n=2,5$, and 10 rectangles.

1. $f(x)=\sqrt{x} ;[a, b]=[0,1]$
2. $f(x)=\frac{1}{x+1} ;[a, b]=[0,1]$
3. $f(x)=\sin x ;[a, b]=[0, \pi]$
4. $f(x)=\cos x ;[a, b]=[0, \pi / 2]$
5. $f(x)=\frac{1}{x} ;[a, b]=[1,2]$
6. $f(x)=\cos x ;[a, b]=[-\pi / 2, \pi / 2]$
7. $f(x)=\sqrt{1-x^{2}} ;[a, b]=[0,1]$
8. $f(x)=\sqrt{1-x^{2}} ;[a, b]=[-1,1]$
9. $f(x)=e^{x} ;[a, b]=[-1,1]$
10. $f(x)=\ln x ;[a, b]=[1,2]$
11. $f(x)=\sin ^{-1} x ;[a, b]=[0,1]$
12. $f(x)=\tan ^{-1} x ;[a, b]=[0,1]$

13-18 Graph each function over the specified interval. Then use simple area formulas from geometry to find the area function $A(x)$ that gives the area between the graph of the specified function $f$ and the interval $[a, x]$. Confirm that $A^{\prime}(x)=f(x)$ in every case.
13. $f(x)=3 ;[a, x]=[1, x]$
14. $f(x)=5 ;[a, x]=[2, x]$
15. $f(x)=2 x+2 ;[a, x]=[0, x]$
16. $f(x)=3 x-3 ;[a, x]=[1, x]$
17. $f(x)=2 x+2 ;[a, x]=[1, x]$
18. $f(x)=3 x-3 ;[a, x]=[2, x]$

19-22 True-False Determine whether the statement is true or false. Explain your answer.
19. If $A(n)$ denotes the area of a regular $n$-sided polygon inscribed in a circle of radius 2 , then $\lim _{n \rightarrow+\infty} A(n)=2 \pi$.
20. If the area under the curve $y=x^{2}$ over an interval is approximated by the total area of a collection of rectangles, the approximation will be too large.
21. If $A(x)$ is the area under the graph of a nonnegative continuous function $f$ over an interval $[a, x]$, then $A^{\prime}(x)=f(x)$.
22. If $A(x)$ is the area under the graph of a nonnegative continuous function $f$ over an interval $[a, x]$, then $A(x)$ will be a continuous function.

## FOCUS ON CONCEPTS

23. Explain how to use the formula for $A(x)$ found in the solution to Example 2 to determine the area between the graph of $y=x^{2}$ and the interval $[3,6]$.
24. Repeat Exercise 23 for the interval $[-3,9]$.
25. Let $A$ denote the area between the graph of $f(x)=\sqrt{x}$ and the interval $[0,1]$, and let $B$ denote the area between the graph of $f(x)=x^{2}$ and the interval $[0,1]$. Explain geometrically why $A+B=1$.
26. Let $A$ denote the area between the graph of $f(x)=1 / x$ and the interval $[1,2]$, and let $B$ denote the area between the graph of $f$ and the interval $\left[\frac{1}{2}, 1\right]$. Explain geometrically why $A=B$.

27-28 The area $A(x)$ under the graph of $f$ and over the interval [ $a, x]$ is given. Find the function $f$ and the value of $a$.
27. $A(x)=x^{2}-4$
28. $A(x)=x^{2}-x$
29. Writing Compare and contrast the rectangle method and the antiderivative method.
30. Writing Suppose that $f$ is a nonnegative continuous function on an interval $[a, b]$ and that $g(x)=f(x)+C$, where $C$ is a positive constant. What will be the area of the region between the graphs of $f$ and $g$ ?

QUICK CHECK ANSWERS 5.1

1. (a) $\frac{\pi}{2}$
(b) $1+\frac{\sqrt{3}}{2}$
2. 2
3. 9
4. $A(x)=\frac{x^{2}}{2} ; A^{\prime}(x)=\frac{2 x}{2}=x=f(x)$
5. $e^{x}+1$

### 5.2 THE INDEFINITE INTEGRAL

In the last section we saw how antidifferentiation could be used to find exact areas. In this section we will develop some fundamental results about antidifferentiation.

## ANTIDERIVATIVES

5.2.1 DEFINITION A function $F$ is called an antiderivative of a function $f$ on a given open interval if $F^{\prime}(x)=f(x)$ for all $x$ in the interval.

For example, the function $F(x)=\frac{1}{3} x^{3}$ is an antiderivative of $f(x)=x^{2}$ on the interval $(-\infty,+\infty)$ because for each $x$ in this interval

$$
F^{\prime}(x)=\frac{d}{d x}\left[\frac{1}{3} x^{3}\right]=x^{2}=f(x)
$$

However, $F(x)=\frac{1}{3} x^{3}$ is not the only antiderivative of $f$ on this interval. If we add any constant $C$ to $\frac{1}{3} x^{3}$, then the function $G(x)=\frac{1}{3} x^{3}+C$ is also an antiderivative of $f$ on $(-\infty,+\infty)$, since

$$
G^{\prime}(x)=\frac{d}{d x}\left[\frac{1}{3} x^{3}+C\right]=x^{2}+0=f(x)
$$

In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative. Thus,

$$
\frac{1}{3} x^{3}, \quad \frac{1}{3} x^{3}+2, \quad \frac{1}{3} x^{3}-5, \quad \frac{1}{3} x^{3}+\sqrt{2}
$$

are all antiderivatives of $f(x)=x^{2}$.

It is reasonable to ask if there are antiderivatives of a function $f$ that cannot be obtained by adding some constant to a known antiderivative $F$. The answer is no-once a single antiderivative of $f$ on an open interval is known, all other antiderivatives on that interval are obtainable by adding constants to the known antiderivative. This is so because Theorem 4.8.3 tells us that if two functions have the same derivative on an open interval, then the functions differ by a constant on the interval. The following theorem summarizes these observations.
5.2.2 THEOREM If $F(x)$ is any antiderivative of $f(x)$ on an open interval, then for any constant $C$ the function $F(x)+C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval can be expressed in the form $F(x)+C$ by choosing the constant $C$ appropriately.

## THE INDEFINITE INTEGRAL



Reproduced from C. I. Gerhardt's "Briefwechsel von G. W. Leibniz mit Mathematikern (1899)."

Extract from the manuscript of Leibniz dated October 29, 1675 in which the integral sign first appeared (see yellow highlight).

The process of finding antiderivatives is called antidifferentiation or integration. Thus, if

$$
\begin{equation*}
\frac{d}{d x}[F(x)]=f(x) \tag{1}
\end{equation*}
$$

then integrating (or antidifferentiating) the function $f(x)$ produces an antiderivative of the form $F(x)+C$. To emphasize this process, Equation (1) is recast using integral notation,

$$
\begin{equation*}
\int f(x) d x=F(x)+C \tag{2}
\end{equation*}
$$

where $C$ is understood to represent an arbitrary constant. It is important to note that (1) and (2) are just different notations to express the same fact. For example,

$$
\int x^{2} d x=\frac{1}{3} x^{3}+C \quad \text { is equivalent to } \quad \frac{d}{d x}\left[\frac{1}{3} x^{3}\right]=x^{2}
$$

Note that if we differentiate an antiderivative of $f(x)$, we obtain $f(x)$ back again. Thus,

$$
\begin{equation*}
\frac{d}{d x}\left[\int f(x) d x\right]=f(x) \tag{3}
\end{equation*}
$$

The expression $\int f(x) d x$ is called an indefinite integral. The adjective "indefinite" emphasizes that the result of antidifferentiation is a "generic" function, described only up to a constant term. The "elongated s" that appears on the left side of (2) is called an integral sign, , the function $f(x)$ is called the integrand, and the constant $C$ is called the constant of integration. Equation (2) should be read as:

The integral of $f(x)$ with respect to $x$ is equal to $F(x)$ plus a constant.

The differential symbol, $d x$, in the differentiation and antidifferentiation operations

$$
\frac{d}{d x}[] \text { and } \int[] d x
$$

[^0]serves to identify the independent variable. If an independent variable other than $x$ is used, say $t$, then the notation must be adjusted appropriately. Thus,
$$
\frac{d}{d t}[F(t)]=f(t) \quad \text { and } \quad \int f(t) d t=F(t)+C
$$
are equivalent statements. Here are some examples of derivative formulas and their equivalent integration formulas:

| DERIVATIVE <br> FORMULA | EQUIVALENT <br> INTEGRATION FORMULA |
| :--- | :---: |
| $\frac{d}{d x}\left[x^{3}\right]=3 x^{2}$ | $\int 3 x^{2} d x=x^{3}+C$ |
| $\frac{d}{d x}[\sqrt{x}]=\frac{1}{2 \sqrt{x}}$ | $\int \frac{1}{2 \sqrt{x}} d x=\sqrt{x}+C$ |
| $\frac{d}{d t}[\tan t]=\sec ^{2} t$ | $\int \sec ^{2} t d t=\tan t+C$ |
| $\frac{d}{d u}\left[u^{3 / 2}\right]=\frac{3}{2} u^{1 / 2}$ | $\int \frac{3}{2} u^{1 / 2} d u=u^{3 / 2}+C$ |

For simplicity, the $d x$ is sometimes absorbed into the integrand. For example,

$$
\begin{aligned}
& \int 1 d x \quad \text { can be written as } \int d x \\
& \int \frac{1}{x^{2}} d x \quad \text { can be written as } \int \frac{d x}{x^{2}}
\end{aligned}
$$

## INTEGRATION FORMULAS

Integration is essentially educated guesswork-given the derivative $f$ of a function $F$, one tries to guess what the function $F$ is. However, many basic integration formulas can be obtained directly from their companion differentiation formulas. Some of the most important are given in Table 5.2.1.

Table 5.2.1
INTEGRATION FORMULAS

| DIFFERENTIATION FORMULA | INTEGRATION FORMULA | DIFFERENTIATION FORMULA | INTEGRATION FORMULA |
| :--- | :--- | :--- | :--- |
| 1. $\frac{d}{d x}[x]=1$ | $\int d x=x+C$ | 8. $\frac{d}{d x}[-\csc x]=\csc x \cot x$ | $\int \csc x \cot x d x=-\csc x+C$ |
| 2. $\frac{d}{d x}\left[\frac{x^{r+1}}{r+1}\right]=x^{r} \quad(r \neq-1)$ | $\int x^{r} d x=\frac{x^{r+1}}{r+1}+C \quad(r \neq-1)$ | 9. $\frac{d}{d x}\left[e^{x}\right]=e^{x}$ | $\int e^{x} d x=e^{x}+C$ |
| 3. $\frac{d}{d x}[\sin x]=\cos x$ | $\int \cos x d x=\sin x+C$ | 10. $\frac{d}{d x}\left[\frac{\left.b^{x}\right]=b^{x}}{\ln b}(0<b, b \neq 1) \int b^{x} d x=\frac{b^{x}}{\ln b}+C \quad(0<b, b \neq 1)\right.$ |  |
| 4. $\frac{d}{d x}[-\cos x]=\sin x$ | $\int \sin x d x=-\cos x+C$ | 11. $\frac{d}{d x}[\ln \|x\|]=\frac{1}{x}$ | $\int \frac{1}{x} d x=\ln \|x\|+C$ |
| 5. $\frac{d}{d x}[\tan x]=\sec ^{2} x$ | $\int \sec ^{2} x d x=\tan x+C$ | 12. $\frac{d}{d x}\left[\tan ^{-1} x\right]=\frac{1}{1+x^{2}}$ | $\int \frac{1}{1+x^{2}} d x=\tan ^{-1} x+C$ |
| 6. $\frac{d}{d x}[-\cot x]=\csc ^{2} x$ | $\int \csc ^{2} x d x=-\cot x+C$ | 13. $\frac{d}{d x}\left[\sin ^{-1} x\right]=\frac{1}{\sqrt{1-x^{2}}}$ | $\int \frac{1}{\sqrt{1-x^{2}}} d x=\sin ^{-1} x+C$ |
| 7. $\frac{d}{d x}[\sec x]=\sec x \tan x$ | $\int \sec x \tan x d x=\sec x+C$ | 14. $\frac{d}{d x}\left[\sec ^{-1}\|x\|\right]=\frac{1}{x \sqrt{x^{2}-1}}$ | $\int \frac{1}{x \sqrt{x^{2}-1}} d x=\sec ^{-1}\|x\|+C$ |

See Exercise 72 for a justification of Formula 14 in Table 5.2.1.

Although Formula 2 in Table 5.2.1 is not applicable to integrating $x^{-1}$, this function can be integrated by rewriting the integral in Formula 11 as

$$
\int \frac{1}{x} d x=\int x^{-1} d x=\ln |x|+C
$$

Example 1 The second integration formula in Table 5.2 .1 will be easier to remember if you express it in words:

To integrate a power of $x$ (other than -1 ), add 1 to the exponent and divide by the new exponent.

Here are some examples:

$$
\begin{aligned}
& \int x^{2} d x=\frac{x^{3}}{3}+C \\
& \int x^{3} d x=\frac{x^{4}}{4}+C \\
& \int \frac{1}{x^{5}} d x=\int x^{-5} d x=\frac{x^{-5+1}}{-5+1}+C=-\frac{1}{4 x^{4}}+C \\
& \int \sqrt{x} d x=\int x^{\frac{1}{2}} d x=\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C=\frac{2}{3} x^{\frac{3}{2}}+C=\frac{2}{3}(\sqrt{x})^{3}+C
\end{aligned}
$$

## PROPERTIES OF THE INDEFINITE INTEGRAL

Our first properties of antiderivatives follow directly from the simple constant factor, sum, and difference rules for derivatives.
5.2.3 THEOREM Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, and that $c$ is a constant. Then:
(a) A constant factor can be moved through an integral sign; that is,

$$
\int c f(x) d x=c F(x)+C
$$

(b) An antiderivative of a sum is the sum of the antiderivatives; that is,

$$
\int[f(x)+g(x)] d x=F(x)+G(x)+C
$$

(c) An antiderivative of a difference is the difference of the antiderivatives; that is,

$$
\int[f(x)-g(x)] d x=F(x)-G(x)+C
$$

PROOF In general, to establish the validity of an equation of the form

$$
\int h(x) d x=H(x)+C
$$

one must show that

$$
\frac{d}{d x}[H(x)]=h(x)
$$

We are given that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, so we know that

$$
\frac{d}{d x}[F(x)]=f(x) \quad \text { and } \quad \frac{d}{d x}[G(x)]=g(x)
$$

Thus,

$$
\begin{aligned}
& \frac{d}{d x}[c F(x)]=c \frac{d}{d x}[F(x)]=c f(x) \\
& \frac{d}{d x}[F(x)+G(x)]=\frac{d}{d x}[F(x)]+\frac{d}{d x}[G(x)]=f(x)+g(x) \\
& \frac{d}{d x}[F(x)-G(x)]=\frac{d}{d x}[F(x)]-\frac{d}{d x}[G(x)]=f(x)-g(x)
\end{aligned}
$$

which proves the three statements of the theorem.
The statements in Theorem 5.2.3 can be summarized by the following formulas:

$$
\begin{align*}
& \int c f(x) d x=c \int f(x) d x  \tag{4}\\
& \int[f(x)+g(x)] d x=\int f(x) d x+\int g(x) d x  \tag{5}\\
& \int[f(x)-g(x)] d x=\int f(x) d x-\int g(x) d x \tag{6}
\end{align*}
$$

However, these equations must be applied carefully to avoid errors and unnecessary complexities arising from the constants of integration. For example, if you use (4) to integrate $2 x$ by writing

$$
\int 2 x d x=2 \int x d x=2\left(\frac{x^{2}}{2}+C\right)=x^{2}+2 C
$$

then you will have an unnecessarily complicated form of the arbitrary constant. This kind of problem can be avoided by inserting the constant of integration in the final result rather than in intermediate calculations. Exercises 65 and 66 explore how careless application of these formulas can lead to errors.

## - Example 2 Evaluate

$$
\text { (a) } \int 4 \cos x d x \quad \text { (b) } \int\left(x+x^{2}\right) d x
$$

Solution (a). Since $F(x)=\sin x$ is an antiderivative for $f(x)=\cos x$ (Table 5.2.1), we obtain

$$
\int 4 \cos x d x=4 \int \cos x d x=4 \sin x+C
$$

(4)

Solution (b). From Table 5.2.1 we obtain

$$
\int\left(x+x^{2}\right) d x \underset{=}{=} x d x+\int x^{2} d x=\frac{x^{2}}{2}+\frac{x^{3}}{3}+C
$$

Parts (b) and (c) of Theorem 5.2.3 can be extended to more than two functions, which in combination with part $(a)$ results in the following general formula:

$$
\begin{align*}
& \int\left[c_{1} f_{1}(x)+c_{2} f_{2}(x)+\cdots+c_{n} f_{n}(x)\right] d x \\
& =c_{1} \int f_{1}(x) d x+c_{2} \int f_{2}(x) d x+\cdots+c_{n} \int f_{n}(x) d x \tag{7}
\end{align*}
$$

Perform the integration in part (c) by first performing a long division on the integrand.

## Example 3

$$
\begin{aligned}
\int\left(3 x^{6}-2 x^{2}+7 x+1\right) d x=3 \int x^{6} d x & -2 \int x^{2} d x+7 \int x d x+\int 1 d x \\
& =\frac{3 x^{7}}{7}-\frac{2 x^{3}}{3}+\frac{7 x^{2}}{2}+x+C
\end{aligned}
$$

Sometimes it is useful to rewrite an integrand in a different form before performing the integration. This is illustrated in the following example.

- Example 4 Evaluate
(a) $\int \frac{\cos x}{\sin ^{2} x} d x$
(b) $\int \frac{t^{2}-2 t^{4}}{t^{4}} d t$
(c) $\int \frac{x^{2}}{x^{2}+1} d x$

Solution (a).

$$
\begin{array}{r}
\int \frac{\cos x}{\sin ^{2} x} d x=\int \frac{1}{\sin x} \frac{\cos x}{\sin x} d x=\int \csc x \cot x d x=-\csc x+C \\
\text { Formula } 8 \text { in Table 5.2.1 }
\end{array}
$$

## Solution (b).

$$
\begin{aligned}
\int \frac{t^{2}-2 t^{4}}{t^{4}} d t & =\int\left(\frac{1}{t^{2}}-2\right) d t=\int\left(t^{-2}-2\right) d t \\
& =\frac{t^{-1}}{-1}-2 t+C=-\frac{1}{t}-2 t+C
\end{aligned}
$$

Solution (c). By adding and subtracting 1 from the numerator of the integrand, we can rewrite the integral in a form in which Formulas 1 and 12 of Table 5.2.1 can be applied:

$$
\begin{aligned}
\int \frac{x^{2}}{x^{2}+1} d x & =\int\left(\frac{x^{2}+1}{x^{2}+1}-\frac{1}{x^{2}+1}\right) d x \\
& =\int\left(1-\frac{1}{x^{2}+1}\right) d x=x-\tan ^{-1} x+C
\end{aligned}
$$

## INTEGRAL CURVES

Graphs of antiderivatives of a function $f$ are called integral curves of $f$. We know from Theorem 5.2.2 that if $y=F(x)$ is any integral curve of $f(x)$, then all other integral curves are vertical translations of this curve, since they have equations of the form $y=F(x)+C$. For example, $y=\frac{1}{3} x^{3}$ is one integral curve for $f(x)=x^{2}$, so all the other integral curves have equations of the form $y=\frac{1}{3} x^{3}+C$; conversely, the graph of any equation of this form is an integral curve (Figure 5.2.1).

In many problems one is interested in finding a function whose derivative satisfies specified conditions. The following example illustrates a geometric problem of this type.

- Example 5 Suppose that a curve $y=f(x)$ in the $x y$-plane has the property that at each point $(x, y)$ on the curve, the tangent line has slope $x^{2}$. Find an equation for the curve given that it passes through the point $(2,1)$.

Solution. Since the slope of the line tangent to $y=f(x)$ is $d y / d x$, we have $d y / d x=x^{2}$, and

$$
y=\int x^{2} d x=\frac{1}{3} x^{3}+C
$$

In Example 5, the requirement that the graph of $f$ pass through the point $(2,1)$ selects the single integral curve $y=\frac{1}{3} x^{3}-\frac{5}{3}$ from the family of curves $y=\frac{1}{3} x^{3}+C$ (Figure 5.2.2).

Since the curve passes through $(2,1)$, a specific value for $C$ can be found by using the fact that $y=1$ if $x=2$. Substituting these values in the above equation yields

$$
1=\frac{1}{3}\left(2^{3}\right)+C \quad \text { or } \quad C=-\frac{5}{3}
$$

so an equation of the curve is

$$
y=\frac{1}{3} x^{3}-\frac{5}{3}
$$

(Figure 5.2.2).

$\Delta$ Figure 5.2.2

## INTEGRATION FROM THE VIEWPOINT OF DIFFERENTIAL EQUATIONS

We will now consider another way of looking at integration that will be useful in our later work. Suppose that $f(x)$ is a known function and we are interested in finding a function $F(x)$ such that $y=F(x)$ satisfies the equation

$$
\begin{equation*}
\frac{d y}{d x}=f(x) \tag{8}
\end{equation*}
$$

The solutions of this equation are the antiderivatives of $f(x)$, and we know that these can be obtained by integrating $f(x)$. For example, the solutions of the equation

$$
\begin{equation*}
\frac{d y}{d x}=x^{2} \tag{9}
\end{equation*}
$$

are

$$
y=\int x^{2} d x=\frac{x^{3}}{3}+C
$$

Equation (8) is called a differential equation because it involves a derivative of an unknown function. Differential equations are different from the kinds of equations we have encountered so far in that the unknown is a function and not a number as in an equation such as $x^{2}+5 x-6=0$.

Sometimes we will not be interested in finding all of the solutions of (8), but rather we will want only the solution whose graph passes through a specified point $\left(x_{0}, y_{0}\right)$. For example, in Example 5 we solved (9) for the integral curve that passed through the point $(2,1)$.

For simplicity, it is common in the study of differential equations to denote a solution of $d y / d x=f(x)$ as $y(x)$ rather than $F(x)$, as earlier. With this notation, the problem of finding a function $y(x)$ whose derivative is $f(x)$ and whose graph passes through the point $\left(x_{0}, y_{0}\right)$ is expressed as

$$
\begin{equation*}
\frac{d y}{d x}=f(x), \quad y\left(x_{0}\right)=y_{0} \tag{10}
\end{equation*}
$$

This is called an initial-value problem, and the requirement that $y\left(x_{0}\right)=y_{0}$ is called the initial condition for the problem.

- Example 6 Solve the initial-value problem

$$
\frac{d y}{d x}=\cos x, \quad y(0)=1
$$

Solution. The solution of the differential equation is

$$
\begin{equation*}
y=\int \cos x d x=\sin x+C \tag{11}
\end{equation*}
$$

The initial condition $y(0)=1$ implies that $y=1$ if $x=0$; substituting these values in (11) yields

$$
1=\sin (0)+C \quad \text { or } \quad C=1
$$

Thus, the solution of the initial-value problem is $y=\sin x+1$.

## SLOPE FIELDS

If we interpret $d y / d x$ as the slope of a tangent line, then at a point $(x, y)$ on an integral curve of the equation $d y / d x=f(x)$, the slope of the tangent line is $f(x)$. What is interesting about this is that the slopes of the tangent lines to the integral curves can be obtained without actually solving the differential equation. For example, if

$$
\frac{d y}{d x}=\sqrt{x^{2}+1}
$$

then we know without solving the equation that at the point where $x=1$ the tangent line to an integral curve has slope $\sqrt{1^{2}+1}=\sqrt{2}$; and more generally, at a point where $x=a$, the tangent line to an integral curve has slope $\sqrt{a^{2}+1}$.

A geometric description of the integral curves of a differential equation $d y / d x=f(x)$ can be obtained by choosing a rectangular grid of points in the $x y$-plane, calculating the slopes of the tangent lines to the integral curves at the gridpoints, and drawing small portions of the tangent lines through those points. The resulting picture, which is called a slope field or direction field for the equation, shows the "direction" of the integral curves at the gridpoints. With sufficiently many gridpoints it is often possible to visualize the integral curves themselves; for example, Figure 5.2.3a shows a slope field for the differential equation $d y / d x=x^{2}$, and Figure $5.2 .3 b$ shows that same field with the integral curves imposed on it-the more gridpoints that are used, the more completely the slope field reveals the shape of the integral curves. However, the amount of computation can be considerable, so computers are usually used when slope fields with many gridpoints are needed.

Slope fields will be studied in more detail later in the text.

(a)

(b)

## QUICK CHECK EXERCISES 5.2 (See page 332 for answers.)

1. A function $F$ is an antiderivative of a function $f$ on an interval if $\qquad$ for all $x$ in the interval.
2. Write an equivalent integration formula for each given derivative formula.
(a) $\frac{d}{d x}[\sqrt{x}]=\frac{1}{2 \sqrt{x}}$
(b) $\frac{d}{d x}\left[e^{4 x}\right]=4 e^{4 x}$
3. Evaluate the integrals.
(a) $\int\left[x^{3}+x+5\right] d x$
(b) $\int\left[\sec ^{2} x-\csc x \cot x\right] d x$
4. The graph of $y=x^{2}+x$ is an integral curve for the func-
tion $f(x)=$ $\qquad$ If $G$ is a function whose graph is also an integral curve for $f$, and if $G(1)=5$, then $G(x)=$ $\qquad$
5. A slope field for the differential equation

$$
\frac{d y}{d x}=\frac{2 x}{x^{2}-4}
$$

has a line segment with slope $\qquad$ through the point $(0,5)$ and has a line segment with slope $\qquad$ through the point $(-4,1)$.

1. In each part, confirm that the formula is correct, and state a corresponding integration formula.
(a) $\frac{d}{d x}\left[\sqrt{1+x^{2}}\right]=\frac{x}{\sqrt{1+x^{2}}}$
(b) $\frac{d}{d x}\left[x e^{x}\right]=(x+1) e^{x}$
2. In each part, confirm that the stated formula is correct by differentiating.
(a) $\int x \sin x d x=\sin x-x \cos x+C$
(b) $\int \frac{d x}{\left(1-x^{2}\right)^{3 / 2}}=\frac{x}{\sqrt{1-x^{2}}}+C$

## FOCUS ON CONCEPTS

3. What is a constant of integration? Why does an answer to an integration problem involve a constant of integration?
4. What is an integral curve of a function $f$ ? How are two integral curves of a function $f$ related?

5-8 Find the derivative and state a corresponding integration formula.
5. $\frac{d}{d x}\left[\sqrt{x^{3}+5}\right]$
6. $\frac{d}{d x}\left[\frac{x}{x^{2}+3}\right]$
7. $\frac{d}{d x}[\sin (2 \sqrt{x})]$
8. $\frac{d}{d x}[\sin x-x \cos x]$

9-10 Evaluate the integral by rewriting the integrand appropriately, if required, and applying the power rule (Formula 2 in Table 5.2.1).
9. (a) $\int x^{8} d x$
(b) $\int x^{5 / 7} d x$
(c) $\int x^{3} \sqrt{x} d x$
10. (a) $\int \sqrt[3]{x^{2}} d x$
(b) $\int \frac{1}{x^{6}} d x$
(c) $\int x^{-7 / 8} d x$

11-14 Evaluate each integral by applying Theorem 5.2.3 and Formula 2 in Table 5.2.1 appropriately.
11. $\int\left[5 x+\frac{2}{3 x^{5}}\right] d x$
12. $\int\left[x^{-1 / 2}-3 x^{7 / 5}+\frac{1}{9}\right] d x$
13. $\int\left[x^{-3}-3 x^{1 / 4}+8 x^{2}\right] d x$
14. $\int\left[\frac{10}{y^{3 / 4}}-\sqrt[3]{y}+\frac{4}{\sqrt{y}}\right] d y$

15-34 Evaluate the integral and check your answer by differentiating.
15. $\int x\left(1+x^{3}\right) d x$
16. $\int\left(2+y^{2}\right)^{2} d y$
17. $\int x^{1 / 3}(2-x)^{2} d x$
18. $\int\left(1+x^{2}\right)(2-x) d x$
19. $\int \frac{x^{5}+2 x^{2}-1}{x^{4}} d x$
20. $\int \frac{1-2 t^{3}}{t^{3}} d t$
21. $\int\left[\frac{2}{x}+3 e^{x}\right] d x$
22. $\int\left[\frac{1}{2 t}-\sqrt{2} e^{t}\right] d t$
23. $\int\left[3 \sin x-2 \sec ^{2} x\right] d x$
24. $\int\left[\csc ^{2} t-\sec t \tan t\right] d t$
25. $\int \sec x(\sec x+\tan x) d x$
26. $\int \csc x(\sin x+\cot x) d x$
27. $\int \frac{\sec \theta}{\cos \theta} d \theta$
28. $\int \frac{d y}{\csc y}$
29. $\int \frac{\sin x}{\cos ^{2} x} d x$
30. $\int\left[\phi+\frac{2}{\sin ^{2} \phi}\right] d \phi$
31. $\int\left[1+\sin ^{2} \theta \csc \theta\right] d \theta$
32. $\int \frac{\sec x+\cos x}{2 \cos x} d x$
33. $\int\left[\frac{1}{2 \sqrt{1-x^{2}}}-\frac{3}{1+x^{2}}\right] d x$
34. $\int\left[\frac{4}{x \sqrt{x^{2}-1}}+\frac{1+x+x^{3}}{1+x^{2}}\right] d x$
35. Evaluate the integral

$$
\int \frac{1}{1+\sin x} d x
$$

by multiplying the numerator and denominator by an appropriate expression.
36. Use the double-angle formula $\cos 2 x=2 \cos ^{2} x-1$ to evaluate the integral

$$
\int \frac{1}{1+\cos 2 x} d x
$$

37-40 True-False Determine whether the statement is true or false. Explain your answer.
37. If $F(x)$ is an antiderivative of $f(x)$, then

$$
\int f(x) d x=F(x)+C
$$

38. If $C$ denotes a constant of integration, the two formulas

$$
\begin{aligned}
& \int \cos x d x=\sin x+C \\
& \int \cos x d x=(\sin x+\pi)+C
\end{aligned}
$$

are both correct equations.
39. The function $f(x)=e^{-x}+1$ is a solution to the initialvalue problem

$$
\frac{d y}{d x}=-\frac{1}{e^{x}}, \quad y(0)=1
$$

40. Every integral curve of the slope field

$$
\frac{d y}{d x}=\frac{1}{\sqrt{x^{2}+1}}
$$

is the graph of an increasing function of $x$.41. Use a graphing utility to generate some representative integral curves of the function $f(x)=5 x^{4}-\sec ^{2} x$ over the interval $(-\pi / 2, \pi / 2)$.42. Use a graphing utility to generate some representative integral curves of the function $f(x)=(x-1) / x$ over the interval $(0,5)$.

43-46 Solve the initial-value problems.
43. (a) $\frac{d y}{d x}=\sqrt[3]{x}, y(1)=2$
(b) $\frac{d y}{d t}=\sin t+1, y\left(\frac{\pi}{3}\right)=\frac{1}{2}$
(c) $\frac{d y}{d x}=\frac{x+1}{\sqrt{x}}, y(1)=0$
44. (a) $\frac{d y}{d x}=\frac{1}{(2 x)^{3}}, y(1)=0$
(b) $\frac{d y}{d t}=\sec ^{2} t-\sin t, y\left(\frac{\pi}{4}\right)=1$
(c) $\frac{d y}{d x}=x^{2} \sqrt{x^{3}}, y(0)=0$
45. (a) $\frac{d y}{d x}=4 e^{x}, y(0)=1 \quad$ (b) $\frac{d y}{d t}=\frac{1}{t}, y(-1)=5$
46. (a) $\frac{d y}{d t}=\frac{3}{\sqrt{1-t^{2}}}, y\left(\frac{\sqrt{3}}{2}\right)=0$
(b) $\frac{d y}{d x}=\frac{x^{2}-1}{x^{2}+1}, y(1)=\frac{\pi}{2}$

47-50 A particle moves along an $s$-axis with position function $s=s(t)$ and velocity function $v(t)=s^{\prime}(t)$. Use the given information to find $s(t)$.
47. $v(t)=32 t ; \quad s(0)=20$
48. $v(t)=\cos t ; \quad s(0)=2$
49. $v(t)=3 \sqrt{t} ; \quad s(4)=1$
50. $v(t)=3 e^{t} ; \quad s(1)=0$
51. Find the general form of a function whose second derivative is $\sqrt{x}$. [Hint: Solve the equation $f^{\prime \prime}(x)=\sqrt{x}$ for $f(x)$ by integrating both sides twice.]
52. Find a function $f$ such that $f^{\prime \prime}(x)=x+\cos x$ and such that $f(0)=1$ and $f^{\prime}(0)=2$. [Hint: Integrate both sides of the equation twice.]

53-57 Find an equation of the curve that satisfies the given conditions.
53. At each point $(x, y)$ on the curve the slope is $2 x+1$; the curve passes through the point $(-3,0)$.
54. At each point $(x, y)$ on the curve the slope is $(x+1)^{2}$; the curve passes through the point $(-2,8)$.
55. At each point $(x, y)$ on the curve the slope is $-\sin x$; the curve passes through the point $(0,2)$.
56. At each point $(x, y)$ on the curve the slope equals the square of the distance between the point and the $y$-axis; the point $(-1,2)$ is on the curve.
57. At each point $(x, y)$ on the curve, $y$ satisfies the condition $d^{2} y / d x^{2}=6 x$; the line $y=5-3 x$ is tangent to the curve at the point where $x=1$.
c 58. In each part, use a CAS to solve the initial-value problem.
(a) $\frac{d y}{d x}=x^{2} \cos 3 x, y(\pi / 2)=-1$
(b) $\frac{d y}{d x}=\frac{x^{3}}{\left(4+x^{2}\right)^{3 / 2}}, y(0)=-2$
59. (a) Use a graphing utility to generate a slope field for the differential equation $d y / d x=x$ in the region $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$.
(b) Graph some representative integral curves of the function $f(x)=x$.
(c) Find an equation for the integral curve that passes through the point $(2,1)$.
60. (a) Use a graphing utility to generate a slope field for the differential equation $d y / d x=e^{x} / 2$ in the region $-1 \leq x \leq 4$ and $-1 \leq y \leq 4$.
(b) Graph some representative integral curves of the function $f(x)=e^{x} / 2$.
(c) Find an equation for the integral curve that passes through the point $(0,1)$.

61-64 The given slope field figure corresponds to one of the differential equations below. Identify the differential equation that matches the figure, and sketch solution curves through the highlighted points.
(a) $\frac{d y}{d x}=2$
(b) $\frac{d y}{d x}=-x$
(c) $\frac{d y}{d x}=x^{2}-4$
(d) $\frac{d y}{d x}=e^{x / 3}$
61.

62.

63.

64.


## FOCUS ON CONCEPTS

65. Critique the following "proof" that an arbitrary constant must be zero:

$$
C=\int 0 d x=\int 0 \cdot 0 d x=0 \int 0 d x=0
$$

66. Critique the following "proof" that an arbitrary constant must be zero:

$$
\begin{aligned}
0 & =\left(\int x d x\right)-\left(\int x d x\right) \\
& =\int(x-x) d x=\int 0 d x=C
\end{aligned}
$$

67. (a) Show that

$$
F(x)=\tan ^{-1} x \quad \text { and } \quad G(x)=-\tan ^{-1}(1 / x)
$$

differ by a constant on the interval $(0,+\infty)$ by showing that they are antiderivatives of the same function.
(b) Find the constant $C$ such that $F(x)-G(x)=C$ by evaluating the functions $F(x)$ and $G(x)$ at a particular value of $x$.
(c) Check your answer to part (b) by using trigonometric identities.
68. Let $F$ and $G$ be the functions defined by

$$
F(x)=\frac{x^{2}+3 x}{x} \quad \text { and } \quad G(x)= \begin{cases}x+3, & x>0 \\ x, & x<0\end{cases}
$$

(a) Show that $F$ and $G$ have the same derivative.
(b) Show that $G(x) \neq F(x)+C$ for any constant $C$.
(c) Do parts (a) and (b) contradict Theorem 5.2.2? Explain.

69-70 Use a trigonometric identity to evaluate the integral.
69. $\int \tan ^{2} x d x$
70. $\int \cot ^{2} x d x$
71. Use the identities $\cos 2 \theta=1-2 \sin ^{2} \theta=2 \cos ^{2} \theta-1$ to help evaluate the integrals
(a) $\int \sin ^{2}(x / 2) d x$
(b) $\int \cos ^{2}(x / 2) d x$
72. Recall that

$$
\frac{d}{d x}\left[\sec ^{-1} x\right]=\frac{1}{|x| \sqrt{x^{2}-1}}
$$

Use this to verify Formula 14 in Table 5.2.1.
73. The speed of sound in air at $0^{\circ} \mathrm{C}$ (or 273 K on the Kelvin scale) is $1087 \mathrm{ft} / \mathrm{s}$, but the speed $v$ increases as the temperature $T$ rises. Experimentation has shown that the rate of change of $v$ with respect to $T$ is

$$
\frac{d v}{d T}=\frac{1087}{2 \sqrt{273}} T^{-1 / 2}
$$

where $v$ is in feet per second and $T$ is in kelvins (K). Find a formula that expresses $v$ as a function of $T$.
74. Suppose that a uniform metal rod 50 cm long is insulated laterally, and the temperatures at the exposed ends are maintained at $25^{\circ} \mathrm{C}$ and $85^{\circ} \mathrm{C}$, respectively. Assume that an $x$ axis is chosen as in the accompanying figure and that the temperature $T(x)$ satisfies the equation

$$
\frac{d^{2} T}{d x^{2}}=0
$$

Find $T(x)$ for $0 \leq x \leq 50$.

75. Writing What is an initial-value problem? Describe the sequence of steps for solving an initial-value problem.
76. Writing What is a slope field? How are slope fields and integral curves related?

## QUICK CHECK ANSWERS 5.2

1. $F^{\prime}(x)=f(x)$
2. (a) $\int \frac{1}{2 \sqrt{x}} d x=\sqrt{x}+C$
(b) $\int 4 e^{4 x} d x=e^{4 x}+C$
3. (a) $\frac{1}{4} x^{4}+\frac{1}{2} x^{2}+5 x+C$
(b) $\tan x+\csc x+C$
4. $2 x+1 ; x^{2}+x+3$
5. $0 ;-\frac{2}{3}$

### 5.3 INTEGRATION BY SUBSTITUTION

In this section we will study a technique, called substitution, that can often be used to transform complicated integration problems into simpler ones.

## u-SUBSTITUTION

The method of substitution can be motivated by examining the chain rule from the viewpoint of antidifferentiation. For this purpose, suppose that $F$ is an antiderivative of $f$ and that $g$ is a differentiable function. The chain rule implies that the derivative of $F(g(x))$ can be expressed as

$$
\frac{d}{d x}[F(g(x))]=F^{\prime}(g(x)) g^{\prime}(x)
$$

which we can write in integral form as

$$
\begin{equation*}
\int F^{\prime}(g(x)) g^{\prime}(x) d x=F(g(x))+C \tag{1}
\end{equation*}
$$

or since $F$ is an antiderivative of $f$,

$$
\begin{equation*}
\int f(g(x)) g^{\prime}(x) d x=F(g(x))+C \tag{2}
\end{equation*}
$$

For our purposes it will be useful to let $u=g(x)$ and to write $d u / d x=g^{\prime}(x)$ in the differential form $d u=g^{\prime}(x) d x$. With this notation (2) can be expressed as

$$
\begin{equation*}
\int f(u) d u=F(u)+C \tag{3}
\end{equation*}
$$

The process of evaluating an integral of form (2) by converting it into form (3) with the substitution

$$
u=g(x) \quad \text { and } \quad d u=g^{\prime}(x) d x
$$

is called the method of $\boldsymbol{u}$-substitution. Here our emphasis is not on the interpretation of the expression $d u=g^{\prime}(x) d x$. Rather, the differential notation serves primarily as a useful "bookkeeping" device for the method of $u$-substitution. The following example illustrates how the method works.

Example 1 Evaluate $\int\left(x^{2}+1\right)^{50} \cdot 2 x d x$.
Solution. If we let $u=x^{2}+1$, then $d u / d x=2 x$, which implies that $d u=2 x d x$. Thus, the given integral can be written as

$$
\int\left(x^{2}+1\right)^{50} \cdot 2 x d x=\int u^{50} d u=\frac{u^{51}}{51}+C=\frac{\left(x^{2}+1\right)^{51}}{51}+C
$$

It is important to realize that in the method of $u$-substitution you have control over the choice of $u$, but once you make that choice you have no control over the resulting expression for $d u$. Thus, in the last example we chose $u=x^{2}+1$ but $d u=2 x d x$ was computed. Fortunately, our choice of $u$, combined with the computed $d u$, worked out perfectly to produce an integral involving $u$ that was easy to evaluate. However, in general, the method of $u$-substitution will fail if the chosen $u$ and the computed $d u$ cannot be used to produce an integrand in which no expressions involving $x$ remain, or if you cannot evaluate the resulting integral. Thus, for example, the substitution $u=x^{2}, d u=2 x d x$ will not work for the integral

$$
\int 2 x \sin x^{4} d x
$$

because this substitution results in the integral

$$
\int \sin u^{2} d u
$$

which still cannot be evaluated in terms of familiar functions.
In general, there are no hard and fast rules for choosing $u$, and in some problems no choice of $u$ will work. In such cases other methods need to be used, some of which will be discussed later. Making appropriate choices for $u$ will come with experience, but you may find the following guidelines, combined with a mastery of the basic integrals in Table 5.2.1, helpful.

## Guidelines for u-Substitution

Step 1. Look for some composition $f(g(x))$ within the integrand for which the substitution

$$
u=g(x), \quad d u=g^{\prime}(x) d x
$$

produces an integral that is expressed entirely in terms of $u$ and its differential $d u$. This may or may not be possible.

Step 2. If you are successful in Step 1, then try to evaluate the resulting integral in terms of $u$. Again, this may or may not be possible.

Step 3. If you are successful in Step 2, then replace $u$ by $g(x)$ to express your final answer in terms of $x$.

## EASY TO RECOGNIZE SUBSTITUTIONS

The easiest substitutions occur when the integrand is the derivative of a known function, except for a constant added to or subtracted from the independent variable.

## - Example 2

$$
\begin{gathered}
\int \sin (x+9) d x=\int \sin u d u=-\cos u+C=-\cos (x+9)+C \\
\begin{array}{c}
u=x+9 \\
d u=1 \cdot d x=d x
\end{array} \\
\int(x-8)^{23} d x=\int u^{23} d u=\frac{u^{24}}{24}+C=\frac{(x-8)^{24}}{24}+C \\
\begin{array}{c}
u=x-8 \\
d u=1 \cdot d x=d x
\end{array}
\end{gathered}
$$

Another easy $u$-substitution occurs when the integrand is the derivative of a known function, except for a constant that multiplies or divides the independent variable. The following example illustrates two ways to evaluate such integrals.

Example 3 Evaluate $\int \cos 5 x d x$.

## Solution.

$$
\begin{gathered}
\int \cos 5 x d x=\int(\cos u) \cdot \frac{1}{5} d u=\frac{1}{5} \int \cos u d u=\frac{1}{5} \sin u+C=\frac{1}{5} \sin 5 x+C \\
\begin{array}{c}
u=5 x \\
d u=5 d x \text { or } d x=\frac{1}{5} d u
\end{array} \\
\end{gathered}
$$

Alternative Solution. There is a variation of the preceding method that some people prefer. The substitution $u=5 x$ requires $d u=5 d x$. If there were a factor of 5 in the integrand, then we could group the 5 and $d x$ together to form the $d u$ required by the substitution. Since there is no factor of 5 , we will insert one and compensate by putting a factor of $\frac{1}{5}$ in front of the integral. The computations are as follows:

$$
\begin{aligned}
& \int \cos 5 x d x=\frac{1}{5} \int \cos 5 x \cdot 5 d x=\frac{1}{5} \int \cos u d u=\frac{1}{5} \sin u+C=\frac{1}{5} \sin 5 x+C \\
& \begin{aligned}
u & =5 x \\
d u & =5 d x
\end{aligned}
\end{aligned}
$$

More generally, if the integrand is a composition of the form $f(a x+b)$, where $f(x)$ is an easy to integrate function, then the substitution $u=a x+b, d u=a d x$ will work.

## - Example 4

$$
\begin{gathered}
\int \frac{d x}{\left(\frac{1}{3} x-8\right)^{5}}=\int \frac{3 d u}{u^{5}}=3 \int u^{-5} d u=-\frac{3}{4} u^{-4}+C=-\frac{3}{4}\left(\frac{1}{3} x-8\right)^{-4}+C \\
\begin{array}{c}
u=\frac{1}{3} x-8 \\
d u=\frac{1}{3} d x \text { or } d x=3 d u
\end{array}
\end{gathered}
$$

Example 5 Evaluate $\int \frac{d x}{1+3 x^{2}}$.
Solution. Substituting

$$
u=\sqrt{3} x, \quad d u=\sqrt{3} d x
$$

yields

$$
\int \frac{d x}{1+3 x^{2}}=\frac{1}{\sqrt{3}} \int \frac{d u}{1+u^{2}}=\frac{1}{\sqrt{3}} \tan ^{-1} u+C=\frac{1}{\sqrt{3}} \tan ^{-1}(\sqrt{3} x)+C
$$

With the help of Theorem 5.2.3, a complicated integral can sometimes be computed by expressing it as a sum of simpler integrals.

## Example 6

$$
\begin{aligned}
& \int\left(\frac{1}{x}+\sec ^{2} \pi x\right) d x=\int \frac{d x}{x}+\int \sec ^{2} \pi x d x \\
&=\ln |x|+\int \sec ^{2} \pi x d x \\
&=\ln |x|+\frac{1}{\pi} \int \sec ^{2} u d u \\
& \begin{array}{c}
u=\pi x \\
d u=\pi d x \text { or } d x=\frac{1}{\pi} d u
\end{array} \\
&=\ln |x|+\frac{1}{\pi} \tan u+C=\ln |x|+\frac{1}{\pi} \tan \pi x+C
\end{aligned}
$$

The next four examples illustrate a substitution $u=g(x)$ where $g(x)$ is a nonlinear function.

Example 7 Evaluate $\int \sin ^{2} x \cos x d x$.
Solution. If we let $u=\sin x$, then

$$
\frac{d u}{d x}=\cos x, \quad \text { so } \quad d u=\cos x d x
$$

Thus,

$$
\int \sin ^{2} x \cos x d x=\int u^{2} d u=\frac{u^{3}}{3}+C=\frac{\sin ^{3} x}{3}+C
$$

$\overline{\text { Example } 8}$ Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} d x$.
Solution. If we let $u=\sqrt{x}$, then

$$
\frac{d u}{d x}=\frac{1}{2 \sqrt{x}}, \quad \text { so } \quad d u=\frac{1}{2 \sqrt{x}} d x \quad \text { or } \quad 2 d u=\frac{1}{\sqrt{x}} d x
$$

Thus,

$$
\int \frac{e^{\sqrt{x}}}{\sqrt{x}} d x=\int 2 e^{u} d u=2 \int e^{u} d u=2 e^{u}+C=2 e^{\sqrt{x}}+C
$$

Example 9 Evaluate $\int t^{4} \sqrt[3]{3-5 t^{5}} d t$
Solution.

$$
\begin{aligned}
\int t^{4} \sqrt[3]{3-5 t^{5}} d t & =-\frac{1}{25} \int \sqrt[3]{u} d u=-\frac{1}{25} \int u^{1 / 3} d u \\
& =-\frac{1}{25} \frac{u^{4 / 3}}{4 / 3}+C=-\frac{3}{100}\left(3-5 t^{5}\right)^{4 / 3}+C
\end{aligned}
$$

$\overline{-}$ Example 10 Evaluate $\int \frac{e^{x}}{\sqrt{1-e^{2 x}}} d x$
Solution. Substituting

$$
u=e^{x}, \quad d u=e^{x} d x
$$

yields

$$
\int \frac{e^{x}}{\sqrt{1-e^{2 x}}} d x=\int \frac{d u}{\sqrt{1-u^{2}}}=\sin ^{-1} u+C=\sin ^{-1}\left(e^{x}\right)+C
$$

## LESS APPARENT SUBSTITUTIONS

The method of substitution is relatively straightforward, provided the integrand contains an easily recognized composition $f(g(x))$ and the remainder of the integrand is a constant multiple of $g^{\prime}(x)$. If this is not the case, the method may still apply but may require more computation.

Example 11 Evaluate $\int x^{2} \sqrt{x-1} d x$
Solution. The composition $\sqrt{x-1}$ suggests the substitution

$$
\begin{equation*}
u=x-1 \quad \text { so that } \quad d u=d x \tag{4}
\end{equation*}
$$

From the first equality in (4)

$$
x^{2}=(u+1)^{2}=u^{2}+2 u+1
$$

so that

$$
\begin{aligned}
\int x^{2} \sqrt{x-1} d x & =\int\left(u^{2}+2 u+1\right) \sqrt{u} d u=\int\left(u^{5 / 2}+2 u^{3 / 2}+u^{1 / 2}\right) d u \\
& =\frac{2}{7} u^{7 / 2}+\frac{4}{5} u^{5 / 2}+\frac{2}{3} u^{3 / 2}+C \\
& =\frac{2}{7}(x-1)^{7 / 2}+\frac{4}{5}(x-1)^{5 / 2}+\frac{2}{3}(x-1)^{3 / 2}+C
\end{aligned}
$$

Example 12 Evaluate $\int \cos ^{3} x d x$
Solution. The only compositions in the integrand that suggest themselves are

$$
\cos ^{3} x=(\cos x)^{3} \quad \text { and } \quad \cos ^{2} x=(\cos x)^{2}
$$

However, neither the substitution $u=\cos x$ nor the substitution $u=\cos ^{2} x$ work (verify). In this case, an appropriate substitution is not suggested by the composition contained in the integrand. On the other hand, note from Equation (2) that the derivative $g^{\prime}(x)$ appears as a factor in the integrand. This suggests that we write

$$
\int \cos ^{3} x d x=\int \cos ^{2} x \cos x d x
$$

and solve the equation $d u=\cos x d x$ for $u=\sin x$. Since $\sin ^{2} x+\cos ^{2} x=1$, we then have

$$
\begin{aligned}
\int \cos ^{3} x d x & =\int \cos ^{2} x \cos x d x=\int\left(1-\sin ^{2} x\right) \cos x d x=\int\left(1-u^{2}\right) d u \\
& =u-\frac{u^{3}}{3}+C=\sin x-\frac{1}{3} \sin ^{3} x+C
\end{aligned}
$$

Example 13 Evaluate $\int \frac{d x}{a^{2}+x^{2}} d x$, where $a \neq 0$ is a constant.
Solution. Some simple algebra and an appropriate $u$-substitution will allow us to use Formula 12 in Table 5.2.1.

$$
\begin{aligned}
\int \frac{d x}{a^{2}+x^{2}} & =\int \frac{a(d x / a)}{a^{2}\left(1+(x / a)^{2}\right)}=\frac{1}{a} \int \frac{d x / a}{1+(x / a)^{2}} \quad \begin{array}{c}
u=x / a \\
d u=d x / a
\end{array} \\
& =\frac{1}{a} \int \frac{d u}{1+u^{2}}=\frac{1}{a} \tan ^{-1} u+C=\frac{1}{a} \tan ^{-1} \frac{x}{a}+C
\end{aligned}
$$

The method of Example 13 leads to the following generalizations of Formulas 12, 13, and 14 in Table 5.2.1 for $a>0$ :

$$
\begin{align*}
& \int \frac{d u}{a^{2}+u^{2}}=\frac{1}{a} \tan ^{-1} \frac{u}{a}+C  \tag{5}\\
& \int \frac{d u}{\sqrt{a^{2}-u^{2}}}=\sin ^{-1} \frac{u}{a}+C  \tag{6}\\
& \int \frac{d u}{u \sqrt{u^{2}-a^{2}}}=\frac{1}{a} \sec ^{-1}\left|\frac{u}{a}\right|+C \tag{7}
\end{align*}
$$

$\overline{-}$ Example 14 Evaluate $\int \frac{d x}{\sqrt{2-x^{2}}}$.
Solution. Applying (6) with $u=x$ and $a=\sqrt{2}$ yields

$$
\int \frac{d x}{\sqrt{2-x^{2}}}=\sin ^{-1} \frac{x}{\sqrt{2}}+C
$$

## TECHNOLOGY MASTERY

If you have a CAS, use it to calculate the integrals in the examples in this section. If your CAS produces an answer that is different from the one in the text, then confirm algebraically that the two answers agree. Also, explore the effect of using the CAS to simplify the expressions it produces for the integrals.

## INTEGRATION USING COMPUTER ALGEBRA SYSTEMS

The advent of computer algebra systems has made it possible to evaluate many kinds of integrals that would be laborious to evaluate by hand. For example, a handheld calculator evaluated the integral

$$
\int \frac{5 x^{2}}{(1+x)^{1 / 3}} d x=\frac{3(x+1)^{2 / 3}\left(5 x^{2}-6 x+9\right)}{8}+C
$$

in about a second. The computer algebra system Mathematica, running on a personal computer, required even less time to evaluate this same integral. However, just as one would not want to rely on a calculator to compute $2+2$, so one would not want to use a CAS to integrate a simple function such as $f(x)=x^{2}$. Thus, even if you have a CAS, you will want to develop a reasonable level of competence in evaluating basic integrals. Moreover, the mathematical techniques that we will introduce for evaluating basic integrals are precisely the techniques that computer algebra systems use to evaluate more complicated integrals.

## QUICK CHECK EXERCISES 5.3 (See page 340 for answers.)

1. Indicate the $u$-substitution.
(a) $\int 3 x^{2}\left(1+x^{3}\right)^{25} d x=\int u^{25} d u \quad$ if $u=$ $\qquad$ and $d u=$ $\qquad$
(b) $\int 2 x \sin x^{2} d x=\int \sin u d u \quad$ if $u=$ $\qquad$ and $d u=$ $\qquad$
(c) $\int \frac{18 x}{1+9 x^{2}} d x=\int \frac{1}{u} d u$ if $u=$ $\qquad$ and $d u=$
(d) $\int \frac{3}{1+9 x^{2}} d x=\int \frac{1}{1+u^{2}} d u$ if $u=$ $\qquad$ and $d u=$ $\qquad$
2. Supply the missing integrand corresponding to the indicated $u$-substitution.
(a) $\int 5(5 x-3)^{-1 / 3} d x=\int \longrightarrow d u ; u=5 x-3$
(b) $\int(3-\tan x) \sec ^{2} x d x=\int \square d u$;

$$
u=3-\tan x
$$

(c) $\int \frac{\sqrt[3]{8+\sqrt{x}}}{\sqrt{x}} d x=\int \longrightarrow d u ; u=8+\sqrt{x}$
(d) $\int e^{3 x} d x=\int \longrightarrow d u ; u=3 x$

1-12 Evaluate the integrals using the indicated substitutions.

1. (a) $\int 2 x\left(x^{2}+1\right)^{23} d x ; u=x^{2}+1$
(b) $\int \cos ^{3} x \sin x d x ; u=\cos x$
2. (a) $\int \frac{1}{\sqrt{x}} \sin \sqrt{x} d x ; u=\sqrt{x}$
(b) $\int \frac{3 x d x}{\sqrt{4 x^{2}+5}} ; u=4 x^{2}+5$
3. (a) $\int \sec ^{2}(4 x+1) d x ; u=4 x+1$
(b) $\int y \sqrt{1+2 y^{2}} d y ; u=1+2 y^{2}$
4. (a) $\int \sqrt{\sin \pi \theta} \cos \pi \theta d \theta ; u=\sin \pi \theta$
(b) $\int(2 x+7)\left(x^{2}+7 x+3\right)^{4 / 5} d x ; u=x^{2}+7 x+3$
5. (a) $\int \cot x \csc ^{2} x d x ; u=\cot x$
(b) $\int(1+\sin t)^{9} \cos t d t ; u=1+\sin t$
6. (a) $\int \cos 2 x d x ; u=2 x \quad$ (b) $\int x \sec ^{2} x^{2} d x ; u=x^{2}$
7. (a) $\int x^{2} \sqrt{1+x} d x ; u=1+x$
(b) $\int[\csc (\sin x)]^{2} \cos x d x ; u=\sin x$
8. (a) $\int \sin (x-\pi) d x ; u=x-\pi$
(b) $\int \frac{5 x^{4}}{\left(x^{5}+1\right)^{2}} d x ; u=x^{5}+1$
9. (a) $\int \frac{d x}{x \ln x} ; u=\ln x$
(b) $\int e^{-5 x} d x ; u=-5 x$
10. (a) $\int \frac{\sin 3 \theta}{1+\cos 3 \theta} d \theta ; u=1+\cos 3 \theta$
(b) $\int \frac{e^{x}}{1+e^{x}} d x ; u=1+e^{x}$
11. (a) $\int \frac{x^{2} d x}{1+x^{6}} ; u=x^{3}$
(b) $\int \frac{d x}{x \sqrt{1-(\ln x)^{2}}} ; u=\ln x$
12. (a) $\int \frac{d x}{x \sqrt{9 x^{2}-1}} ; u=3 x$
(b) $\int \frac{d x}{\sqrt{x}(1+x)} ; u=\sqrt{x}$

## FOCUS ON CONCEPTS

13. Explain the connection between the chain rule for differentiation and the method of $u$-substitution for integration.
14. Explain how the substitution $u=a x+b$ helps to perform an integration in which the integrand is $f(a x+b)$, where $f(x)$ is an easy to integrate function.

15-56 Evaluate the integrals using appropriate substitutions.
15. $\int(4 x-3)^{9} d x$
16. $\int x^{3} \sqrt{5+x^{4}} d x$
17. $\int \sin 7 x d x$
18. $\int \cos \frac{x}{3} d x$
19. $\int \sec 4 x \tan 4 x d x$
20. $\int \sec ^{2} 5 x d x$
21. $\int e^{2 x} d x$
23. $\int \frac{d x}{\sqrt{1-4 x^{2}}}$
22. $\int \frac{d x}{2 x}$
25. $\int t \sqrt{7 t^{2}+12} d t$
24. $\int \frac{d x}{1+16 x^{2}}$
27. $\int \frac{6}{(1-2 x)^{3}} d x$
26. $\int \frac{x}{\sqrt{4-5 x^{2}}} d x$
28. $\int \frac{x^{2}+1}{\sqrt{x^{3}+3 x}} d x$
29. $\int \frac{x^{3}}{\left(5 x^{4}+2\right)^{3}} d x$
31. $\int e^{\sin x} \cos x d x$
33. $\int x^{2} e^{-2 x^{3}} d x$
30. $\int \frac{\sin (1 / x)}{3 x^{2}} d x$
32. $\int x^{3} e^{x^{4}} d x$
34. $\int \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}} d x$
35. $\int \frac{e^{x}}{1+e^{2 x}} d x$
36. $\int \frac{t}{t^{4}+1} d t$
37. $\int \frac{\sin (5 / x)}{x^{2}} d x$
38. $\int \frac{\sec ^{2}(\sqrt{x})}{\sqrt{x}} d x$
39. $\int \cos ^{4} 3 t \sin 3 t d t$
40. $\int \cos 2 t \sin ^{5} 2 t d t$
41. $\int x \sec ^{2}\left(x^{2}\right) d x$
42. $\int \frac{\cos 4 \theta}{(1+2 \sin 4 \theta)^{4}} d \theta$
43. $\int \cos 4 \theta \sqrt{2-\sin 4 \theta} d \theta$
44. $\int \tan ^{3} 5 x \sec ^{2} 5 x d x$
45. $\int \frac{\sec ^{2} x d x}{\sqrt{1-\tan ^{2} x}}$
46. $\int \frac{\sin \theta}{\cos ^{2} \theta+1} d \theta$
47. $\int \sec ^{3} 2 x \tan 2 x d x$
48. $\int[\sin (\sin \theta)] \cos \theta d \theta$
49. $\int \frac{d x}{e^{x}}$
50. $\int \sqrt{e^{x}} d x$
51. $\int \frac{d x}{\sqrt{x} e^{(2 \sqrt{x})}}$
52. $\int \frac{e^{\sqrt{2 y+1}}}{\sqrt{2 y+1}} d y$
53. $\int \frac{y}{\sqrt{2 y+1}} d y$
54. $\int x \sqrt{4-x} d x$
55. $\int \sin ^{3} 2 \theta d \theta$
56. $\int \sec ^{4} 3 \theta d \theta$ [Hint: Apply a trigonometric identity.]

57-60 Evaluate each integral by first modifying the form of the integrand and then making an appropriate substitution, if needed.
57. $\int \frac{t+1}{t} d t$
58. $\int e^{2 \ln x} d x$
59. $\int\left[\ln \left(e^{x}\right)+\ln \left(e^{-x}\right)\right] d x$
60. $\int \cot x d x$

61-62 Evaluate the integrals with the aid of Formulas (5), (6), and (7).
61. (a) $\int \frac{d x}{\sqrt{9-x^{2}}}$
(b) $\int \frac{d x}{5+x^{2}}$
(c) $\int \frac{d x}{x \sqrt{x^{2}-\pi}}$
62. (a) $\int \frac{e^{x}}{4+e^{2 x}} d x$
(b) $\int \frac{d x}{\sqrt{9-4 x^{2}}}$
(c) $\int \frac{d y}{y \sqrt{5 y^{2}-3}}$

63-65 Evaluate the integrals assuming that $n$ is a positive integer and $b \neq 0$.
63. $\int(a+b x)^{n} d x$
64. $\int \sqrt[n]{a+b x} d x$
65. $\int \sin ^{n}(a+b x) \cos (a+b x) d x$
66. Use a CAS to check the answers you obtained in Exercises 63-65. If the answer produced by the CAS does not match yours, show that the two answers are equivalent. [Suggestion: Mathematica users may find it helpful to apply the Simplify command to the answer.]

## FOCUS ON CONCEPTS

67. (a) Evaluate the integral $\int \sin x \cos x d x$ by two methods: first by letting $u=\sin x$, and then by letting $u=\cos x$.
(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.
68. (a) Evaluate the integral $\int(5 x-1)^{2} d x$ by two methods: first square and integrate, then let $u=5 x-1$.
(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

69-72 Solve the initial-value problems.
69. $\frac{d y}{d x}=\sqrt{5 x+1}, y(3)=-2$
70. $\frac{d y}{d x}=2+\sin 3 x, y(\pi / 3)=0$
71. $\frac{d y}{d t}=-e^{2 t}, y(0)=6$
72. $\frac{d y}{d t}=\frac{1}{25+9 t^{2}}, \quad y\left(-\frac{5}{3}\right)=\frac{\pi}{30}$73. (a) Evaluate $\int\left[x / \sqrt{x^{2}+1}\right] d x$.
(b) Use a graphing utility to generate some typical integral curves of $f(x)=x / \sqrt{x^{2}+1}$ over the interval $(-5,5)$.74. (a) Evaluate $\int\left[x /\left(x^{2}+1\right)\right] d x$.
(b) Use a graphing utility to generate some typical integral curves of $f(x)=x /\left(x^{2}+1\right)$ over the interval $(-5,5)$.
75. Find a function $f$ such that the slope of the tangent line at a point $(x, y)$ on the curve $y=f(x)$ is $\sqrt{3 x+1}$ and the curve passes through the point $(0,1)$.
76. A population of minnows in a lake is estimated to be 100,000 at the beginning of the year 2005. Suppose that $t$ years after the beginning of 2005 the rate of growth of the population $p(t)$ (in thousands) is given by $p^{\prime}(t)=(3+0.12 t)^{3 / 2}$. Estimate the projected population at the beginning of the year 2010.
77. Derive integration Formula (6).
78. Derive integration Formula (7).
79. Writing If you want to evaluate an integral by $u$-substitution, how do you decide what part of the integrand to choose for $u$ ?
80. Writing The evaluation of an integral can sometimes result in apparently different answers (Exercises 67 and 68). Explain why this occurs and give an example. How might you show that two apparently different answers are actually equivalent?

## QUICK CHECK ANSWERS 5.3

1. (a) $1+x^{3} ; 3 x^{2} d x$
(b) $x^{2} ; 2 x d x$
(c) $1+9 x^{2} ; 18 x d x$
(d) $3 x ; 3 d x$
2. (a) $u^{-1 / 3}$
(b) $-u$
(c) $2 \sqrt[3]{u}$
(d) $\frac{1}{3} e^{u}$

### 5.4 THE DEFINITION OF AREA AS A LIMIT; SIGMA NOTATION

Our main goal in this section is to use the rectangle method to give a precise mathematical definition of the "area under a curve."

## SIGMA NOTATION

To simplify our computations, we will begin by discussing a useful notation for expressing lengthy sums in a compact form. This notation is called sigma notation or summation notation because it uses the uppercase Greek letter $\Sigma$ (sigma) to denote various kinds of sums. To illustrate how this notation works, consider the sum

$$
1^{2}+2^{2}+3^{2}+4^{2}+5^{2}
$$

in which each term is of the form $k^{2}$, where $k$ is one of the integers from 1 to 5 . In sigma notation this sum can be written as

$$
\sum_{k=1}^{5} k^{2}
$$

which is read "the summation of $k^{2}$, where $k$ runs from 1 to 5 ." The notation tells us to form the sum of the terms that result when we substitute successive integers for $k$ in the expression $k^{2}$, starting with $k=1$ and ending with $k=5$.

More generally, if $f(k)$ is a function of $k$, and if $m$ and $n$ are integers such that $m \leq n$, then

$$
\begin{equation*}
\sum_{k=m}^{n} f(k) \tag{1}
\end{equation*}
$$

denotes the sum of the terms that result when we substitute successive integers for $k$, starting with $k=m$ and ending with $k=n$ (Figure 5.4.1).

## Example 1

$$
\begin{aligned}
& \sum_{k=4}^{8} k^{3}=4^{3}+5^{3}+6^{3}+7^{3}+8^{3} \\
& \sum_{k=1}^{5} 2 k=2 \cdot 1+2 \cdot 2+2 \cdot 3+2 \cdot 4+2 \cdot 5=2+4+6+8+10 \\
& \sum_{k=0}^{5}(2 k+1)=1+3+5+7+9+11 \\
& \sum_{k=0}^{5}(-1)^{k}(2 k+1)=1-3+5-7+9-11 \\
& \sum_{k=-3}^{1} k^{3}=(-3)^{3}+(-2)^{3}+(-1)^{3}+0^{3}+1^{3}=-27-8-1+0+1 \\
& \sum_{k=1}^{3} k \sin \left(\frac{k \pi}{5}\right)=\sin \frac{\pi}{5}+2 \sin \frac{2 \pi}{5}+3 \sin \frac{3 \pi}{5}
\end{aligned}
$$

The numbers $m$ and $n$ in (1) are called, respectively, the lower and upper limits of summation; and the letter $k$ is called the index of summation. It is not essential to use $k$ as the index of summation; any letter not reserved for another purpose will do. For example,
all denote the sum

$$
\sum_{i=1}^{6} \frac{1}{i}, \quad \sum_{j=1}^{6} \frac{1}{j}, \quad \text { and } \quad \sum_{n=1}^{6} \frac{1}{n}
$$

$$
1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}
$$

If the upper and lower limits of summation are the same, then the "sum" in (1) reduces to a single term. For example,

$$
\sum_{k=2}^{2} k^{3}=2^{3} \quad \text { and } \quad \sum_{i=1}^{1} \frac{1}{i+2}=\frac{1}{1+2}=\frac{1}{3}
$$

In the sums

$$
\sum_{i=1}^{5} 2 \text { and } \sum_{j=0}^{2} x^{3}
$$

the expression to the right of the $\Sigma$ sign does not involve the index of summation. In such cases, we take all the terms in the sum to be the same, with one term for each allowable value of the summation index. Thus,

$$
\sum_{i=1}^{5} 2=2+2+2+2+2 \text { and } \sum_{j=0}^{2} x^{3}=x^{3}+x^{3}+x^{3}
$$

## CHANGING THE LIMITS OF SUMMATION

A sum can be written in more than one way using sigma notation with different limits of summation and correspondingly different summands. For example,

$$
\sum_{i=1}^{5} 2 i=2+4+6+8+10=\sum_{j=0}^{4}(2 j+2)=\sum_{k=3}^{7}(2 k-4)
$$

On occasion we will want to change the sigma notation for a given sum to a sigma notation with different limits of summation.

## PROPERTIES OF SUMS

When stating general properties of sums it is often convenient to use a subscripted letter such as $a_{k}$ in place of the function notation $f(k)$. For example,

$$
\begin{aligned}
& \sum_{k=1}^{5} a_{k}=a_{1}+a_{2}+a_{3}+a_{4}+a_{5}=\sum_{j=1}^{5} a_{j}=\sum_{k=-1}^{3} a_{k+2} \\
& \sum_{k=1}^{n} a_{k}=a_{1}+a_{2}+\cdots+a_{n}=\sum_{j=1}^{n} a_{j}=\sum_{k=-1}^{n-2} a_{k+2}
\end{aligned}
$$

Our first properties provide some basic rules for manipulating sums.
5.4.1 THEOREM
(a) $\sum_{k=1}^{n} c a_{k}=c \sum_{k=1}^{n} a_{k} \quad$ (if $c$ does not depend on $k$ )
(b) $\sum_{k=1}^{n}\left(a_{k}+b_{k}\right)=\sum_{k=1}^{n} a_{k}+\sum_{k=1}^{n} b_{k}$
(c) $\sum_{k=1}^{n}\left(a_{k}-b_{k}\right)=\sum_{k=1}^{n} a_{k}-\sum_{k=1}^{n} b_{k}$

We will prove parts $(a)$ and $(b)$ and leave part $(c)$ as an exercise.
PROOF (a)

$$
\sum_{k=1}^{n} c a_{k}=c a_{1}+c a_{2}+\cdots+c a_{n}=c\left(a_{1}+a_{2}+\cdots+a_{n}\right)=c \sum_{k=1}^{n} a_{k}
$$

PROOF (b)

$$
\begin{aligned}
\sum_{k=1}^{n}\left(a_{k}+b_{k}\right) & =\left(a_{1}+b_{1}\right)+\left(a_{2}+b_{2}\right)+\cdots+\left(a_{n}+b_{n}\right) \\
& =\left(a_{1}+a_{2}+\cdots+a_{n}\right)+\left(b_{1}+b_{2}+\cdots+b_{n}\right)=\sum_{k=1}^{n} a_{k}+\sum_{k=1}^{n} b_{k}
\end{aligned}
$$

Restating Theorem 5.4.1 in words:
(a) A constant factor can be moved through a sigma sign.
(b) Sigma distributes across sums.
(c) Sigma distributes across differences.

## SUMMATION FORMULAS

The following theorem lists some useful formulas for sums of powers of integers. The derivations of these formulas are given in Appendix D.

## TECHNOLOGY MASTERY

If you have access to a CAS, it will provide a method for finding closed forms such as those in Theorem 5.4.2. Use your CAS to confirm the formulas in that theorem, and then find closed forms for

$$
\sum_{k=1}^{n} k^{4} \text { and } \sum_{k=1}^{n} k^{5}
$$

### 5.4.2 THEOREM

(a) $\sum_{k=1}^{n} k=1+2+\cdots+n=\frac{n(n+1)}{2}$
(b) $\sum_{k=1}^{n} k^{2}=1^{2}+2^{2}+\cdots+n^{2}=\frac{n(n+1)(2 n+1)}{6}$
(c) $\sum_{k=1}^{n} k^{3}=1^{3}+2^{3}+\cdots+n^{3}=\left[\frac{n(n+1)}{2}\right]^{2}$
$\overline{\text { Example } 2}$ Evaluate $\sum_{k=1}^{30} k(k+1)$.

## Solution.

$$
\begin{aligned}
\sum_{k=1}^{30} k(k+1) & =\sum_{k=1}^{30}\left(k^{2}+k\right)=\sum_{k=1}^{30} k^{2}+\sum_{k=1}^{30} k \\
& =\frac{30(31)(61)}{6}+\frac{30(31)}{2}=9920
\end{aligned}
$$

Theorem 5.4.2(a), (b) $<$

In formulas such as

$$
\sum_{k=1}^{n} k=\frac{n(n+1)}{2} \quad \text { or } \quad 1+2+\cdots+n=\frac{n(n+1)}{2}
$$

the left side of the equality is said to express the sum in open form and the right side is said to express it in closed form. The open form indicates the summands and the closed form is an explicit formula for the sum.
$\overline{\text { Example } 3}$ Express $\sum_{k=1}^{n}(3+k)^{2}$ in closed form.
Solution.

$$
\begin{aligned}
\sum_{k=1}^{n}(3+k)^{2} & =4^{2}+5^{2}+\cdots+(3+n)^{2} \\
& =\left[1^{2}+2^{2}+3^{3}+4^{2}+5^{2}+\cdots+(3+n)^{2}\right]-\left[1^{2}+2^{2}+3^{2}\right] \\
& =\left(\sum_{k=1}^{3+n} k^{2}\right)-14 \\
& =\frac{(3+n)(4+n)(7+2 n)}{6}-14=\frac{1}{6}\left(73 n+21 n^{2}+2 n^{3}\right)
\end{aligned}
$$

## A DEFINITION OF AREA

We now turn to the problem of giving a precise definition of what is meant by the "area under a curve." Specifically, suppose that the function $f$ is continuous and nonnegative on the interval $[a, b]$, and let $R$ denote the region bounded below by the $x$-axis, bounded on the sides by the vertical lines $x=a$ and $x=b$, and bounded above by the curve $y=f(x)$ (Figure 5.4.2). Using the rectangle method of Section 5.1, we can motivate a definition for the area of $R$ as follows:

$\Delta$ Figure 5.4.3

$\triangle$ Figure 5.4.4

The limit in (2) is interpreted to mean that given any number $\epsilon>0$ the inequality

$$
\left|A-\sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x\right|<\epsilon
$$

holds when $n$ is sufficiently large, no matter how the points $x_{k}^{*}$ are selected.


Figure 5.4.5 $\operatorname{area}\left(R_{n}\right) \approx \operatorname{area}(R)$

- Divide the interval $[a, b]$ into $n$ equal subintervals by inserting $n-1$ equally spaced points between $a$ and $b$, and denote those points by

$$
x_{1}, x_{2}, \ldots, x_{n-1}
$$

(Figure 5.4.3). Each of these subintervals has width $(b-a) / n$, which is customarily denoted by

$$
\Delta x=\frac{b-a}{n}
$$

- Over each subinterval construct a rectangle whose height is the value of $f$ at an arbitrarily selected point in the subinterval. Thus, if

$$
x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}
$$

denote the points selected in the subintervals, then the rectangles will have heights $f\left(x_{1}^{*}\right), f\left(x_{2}^{*}\right), \ldots, f\left(x_{n}^{*}\right)$ and areas

$$
f\left(x_{1}^{*}\right) \Delta x, \quad f\left(x_{2}^{*}\right) \Delta x, \ldots, \quad f\left(x_{n}^{*}\right) \Delta x
$$

(Figure 5.4.4).

- The union of the rectangles forms a region $R_{n}$ whose area can be regarded as an approximation to the area $A$ of the region $R$; that is,

$$
A=\operatorname{area}(R) \approx \operatorname{area}\left(R_{n}\right)=f\left(x_{1}^{*}\right) \Delta x+f\left(x_{2}^{*}\right) \Delta x+\cdots+f\left(x_{n}^{*}\right) \Delta x
$$

(Figure 5.4.5). This can be expressed more compactly in sigma notation as

$$
A \approx \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x
$$

- Repeat the process using more and more subdivisions, and define the area of $R$ to be the "limit" of the areas of the approximating regions $R_{n}$ as $n$ increases without bound. That is, we define the area $A$ as

$$
A=\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x
$$

In summary, we make the following definition.
5.4.3 DEFINITION (Area Under a Curve) If the function $f$ is continuous on $[a, b]$ and if $f(x) \geq 0$ for all $x$ in $[a, b]$, then the area $A$ under the curve $y=f(x)$ over the interval $[a, b]$ is defined by

$$
\begin{equation*}
A=\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x \tag{2}
\end{equation*}
$$

There is a difference in interpretation between $\lim _{n \rightarrow+\infty}$ and $\lim _{x \rightarrow+\infty}$, where $n$ represents a positive integer and $x$ represents a real number. Later we will study limits of the type $\lim _{n \rightarrow+\infty}$ in detail, but for now suffice it to say that the computational techniques we have used for limits of type $\lim _{x \rightarrow+\infty}$ will also work for $\lim _{n \rightarrow+\infty}$.

The values of $x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}$ in (2) can be chosen arbitrarily, so it is conceivable that different choices of these values might produce different values of $A$. Were this to happen, then Definition 5.4.3 would not be an acceptable definition of area. Fortunately, this does not happen; it is proved in advanced courses that if $f$ is continuous (as we have assumed), then the same value of $A$ results no matter how the $x_{k}^{*}$ are chosen. In practice they are chosen in some systematic fashion, some common choices being

- the left endpoint of each subinterval
- the right endpoint of each subinterval
- the midpoint of each subinterval

To be more specific, suppose that the interval $[a, b]$ is divided into $n$ equal parts of length $\Delta x=(b-a) / n$ by the points $x_{1}, x_{2}, \ldots, x_{n-1}$, and let $x_{0}=a$ and $x_{n}=b$ (Figure 5.4.6). Then,

$$
x_{k}=a+k \Delta x \quad \text { for } k=0,1,2, \ldots, n
$$

Thus, the left endpoint, right endpoint, and midpoint choices for $x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}$ are given by

$$
\begin{align*}
& x_{k}^{*}=x_{k-1}=a+(k-1) \Delta x \quad \text { Left endpoint }  \tag{3}\\
& x_{k}^{*}=x_{k}=a+k \Delta x \quad \text { Right endpoint }  \tag{4}\\
& x_{k}^{*}=\frac{1}{2}\left(x_{k-1}+x_{k}\right)=a+\left(k-\frac{1}{2}\right) \Delta x \tag{5}
\end{align*}
$$

Figure 5.4.6 $x_{0}$


When applicable, the antiderivative method will be the method of choice for finding exact areas. However, the following examples will help to reinforce the ideas that we have just discussed.

Example 4 Use Definition 5.4.3 with $x_{k}^{*}$ as the right endpoint of each subinterval to find the area between the graph of $f(x)=x^{2}$ and the interval $[0,1]$.

Solution. The length of each subinterval is

$$
\Delta x=\frac{b-a}{n}=\frac{1-0}{n}=\frac{1}{n}
$$

so it follows from (4) that

$$
x_{k}^{*}=a+k \Delta x=\frac{k}{n}
$$

Thus,

$$
\begin{aligned}
\sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x=\sum_{k=1}^{n}\left(x_{k}^{*}\right)^{2} \Delta x & =\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{2} \frac{1}{n}=\frac{1}{n^{3}} \sum_{k=1}^{n} k^{2} \\
& =\frac{1}{n^{3}}\left[\frac{n(n+1)(2 n+1)}{6}\right] \quad \text { Part }(b) \text { of Theorem 5.4.2 } \\
& =\frac{1}{6}\left(\frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{2 n+1}{n}\right)=\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)
\end{aligned}
$$

from which it follows that

$$
A=\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x=\lim _{n \rightarrow+\infty}\left[\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]=\frac{1}{3}
$$

Observe that this is consistent with the results in Table 5.1.2 and the related discussion in Section 5.1.

In the solution to Example 4 we made use of one of the "closed form" summation formulas from Theorem 5.4.2. The next result collects some consequences of Theorem 5.4.2 that can facilitate computations of area using Definition 5.4.3.

What pattern is revealed by parts $(b)-$ (d) of Theorem 5.4.4? Does part (a) also fit this pattern? What would you conjecture to be the value of

$$
\lim _{n \rightarrow+\infty} \frac{1}{n^{m}} \sum_{k=1}^{n} k^{m-1}
$$

### 5.4.4 THEOREM

(a) $\lim _{n \rightarrow+\infty} \frac{1}{n} \sum_{k=1}^{n} 1=1$
(b) $\lim _{n \rightarrow+\infty} \frac{1}{n^{2}} \sum_{k=1}^{n} k=\frac{1}{2}$
(c) $\lim _{n \rightarrow+\infty} \frac{1}{n^{3}} \sum_{k=1}^{n} k^{2}=\frac{1}{3}$
(d) $\lim _{n \rightarrow+\infty} \frac{1}{n^{4}} \sum_{k=1}^{n} k^{3}=\frac{1}{4}$

The proof of Theorem 5.4.4 is left as an exercise for the reader.

- Example 5 Use Definition 5.4.3 with $x_{k}^{*}$ as the midpoint of each subinterval to find the area under the parabola $y=f(x)=9-x^{2}$ and over the interval $[0,3]$.

Solution. Each subinterval has length

$$
\Delta x=\frac{b-a}{n}=\frac{3-0}{n}=\frac{3}{n}
$$

so it follows from (5) that

$$
x_{k}^{*}=a+\left(k-\frac{1}{2}\right) \Delta x=\left(k-\frac{1}{2}\right)\left(\frac{3}{n}\right)
$$

Thus,

$$
\begin{aligned}
f\left(x_{k}^{*}\right) \Delta x & =\left[9-\left(x_{k}^{*}\right)^{2}\right] \Delta x=\left[9-\left(k-\frac{1}{2}\right)^{2}\left(\frac{3}{n}\right)^{2}\right]\left(\frac{3}{n}\right) \\
& =\left[9-\left(k^{2}-k+\frac{1}{4}\right)\left(\frac{9}{n^{2}}\right)\right]\left(\frac{3}{n}\right) \\
& =\frac{27}{n}-\frac{27}{n^{3}} k^{2}+\frac{27}{n^{3}} k-\frac{27}{4 n^{3}}
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
A & =\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x \\
& =\lim _{n \rightarrow+\infty} \sum_{k=1}^{n}\left(\frac{27}{n}-\frac{27}{n^{3}} k^{2}+\frac{27}{n^{3}} k-\frac{27}{4 n^{3}}\right) \\
& =\lim _{n \rightarrow+\infty} 27\left[\frac{1}{n} \sum_{k=1}^{n} 1-\frac{1}{n^{3}} \sum_{k=1}^{n} k^{2}+\frac{1}{n}\left(\frac{1}{n^{2}} \sum_{k=1}^{n} k\right)-\frac{1}{4 n^{2}}\left(\frac{1}{n} \sum_{k=1}^{n} 1\right)\right] \\
& =27\left[1-\frac{1}{3}+0 \cdot \frac{1}{2}-0 \cdot 1\right]=18
\end{aligned}
$$

NUMERICAL APPROXIMATIONS OF AREA
The antiderivative method discussed in Section 5.1 (and to be studied in more detail later) is an appropriate tool for finding the exact area under a curve when an antiderivative of the integrand can be found. However, if an antiderivative cannot be found, then we must resort to approximating the area. Definition 5.4 .3 provides a way of doing this. It follows from this definition that if $n$ is large, then

$$
\begin{equation*}
\sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x=\Delta x \sum_{k=1}^{n} f\left(x_{k}^{*}\right)=\Delta x\left[f\left(x_{1}^{*}\right)+f\left(x_{2}^{*}\right)+\cdots+f\left(x_{n}^{*}\right)\right] \tag{6}
\end{equation*}
$$

will be a good approximation to the area $A$. If one of Formulas (3), (4), or (5) is used to choose the $x_{k}^{*}$ in (6), then the result is called the left endpoint approximation, the right endpoint approximation, or the midpoint approximation, respectively (Figure 5.4.7).


A Figure 5.4.7


A Figure 5.4.8

- Example 6 Find the left endpoint, right endpoint, and midpoint approximations of the area under the curve $y=9-x^{2}$ over the interval $[0,3]$ with $n=10, n=20$, and $n=50$ (Figure 5.4.8). Compare the accuracies of these three methods.

Solution. Details of the computations for the case $n=10$ are shown to six decimal places in Table 5.4.1 and the results of all the computations are given in Table 5.4.2. We showed in Example 5 that the exact area is 18 (i.e., 18 square units), so in this case the midpoint approximation is more accurate than the endpoint approximations. This is also evident geometrically from Figure 5.4.9. You can also see from the figure that in this case the left endpoint approximation overestimates the area and the right endpoint approximation underestimates it. Later in the text we will investigate the error that results when an area is approximated by the midpoint rule.

## NET SIGNED AREA

In Definition 5.4.3 we assumed that $f$ is continuous and nonnegative on the interval $[a, b]$. If $f$ is continuous and attains both positive and negative values on $[a, b]$, then the limit

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x \tag{7}
\end{equation*}
$$

no longer represents the area between the curve $y=f(x)$ and the interval $[a, b]$ on the $x$-axis; rather, it represents a difference of areas-the area of the region that is above the interval $[a, b]$ and below the curve $y=f(x)$ minus the area of the region that is below the interval $[a, b]$ and above the curve $y=f(x)$. We call this the net signed area

Table 5.4.1

| $k$ | LEFT ENDPOINT APPROXIMATION |  | RIGHT ENDPOINT approximation |  | MIDPOINT APPROXIMATION |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $x_{k}^{*}$ | $9-\left(x_{k}^{*}\right)^{2}$ | $x_{k}^{*}$ | $9-\left(x_{k}^{*}\right)^{2}$ | $x_{k}^{*}$ | $9-\left(x_{k}^{*}\right)^{2}$ |
| 1 | 0.0 | 9.000000 | 0.3 | 8.910000 | 0.15 | 8.977500 |
| 2 | 0.3 | 8.910000 | 0.6 | 8.640000 | 0.45 | 8.797500 |
| 3 | 0.6 | 8.640000 | 0.9 | 8.190000 | 0.75 | 8.437500 |
| 4 | 0.9 | 8.190000 | 1.2 | 7.560000 | 1.05 | 7.897500 |
| 5 | 1.2 | 7.560000 | 1.5 | 6.750000 | 1.35 | 7.177500 |
| 6 | 1.5 | 6.750000 | 1.8 | 5.760000 | 1.65 | 6.277500 |
| 7 | 1.8 | 5.760000 | 2.1 | 4.590000 | 1.95 | 5.197500 |
| 8 | 2.1 | 4.590000 | 2.4 | 3.240000 | 2.25 | 3.937500 |
| 9 | 2.4 | 3.240000 | 2.7 | 1.710000 | 2.55 | 2.497500 |
| 10 | 2.7 | 1.710000 | 3.0 | 0.000000 | 2.85 | 0.877500 |
|  |  | 64.350000 |  | 55.350000 |  | 60.075000 |
| $\Delta x \sum_{k=1}^{n} f\left(x_{k}^{*}\right)$ |  | $\begin{array}{r} (0.3)(64.350000) \\ =19.305000 \end{array}$ |  | $\begin{array}{r} (0.3)(55.350000) \\ =16.605000 \end{array}$ |  | $\begin{array}{r} (0.3)(60.075000) \\ =18.022500 \end{array}$ |
|  |  | $=19.305000$ |  | $=16.605000$ |  | $=18.022500$ |

Table 5.4.2

| $n$ | LEFT ENDPOINT <br> APPROXIMATION | RIGHT ENDPOINT <br> APPROXIMATION | MIDPOINT <br> APPROXIMATION |
| :--- | :---: | :---: | :---: |
| 10 | 19.305000 | 16.605000 | 18.022500 |
| 20 | 18.663750 | 17.313750 | 18.005625 |
| 50 | 18.268200 | 17.728200 | 18.000900 |


The left endpoint
approximation
overestimates
the area.

Figure 5.4.9

(a)

(b)
$\triangle$ Figure 5.4.10

As with Definition 5.4.3, it can be proved that the limit in (9) always exists and that the same value of $A$ results no matter how the points in the subintervals are chosen.

$\Delta$ Figure 5.4.11
between the graph of $y=f(x)$ and the interval $[a, b]$. For example, in Figure 5.4.10a, the net signed area between the curve $y=f(x)$ and the interval $[a, b]$ is

$$
\left(A_{I}+A_{I I I}\right)-A_{I I}=[\text { area above }[a, b]]-[\text { area below }[a, b]]
$$

To explain why the limit in (7) represents this net signed area, let us subdivide the interval $[a, b]$ in Figure 5.4.10a into $n$ equal subintervals and examine the terms in the sum

$$
\begin{equation*}
\sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x \tag{8}
\end{equation*}
$$

If $f\left(x_{k}^{*}\right)$ is positive, then the product $f\left(x_{k}^{*}\right) \Delta x$ represents the area of the rectangle with height $f\left(x_{k}^{*}\right)$ and base $\Delta x$ (the pink rectangles in Figure 5.4.10b). However, if $f\left(x_{k}^{*}\right)$ is negative, then the product $f\left(x_{k}^{*}\right) \Delta x$ is the negative of the area of the rectangle with height $\left|f\left(x_{k}^{*}\right)\right|$ and base $\Delta x$ (the green rectangles in Figure 5.4.10b). Thus, (8) represents the total area of the pink rectangles minus the total area of the green rectangles. As $n$ increases, the pink rectangles fill out the regions with areas $A_{I}$ and $A_{I I I}$ and the green rectangles fill out the region with area $A_{I I}$, which explains why the limit in (7) represents the signed area between $y=f(x)$ and the interval $[a, b]$. We formalize this in the following definition.
5.4.5 Definition (Net Signed Area) If the function $f$ is continuous on $[a, b]$, then the net signed area $A$ between $y=f(x)$ and the interval $[a, b]$ is defined by

$$
\begin{equation*}
A=\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x \tag{9}
\end{equation*}
$$

Figure 5.4.11 shows the graph of $f(x)=x-1$ over the interval [0, 2]. It is geometrically evident that the areas $A_{1}$ and $A_{2}$ in that figure are equal, so we expect the net signed area between the graph of $f$ and the interval $[0,2]$ to be zero.

Example 7 Confirm that the net signed area between the graph of $f(x)=x-1$ and the interval $[0,2]$ is zero by using Definition 5.4.5 with $x_{k}^{*}$ chosen to be the left endpoint of each subinterval.

Solution. Each subinterval has length

$$
\Delta x=\frac{b-a}{n}=\frac{2-0}{n}=\frac{2}{n}
$$

so it follows from (3) that

$$
x_{k}^{*}=a+(k-1) \Delta x=(k-1)\left(\frac{2}{n}\right)
$$

Thus,

$$
\begin{aligned}
\sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x=\sum_{k=1}^{n}\left(x_{k}^{*}-1\right) \Delta x & =\sum_{k=1}^{n}\left[(k-1)\left(\frac{2}{n}\right)-1\right]\left(\frac{2}{n}\right) \\
& =\sum_{k=1}^{n}\left[\left(\frac{4}{n^{2}}\right) k-\frac{4}{n^{2}}-\frac{2}{n}\right]
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
A & =\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x=\lim _{n \rightarrow+\infty}\left[4\left(\frac{1}{n^{2}} \sum_{k=1}^{n} k\right)-\frac{4}{n}\left(\frac{1}{n} \sum_{k=1}^{n} 1\right)-2\left(\frac{1}{n} \sum_{k=1}^{n} 1\right)\right] \\
& =4\left(\frac{1}{2}\right)-0 \cdot 1-2 \cdot 1=0 \quad \text { Theorem 5.4.4 }
\end{aligned}
$$

This confirms that the net signed area is zero.

## QUICK CHECK EXERCISES 5.4 (See page 352 for answers.)

1. (a) Write the sum in two ways:

$$
\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}=\sum_{k=1}^{4} \square=\sum_{j=0}^{3}
$$

(b) Express the sum $10+10^{2}+10^{3}+10^{4}+10^{5}$ using sigma notation.
2. Express the sums in closed form.
(a) $\sum_{k=1}^{n} k$
(b) $\sum_{k=1}^{n}(6 k+1)$
(c) $\sum_{k=1}^{n} k^{2}$
3. Divide the interval $[1,3]$ into $n=4$ subintervals of equal length.
(a) Each subinterval has width $\qquad$
(b) The left endpoints of the subintervals are $\qquad$ _.
(c) The midpoints of the subintervals are $\qquad$
(d) The right endpoints of the subintervals are $\qquad$
4. Find the left endpoint approximation for the area between the curve $y=x^{2}$ and the interval $[1,3]$ using $n=4$ equal subdivisions of the interval.
5. The right endpoint approximation for the net signed area between $y=f(x)$ and an interval $[a, b]$ is given by

$$
\sum_{k=1}^{n} \frac{6 k+1}{n^{2}}
$$

Find the exact value of this net signed area.

## EXERCISE SET 5.4 C CAS

1. Evaluate.
(a) $\sum_{k=1}^{3} k^{3}$
(b) $\sum_{j=2}^{6}(3 j-1)$
(c) $\sum_{i=-4}^{1}\left(i^{2}-i\right)$
(d) $\sum_{n=0}^{5} 1$
(e) $\sum_{k=0}^{4}(-2)^{k}$
(f) $\sum_{n=1}^{6} \sin n \pi$

## 2. Evaluate.

(a) $\sum_{k=1}^{4} k \sin \frac{k \pi}{2}$
(b) $\sum_{j=0}^{5}(-1)^{j}$
(c) $\sum_{i=7}^{20} \pi^{2}$
(d) $\sum_{m=3}^{5} 2^{m+1}$
(e) $\sum_{n=1}^{6} \sqrt{n}$
(f) $\sum_{k=0}^{10} \cos k \pi$

3-8 Write each expression in sigma notation but do not evaluate.
3. $1+2+3+\cdots+10$
4. $3 \cdot 1+3 \cdot 2+3 \cdot 3+\cdots+3 \cdot 20$
5. $2+4+6+8+\cdots+20 \quad$ 6. $1+3+5+7+\cdots+15$
7. $1-3+5-7+9-11 \quad$ 8. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$
9. (a) Express the sum of the even integers from 2 to 100 in sigma notation.
(b) Express the sum of the odd integers from 1 to 99 in sigma notation.
10. Express in sigma notation.
(a) $a_{1}-a_{2}+a_{3}-a_{4}+a_{5}$
(b) $-b_{0}+b_{1}-b_{2}+b_{3}-b_{4}+b_{5}$
(c) $a_{0}+a_{1} x+a_{2} x^{2}+\cdots+a_{n} x^{n}$
(d) $a^{5}+a^{4} b+a^{3} b^{2}+a^{2} b^{3}+a b^{4}+b^{5}$

11-16 Use Theorem 5.4.2 to evaluate the sums. Check your answers using the summation feature of a calculating utility.
11. $\sum_{k=1}^{100} k$
12. $\sum_{k=1}^{100}(7 k+1)$
13. $\sum_{k=1}^{20} k^{2}$
14. $\sum_{k=4}^{20} k^{2}$
15. $\sum_{k=1}^{30} k(k-2)(k+2)$
16. $\sum_{k=1}^{6}\left(k-k^{3}\right)$

17-20 Express the sums in closed form.
17. $\sum_{k=1}^{n} \frac{3 k}{n}$
18. $\sum_{k=1}^{n-1} \frac{k^{2}}{n}$
19. $\sum_{k=1}^{n-1} \frac{k^{3}}{n^{2}}$
20. $\sum_{k=1}^{n}\left(\frac{5}{n}-\frac{2 k}{n}\right)$

21-24 True-False Determine whether the statement is true or false. Explain your answer.
21. For all positive integers $n$

$$
1^{3}+2^{3}+\cdots+n^{3}=(1+2+\cdots+n)^{2}
$$

22. The midpoint approximation is the average of the left endpoint approximation and the right endpoint approximation.
23. Every right endpoint approximation for the area under the graph of $y=x^{2}$ over an interval $[a, b]$ will be an overestimate.
24. For any continuous function $f$, the area between the graph of $f$ and an interval $[a, b]$ (on which $f$ is defined) is equal to the absolute value of the net signed area between the graph of $f$ and the interval $[a, b]$.

## FOCUS ON CONCEPTS

25. (a) Write the first three and final two summands in the sum

$$
\sum_{k=1}^{n}\left(2+k \cdot \frac{3}{n}\right)^{4} \frac{3}{n}
$$

Explain why this sum gives the right endpoint approximation for the area under the curve $y=x^{4}$ over the interval [2, 5].
(b) Show that a change in the index range of the sum in part (a) can produce the left endpoint approximation for the area under the curve $y=x^{4}$ over the interval [2, 5].
26. For a function $f$ that is continuous on $[a, b]$, Definition 5.4.5 says that the net signed area $A$ between $y=f(x)$ and the interval $[a, b]$ is

$$
A=\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x
$$

Give geometric interpretations for the symbols $n, x_{k}^{*}$, and $\Delta x$. Explain how to interpret the limit in this definition.

27-30 Divide the specified interval into $n=4$ subintervals of equal length and then compute

$$
\sum_{k=1}^{4} f\left(x_{k}^{*}\right) \Delta x
$$

with $x_{k}^{*}$ as (a) the left endpoint of each subinterval, (b) the midpoint of each subinterval, and (c) the right endpoint of each subinterval. Illustrate each part with a graph of $f$ that includes the rectangles whose areas are represented in the sum.
27. $f(x)=3 x+1 ;[2,6]$
28. $f(x)=1 / x ;[1,9]$
29. $f(x)=\cos x ;[0, \pi]$
30. $f(x)=2 x-x^{2}$; $[-1,3]$
[C 31-34 Use a calculating utility with summation capabilities or a CAS to obtain an approximate value for the area between the curve $y=f(x)$ and the specified interval with $n=10,20$, and 50 subintervals using the (a) left endpoint, (b) midpoint, and (c) right endpoint approximations.
31. $f(x)=1 / x$; $[1,2]$
32. $f(x)=1 / x^{2}$; $[1,3]$
33. $f(x)=\sqrt{x} ;[0,4]$
34. $f(x)=\sin x ;[0, \pi / 2]$

35-40 Use Definition 5.4.3 with $x_{k}^{*}$ as the right endpoint of each subinterval to find the area under the curve $y=f(x)$ over the specified interval.
35. $f(x)=x / 2 ;[1,4]$
36. $f(x)=5-x$; $[0,5]$
37. $f(x)=9-x^{2} ;[0,3]$
38. $f(x)=4-\frac{1}{4} x^{2}$; $[0,3]$
39. $f(x)=x^{3} ;[2,6]$
40. $f(x)=1-x^{3} ;[-3,-1]$

41-44 Use Definition 5.4.3 with $x_{k}^{*}$ as the left endpoint of each subinterval to find the area under the curve $y=f(x)$ over the specified interval.
41. $f(x)=x / 2 ;[1,4]$
42. $f(x)=5-x$; $[0,5]$
43. $f(x)=9-x^{2} ;[0,3]$
44. $f(x)=4-\frac{1}{4} x^{2}$; $[0,3]$

45-48 Use Definition 5.4.3 with $x_{k}^{*}$ as the midpoint of each subinterval to find the area under the curve $y=f(x)$ over the specified interval.
45. $f(x)=2 x$; $[0,4]$
46. $f(x)=6-x$; $[1,5]$
47. $f(x)=x^{2} ;[0,1]$
48. $f(x)=x^{2} ;[-1,1]$

49-52 Use Definition 5.4 .5 with $x_{k}^{*}$ as the right endpoint of each subinterval to find the net signed area between the curve $y=f(x)$ and the specified interval.
49. $f(x)=x$; $[-1,1]$. Verify your answer with a simple geometric argument.
50. $f(x)=x ;[-1,2]$. Verify your answer with a simple geometric argument.
51. $f(x)=x^{2}-1 ;[0,2] \quad$ 52. $f(x)=x^{3} ;[-1,1]$
53. (a) Show that the area under the graph of $y=x^{3}$ and over the interval $[0, b]$ is $b^{4} / 4$.
(b) Find a formula for the area under $y=x^{3}$ over the interval $[a, b]$, where $a \geq 0$.
54. Find the area between the graph of $y=\sqrt{x}$ and the interval [0, 1]. [Hint: Use the result of Exercise 25 of Section 5.1.]
55. An artist wants to create a rough triangular design using uniform square tiles glued edge to edge. She places $n$ tiles in a row to form the base of the triangle and then makes each successive row two tiles shorter than the preceding row. Find a formula for the number of tiles used in the design. [Hint: Your answer will depend on whether $n$ is even or odd.]
56. An artist wants to create a sculpture by gluing together uniform spheres. She creates a rough rectangular base that has 50 spheres along one edge and 30 spheres along the other. She then creates successive layers by gluing spheres in the grooves of the preceding layer. How many spheres will there be in the sculpture?

57-60 Consider the sum

$$
\begin{aligned}
\sum_{k=1}^{4}\left[(k+1)^{3}-k^{3}\right]= & {\left[5^{3}-4^{3}\right]+\left[4^{3}-3^{3}\right] } \\
& +\left[3^{3}-2^{3}\right]+\left[2^{3}-1^{3}\right] \\
= & 5^{3}-1^{3}=124
\end{aligned}
$$

For convenience, the terms are listed in reverse order. Note how cancellation allows the entire sum to collapse like a telescope. A sum is said to telescope when part of each term cancels part of an adjacent term, leaving only portions of the first and last terms uncanceled. Evaluate the telescoping sums in these exercises.
57. $\sum_{k=5}^{17}\left(3^{k}-3^{k-1}\right)$
58. $\sum_{k=1}^{50}\left(\frac{1}{k}-\frac{1}{k+1}\right)$
59. $\sum_{k=2}^{20}\left(\frac{1}{k^{2}}-\frac{1}{(k-1)^{2}}\right)$
60. $\sum_{k=1}^{100}\left(2^{k+1}-2^{k}\right)$
61. (a) Show that

$$
\begin{gathered}
\frac{1}{1 \cdot 3}+\frac{1}{3 \cdot 5}+\cdots+\frac{1}{(2 n-1)(2 n+1)}=\frac{n}{2 n+1} \\
{\left[\text { Hint: } \frac{1}{(2 n-1)(2 n+1)}=\frac{1}{2}\left(\frac{1}{2 n-1}-\frac{1}{2 n+1}\right)\right]}
\end{gathered}
$$

(b) Use the result in part (a) to find

$$
\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} \frac{1}{(2 k-1)(2 k+1)}
$$

62. (a) Show that

$$
\begin{aligned}
& \frac{1}{1 \cdot 2}+\frac{1}{2 \cdot 3}+\frac{1}{3 \cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1} \\
& {\left[\text { Hint: } \frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}\right]}
\end{aligned}
$$

(b) Use the result in part (a) to find

$$
\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)}
$$

63. Let $\bar{x}$ denote the arithmetic average of the $n$ numbers $x_{1}, x_{2}, \ldots, x_{n}$. Use Theorem 5.4.1 to prove that

$$
\sum_{i=1}^{n}\left(x_{i}-\bar{x}\right)=0
$$

64. Let

$$
S=\sum_{k=0}^{n} a r^{k}
$$

Show that $S-r S=a-a r^{n+1}$ and hence that

$$
\sum_{k=0}^{n} a r^{k}=\frac{a-a r^{n+1}}{1-r} \quad(r \neq 1)
$$

(A sum of this form is called a geometric sum.)
65. By writing out the sums, determine whether the following are valid identities.
(a) $\int\left[\sum_{i=1}^{n} f_{i}(x)\right] d x=\sum_{i=1}^{n}\left[\int f_{i}(x) d x\right]$
(b) $\frac{d}{d x}\left[\sum_{i=1}^{n} f_{i}(x)\right]=\sum_{i=1}^{n}\left[\frac{d}{d x}\left[f_{i}(x)\right]\right]$
66. Which of the following are valid identities?
(a) $\sum_{i=1}^{n} a_{i} b_{i}=\sum_{i=1}^{n} a_{i} \sum_{i=1}^{n} b_{i}$
(b) $\sum_{i=1}^{n} a_{i}^{2}=\left(\sum_{i=1}^{n} a_{i}\right)^{2}$
(c) $\sum_{i=1}^{n} \frac{a_{i}}{b_{i}}=\frac{\sum_{i=1}^{n} a_{i}}{\sum_{i=1}^{n} b_{i}}$.
(d) $\sum_{i=1}^{n} a_{i}=\sum_{j=0}^{n-1} a_{j+1}$
67. Prove part (c) of Theorem 5.4.1.
68. Prove Theorem 5.4.4.
69. Writing What is net signed area? How does this concept expand our application of the rectangle method?
70. Writing Based on Example 6, one might conjecture that the midpoint approximation always provides a better approximation than either endpoint approximation. Discuss the merits of this conjecture.

1. (a) $\frac{1}{2 k} ; \frac{1}{2(j+1)}$
(b) $\sum_{k=1}^{5} 10^{k}$
2. (a) $\frac{n(n+1)}{2}$
(b) $3 n(n+1)+n$ (c) $\frac{n(n+1)(2 n+1)}{6}$
3. (a) 0.5 (b) $1,1.5,2,2.5$
(c) $1.25,1.75,2.25,2.75$
(d) $1.5,2,2.5,3$
4. 6.75
5. $\lim _{n \rightarrow+\infty} \frac{3 n^{2}+4 n}{n^{2}}=3$

### 5.5 THE DEFINITE INTEGRAL


$\triangle$ Figure 5.5.1

$\Delta$ Figure 5.5.2

In this section we will introduce the concept of a "definite integral," which will link the concept of area to other important concepts such as length, volume, density, probability, and work.

## RIEMANN SUMS AND THE DEFINITE INTEGRAL

In our definition of net signed area (Definition 5.4.5), we assumed that for each positive number $n$, the interval $[a, b]$ was subdivided into $n$ subintervals of equal length to create bases for the approximating rectangles. For some functions it may be more convenient to use rectangles with different widths (see Making Connections Exercises 2 and 3); however, if we are to "exhaust" an area with rectangles of different widths, then it is important that successive subdivisions be constructed in such a way that the widths of all the rectangles approach zero as $n$ increases (Figure 5.5.1). Thus, we must preclude the kind of situation that occurs in Figure 5.5.2 in which the right half of the interval is never subdivided. If this kind of subdivision were allowed, the error in the approximation would not approach zero as $n$ increased.

A partition of the interval $[a, b]$ is a collection of points

$$
a=x_{0}<x_{1}<x_{2}<\cdots<x_{n-1}<x_{n}=b
$$

that divides $[a, b]$ into $n$ subintervals of lengths

$$
\Delta x_{1}=x_{1}-x_{0}, \quad \Delta x_{2}=x_{2}-x_{1}, \quad \Delta x_{3}=x_{3}-x_{2}, \ldots, \quad \Delta x_{n}=x_{n}-x_{n-1}
$$

The partition is said to be regular provided the subintervals all have the same length

$$
\Delta x_{k}=\Delta x=\frac{b-a}{n}
$$

For a regular partition, the widths of the approximating rectangles approach zero as $n$ is made large. Since this need not be the case for a general partition, we need some way to measure the "size" of these widths. One approach is to let max $\Delta x_{k}$ denote the largest of the subinterval widths. The magnitude $\max \Delta x_{k}$ is called the mesh size of the partition. For example, Figure 5.5 .3 shows a partition of the interval [0, 6] into four subintervals with a mesh size of 2 .
$>$ Figure 5.5.3


If we are to generalize Definition 5.4 .5 so that it allows for unequal subinterval widths, we must replace the constant length $\Delta x$ by the variable length $\Delta x_{k}$. When this is done the sum

$$
\sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x \quad \text { is replaced by } \quad \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k}
$$

We also need to replace the expression $n \rightarrow+\infty$ by an expression that guarantees us that the lengths of all subintervals approach zero. We will use the expression max $\Delta x_{k} \rightarrow 0$ for this purpose. Based on our intuitive concept of area, we would then expect the net signed area $A$ between the graph of $f$ and the interval $[a, b]$ to satisfy the equation

$$
A=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k}
$$

(We will see in a moment that this is the case.) The limit that appears in this expression is one of the fundamental concepts of integral calculus and forms the basis for the following definition.
5.5.1 DEFINITION A function $f$ is said to be integrable on a finite closed interval $[a, b]$ if the limit

$$
\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k}
$$

exists and does not depend on the choice of partitions or on the choice of the points $x_{k}^{*}$ in the subintervals. When this is the case we denote the limit by the symbol

$$
\int_{a}^{b} f(x) d x=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k}
$$

which is called the definite integral of $f$ from $a$ to $b$. The numbers $a$ and $b$ are called the lower limit of integration and the upper limit of integration, respectively, and $f(x)$ is called the integrand.

The notation used for the definite integral deserves some comment. Historically, the expression " $f(x) d x$ " was interpreted to be the "infinitesimal area" of a rectangle with height $f(x)$ and "infinitesimal" width $d x$. By "summing" these infinitesimal areas, the entire area under the curve was obtained. The integral symbol " $\int$ " is an "elongated s" that was used to indicate this summation. For us, the integral symbol " $\int$ " and the symbol " $d x$ " can serve as reminders that the definite integral is actually a limit of a summation as $\Delta x_{k} \rightarrow 0$. The sum that appears in Definition 5.5.1 is called a Riemann sum, and the definite integral is sometimes called the Riemann integral in honor of the German mathematician Bernhard Riemann who formulated many of the basic concepts of integral calculus. (The reason for the similarity in notation between the definite integral and the indefinite integral will become clear in the next section, where we will establish a link between the two types of "integration.") Georg Friedrich Bernhard Riemann (1826-1866) German mathematician. Bernhard Riemann, as he is commonly known, was the son of a Protestant minister. He received his elementary education from his father and showed brilliance in arithmetic at an early age. In 1846 he enrolled at Göttingen University to study theology and philology, but he soon transferred to mathematics. He studied physics under W. E. Weber and mathematics under Carl Friedrich Gauss, whom some people consider to be the greatest mathematician who ever lived. In 1851 Riemann received his Ph.D. under Gauss, after which he remained at Göttingen to teach. In 1862, one month after his marriage, Riemann suffered an attack of pleuritis, and for the remainder of his life was an extremely sick man. He finally succumbed to tuberculosis in 1866 at age 39.

An interesting story surrounds Riemann's work in geometry. For his introductory lecture prior to becoming an associate professor, Riemann submitted three possible topics to Gauss. Gauss surprised

Riemann by choosing the topic Riemann liked the least, the foundations of geometry. The lecture was like a scene from a movie. The old and failing Gauss, a giant in his day, watching intently as his brilliant and youthful protégé skillfully pieced together portions of the old man's own work into a complete and beautiful system. Gauss is said to have gasped with delight as the lecture neared its end, and on the way home he marveled at his student's brilliance. Gauss died shortly thereafter. The results presented by Riemann that day eventually evolved into a fundamental tool that Einstein used some 50 years later to develop relativity theory.

In addition to his work in geometry, Riemann made major contributions to the theory of complex functions and mathematical physics. The notion of the definite integral, as it is presented in most basic calculus courses, is due to him. Riemann's early death was a great loss to mathematics, for his mathematical work was brilliant and of fundamental importance.

The limit that appears in Definition 5.5.1 is somewhat different from the kinds of limits discussed in Chapter 1. Loosely phrased, the expression

$$
\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k}=L
$$

is intended to convey the idea that we can force the Riemann sums to be as close as we please to $L$, regardless of how the values of $x_{k}^{*}$ are chosen, by making the mesh size of the partition sufficiently small. While it is possible to give a more formal definition of this limit, we will simply rely on intuitive arguments when applying Definition 5.5.1.

Although a function need not be continuous on an interval to be integrable on that interval (Exercise 42), we will be interested primarily in definite integrals of continuous functions. The following theorem, which we will state without proof, says that if a function is continuous on a finite closed interval, then it is integrable on that interval, and its definite integral is the net signed area between the graph of the function and the interval.
5.5.2 THEOREM If a function $f$ is continuous on an interval $[a, b]$, then $f$ is integrable on $[a, b]$, and the net signed area $A$ between the graph of $f$ and the interval $[a, b]$ is

$$
\begin{equation*}
A=\int_{a}^{b} f(x) d x \tag{1}
\end{equation*}
$$

In Example 1, it is understood that the units of area are the squared units of length, even though we have not stated the units of length explicitly.

Formula (1) follows from the integrability of $f$, since the integrability allows us to use any partitions to evaluate the integral. In particular, if we use regular partitions of $[a, b]$, then

$$
\Delta x_{k}=\Delta x=\frac{b-a}{n}
$$

for all values of $k$. This implies that max $\Delta x_{k}=(b-a) / n$, from which it follows that $\max \Delta x_{k} \rightarrow 0$ if and only if $n \rightarrow+\infty$. Thus,

$$
\int_{a}^{b} f(x) d x=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k}=\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x=A
$$

In the simplest cases, definite integrals of continuous functions can be calculated using formulas from plane geometry to compute signed areas.

- Example 1 Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.
(a) $\int_{1}^{4} 2 d x$
(b) $\int_{-1}^{2}(x+2) d x$
(c) $\int_{0}^{1} \sqrt{1-x^{2}} d x$

Solution (a). The graph of the integrand is the horizontal line $y=2$, so the region is a rectangle of height 2 extending over the interval from 1 to 4 (Figure 5.5.4a). Thus,

$$
\int_{1}^{4} 2 d x=(\text { area of rectangle })=2(3)=6
$$

Solution (b). The graph of the integrand is the line $y=x+2$, so the region is a trapezoid whose base extends from $x=-1$ to $x=2$ (Figure 5.5.4b). Thus,

$$
\int_{-1}^{2}(x+2) d x=(\text { area of trapezoid })=\frac{1}{2}(1+4)(3)=\frac{15}{2}
$$


(a)

(b)

(c)
$\triangle$ Figure 5.5.4

Solution (c). The graph of $y=\sqrt{1-x^{2}}$ is the upper semicircle of radius 1 , centered at the origin, so the region is the right quarter-circle extending from $x=0$ to $x=1$ (Figure 5.5.4c). Thus,

$$
\int_{0}^{1} \sqrt{1-x^{2}} d x=(\text { area of quarter-circle })=\frac{1}{4} \pi\left(1^{2}\right)=\frac{\pi}{4}
$$



Figure 5.5.5

- Example 2 Evaluate
(a) $\int_{0}^{2}(x-1) d x$
(b) $\int_{0}^{1}(x-1) d x$

Solution. The graph of $y=x-1$ is shown in Figure 5.5.5, and we leave it for you to verify that the shaded triangular regions both have area $\frac{1}{2}$. Over the interval [0, 2] the net signed area is $A_{1}-A_{2}=\frac{1}{2}-\frac{1}{2}=0$, and over the interval [0,1] the net signed area is $-A_{2}=-\frac{1}{2}$. Thus,

$$
\int_{0}^{2}(x-1) d x=0 \quad \text { and } \quad \int_{0}^{1}(x-1) d x=-\frac{1}{2}
$$

(Recall that in Example 7 of Section 5.4, we used Definition 5.4.5 to show that the net signed area between the graph of $y=x-1$ and the interval [ 0,2 ] is zero.)

## PROPERTIES OF THE DEFINITE INTEGRAL

It is assumed in Definition 5.5.1 that $[a, b]$ is a finite closed interval with $a<b$, and hence the upper limit of integration in the definite integral is greater than the lower limit of integration. However, it will be convenient to extend this definition to allow for cases in which the upper and lower limits of integration are equal or the lower limit of integration is greater than the upper limit of integration. For this purpose we make the following special definitions.

### 5.5.3 DEFINITION

(a) If $a$ is in the domain of $f$, we define

$$
\int_{a}^{a} f(x) d x=0
$$

(b) If $f$ is integrable on $[a, b]$, then we define

$$
\int_{b}^{a} f(x) d x=-\int_{a}^{b} f(x) d x
$$


$\Delta$ Figure 5.5.6

Part (a) of this definition is consistent with the intuitive idea that the area between a point on the $x$-axis and a curve $y=f(x)$ should be zero (Figure 5.5.6). Part (b) of the definition is simply a useful convention; it states that interchanging the limits of integration reverses the sign of the integral.

## - Example 3

(a) $\int_{1}^{1} x^{2} d x=0$
(b) $\int_{1}^{0} \sqrt{1-x^{2}} d x=-\int_{0}^{1} \sqrt{1-x^{2}} d x=-\frac{\pi}{4} \longleftarrow$ Example 1(c)

Because definite integrals are defined as limits, they inherit many of the properties of limits. For example, we know that constants can be moved through limit signs and that the limit of a sum or difference is the sum or difference of the limits. Thus, you should not be surprised by the following theorem, which we state without formal proof.
5.5.4 THEOREM If $f$ and $g$ are integrable on $[a, b]$ and if $c$ is a constant, then $c f$, $f+g$, and $f-g$ are integrable on $[a, b]$ and
(a) $\int_{a}^{b} c f(x) d x=c \int_{a}^{b} f(x) d x$
(b) $\int_{a}^{b}[f(x)+g(x)] d x=\int_{a}^{b} f(x) d x+\int_{a}^{b} g(x) d x$
(c) $\int_{a}^{b}[f(x)-g(x)] d x=\int_{a}^{b} f(x) d x-\int_{a}^{b} g(x) d x$

## Example 4 Evaluate

$$
\int_{0}^{1}\left(5-3 \sqrt{1-x^{2}}\right) d x
$$

Solution. From parts (a) and (c) of Theorem 5.5 .4 we can write

$$
\int_{0}^{1}\left(5-3 \sqrt{1-x^{2}}\right) d x=\int_{0}^{1} 5 d x-\int_{0}^{1} 3 \sqrt{1-x^{2}} d x=\int_{0}^{1} 5 d x-3 \int_{0}^{1} \sqrt{1-x^{2}} d x
$$

The first integral in this difference can be interpreted as the area of a rectangle of height 5 and base 1 , so its value is 5 , and from Example 1 the value of the second integral is $\pi / 4$. Thus,

$$
\int_{0}^{1}\left(5-3 \sqrt{1-x^{2}}\right) d x=5-3\left(\frac{\pi}{4}\right)=5-\frac{3 \pi}{4}
$$

Part (b) of Theorem 5.5.4 can be extended to more than two functions. More precisely,

$$
\begin{aligned}
\int_{a}^{b} & {\left[f_{1}(x)+f_{2}(x)+\cdots+f_{n}(x)\right] d x } \\
& =\int_{a}^{b} f_{1}(x) d x+\int_{a}^{b} f_{2}(x) d x+\cdots+\int_{a}^{b} f_{n}(x) d x
\end{aligned}
$$


$\Delta$ Figure 5.5.7

Part (b) of Theorem 5.5.6 states that the direction (sometimes called the sense) of the inequality $f(x) \geq g(x)$ is unchanged if one integrates both sides. Moreover, if $b>a$, then both parts of the theorem remain true if $\geq$ is replaced by $\leq,>$, or $<$ throughout.

$\Delta$ Figure 5.5.8

$\Delta$ Figure 5.5.9

Some properties of definite integrals can be motivated by interpreting the integral as an area. For example, if $f$ is continuous and nonnegative on the interval $[a, b]$, and if $c$ is a point between $a$ and $b$, then the area under $y=f(x)$ over the interval $[a, b]$ can be split into two parts and expressed as the area under the graph from $a$ to $c$ plus the area under the graph from $c$ to $b$ (Figure 5.5.7), that is,

$$
\int_{a}^{b} f(x) d x=\int_{a}^{c} f(x) d x+\int_{c}^{b} f(x) d x
$$

This is a special case of the following theorem about definite integrals, which we state without proof.

### 5.5.5 THEOREM If $f$ is integrable on a closed interval containing the three points

 $a, b$, and $c$, then$$
\int_{a}^{b} f(x) d x=\int_{a}^{c} f(x) d x+\int_{c}^{b} f(x) d x
$$

no matter how the points are ordered.

The following theorem, which we state without formal proof, can also be motivated by interpreting definite integrals as areas.

### 5.5.6 THEOREM

(a) If $f$ is integrable on $[a, b]$ and $f(x) \geq 0$ for all $x$ in $[a, b]$, then

$$
\int_{a}^{b} f(x) d x \geq 0
$$

(b) If $f$ and $g$ are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all $x$ in $[a, b]$, then

$$
\int_{a}^{b} f(x) d x \geq \int_{a}^{b} g(x) d x
$$

Geometrically, part (a) of this theorem states the obvious fact that if $f$ is nonnegative on $[a, b]$, then the net signed area between the graph of $f$ and the interval $[a, b]$ is also nonnegative (Figure 5.5.8). Part (b) has its simplest interpretation when $f$ and $g$ are nonnegative on $[a, b]$, in which case the theorem states that if the graph of $f$ does not go below the graph of $g$, then the area under the graph of $f$ is at least as large as the area under the graph of $g$ (Figure 5.5.9).

## DISCONTINUITIES AND INTEGRABILITY

In the late nineteenth and early twentieth centuries, mathematicians began to investigate conditions under which the limit that defines an integral fails to exist, that is, conditions under which a function fails to be integrable. The matter is quite complex and beyond the scope of this text. However, there are a few basic results about integrability that are important to know; we begin with a definition.

$f$ is bounded on $[a, b]$.
$\Delta$ Figure 5.5.10


- Figure 5.5.11
5.5.7 Definition A function $f$ that is defined on an interval is said to be bounded on the interval if there is a positive number $M$ such that

$$
-M \leq f(x) \leq M
$$

for all $x$ in the interval. Geometrically, this means that the graph of $f$ over the interval lies between the lines $y=-M$ and $y=M$.

For example, a continuous function $f$ is bounded on every finite closed interval because the Extreme-Value Theorem (4.4.2) implies that $f$ has an absolute maximum and an absolute minimum on the interval; hence, its graph will lie between the lines $y=-M$ and $y=M$, provided we make $M$ large enough (Figure 5.5.10). In contrast, a function that has a vertical asymptote inside of an interval is not bounded on that interval because its graph over the interval cannot be made to lie between the lines $y=-M$ and $y=M$, no matter how large we make the value of $M$ (Figure 5.5.11).

The following theorem, which we state without proof, provides some facts about integrability for functions with discontinuities. In the exercises we have included some problems that are concerned with this theorem (Exercises 42, 43, and 44).

### 5.5.8 THEOREM Let $f$ be a function that is defined on the finite closed interval $[a, b]$.

(a) If $f$ has finitely many discontinuities in $[a, b]$ but is bounded on $[a, b]$, then $f$ is integrable on $[a, b]$.
(b) If $f$ is not bounded on $[a, b]$, then $f$ is not integrable on $[a, b]$.

QUICK CHECK EXERCISES 5.5 (See page 362 for answers.)

1. In each part, use the partition of $[2,7]$ in the accompanying figure.

(a) What is $n$, the number of subintervals in this partition?
(b) $x_{0}=\square ; x_{1}=\square ; x_{2}=\square ;$ $x_{3}=$ $\qquad$ ; $x_{4}=$ $\qquad$
(c) $\Delta x_{1}=$ $\qquad$ ; $\Delta x_{2}=$ $\qquad$ ; $\Delta x_{3}=$ $\qquad$ $\Delta x_{4}=$ $\qquad$
(d) The mesh of this partition is $\qquad$ .
2. Let $f(x)=2 x-8$. Use the partition of $[2,7]$ in Quick Check Exercise 1 and the choices $x_{1}^{*}=2, x_{2}^{*}=4, x_{3}^{*}=5$, and $x_{4}^{*}=7$ to evaluate the Riemann sum

$$
\sum_{k=1}^{4} f\left(x_{k}^{*}\right) \Delta x_{k}
$$

3. Use the accompanying figure to evaluate

$$
\int_{2}^{7}(2 x-8) d x
$$

1-4 Find the value of
$\begin{array}{ll}\text { (a) } \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k} & \text { (b) } \max \Delta x_{k} .\end{array}$

1. $f(x)=x+1 ; a=0, b=4 ; n=3$;
$\Delta x_{1}=1, \Delta x_{2}=1, \Delta x_{3}=2$;
$x_{1}^{*}=\frac{1}{3}, x_{2}^{*}=\frac{3}{2}, x_{3}^{*}=3$
2. $f(x)=\cos x ; a=0, b=2 \pi ; n=4$;
$\Delta x_{1}=\pi / 2, \Delta x_{2}=3 \pi / 4, \Delta x_{3}=\pi / 2, \Delta x_{4}=\pi / 4 ;$
$x_{1}^{*}=\pi / 4, x_{2}^{*}=\pi, x_{3}^{*}=3 \pi / 2, x_{4}^{*}=7 \pi / 4$
3. $f(x)=4-x^{2} ; a=-3, b=4 ; n=4$;
$\Delta x_{1}=1, \Delta x_{2}=2, \Delta x_{3}=1, \Delta x_{4}=3$;
$x_{1}^{*}=-\frac{5}{2}, x_{2}^{*}=-1, x_{3}^{*}=\frac{1}{4}, x_{4}^{*}=3$
4. $f(x)=x^{3} ; a=-3, b=3 ; n=4$;
$\Delta x_{1}=2, \Delta x_{2}=1, \Delta x_{3}=1, \Delta x_{4}=2 ;$
$x_{1}^{*}=-2, x_{2}^{*}=0, x_{3}^{*}=0, x_{4}^{*}=2$
5-8 Use the given values of $a$ and $b$ to express the following limits as integrals. (Do not evaluate the integrals.)
5. $\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n}\left(x_{k}^{*}\right)^{2} \Delta x_{k} ; a=-1, b=2$
6. $\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n}\left(x_{k}^{*}\right)^{3} \Delta x_{k} ; a=1, b=2$
7. $\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} 4 x_{k}^{*}\left(1-3 x_{k}^{*}\right) \Delta x_{k} ; \quad a=-3, b=3$
8. $\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n}\left(\sin ^{2} x_{k}^{*}\right) \Delta x_{k} ; a=0, b=\pi / 2$

9-10 Use Definition 5.5.1 to express the integrals as limits of Riemann sums. (Do not evaluate the integrals.)
9. (a) $\int_{1}^{2} 2 x d x$
(b) $\int_{0}^{1} \frac{x}{x+1} d x$
10. (a) $\int_{1}^{2} \sqrt{x} d x$
(b) $\int_{-\pi / 2}^{\pi / 2}(1+\cos x) d x$

## FOCUS ON CONCEPTS

11. Explain informally why Theorem 5.5.4(a) follows from Definition 5.5.1.
12. Explain informally why Theorem 5.5.6(a) follows from Definition 5.5.1.

13-16 Sketch the region whose signed area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry, where needed.
13. (a) $\int_{0}^{3} x d x$
(b) $\int_{-2}^{-1} x d x$
(c) $\int_{-1}^{4} x d x$
(d) $\int_{-5}^{-5} x d x$
14. (a) $\int_{0}^{2}\left(1-\frac{1}{2} x\right) d x$
(b) $\int_{-1}^{1}\left(1-\frac{1}{2} x\right) d x$
(c) $\int_{2}^{3}\left(1-\frac{1}{2} x\right) d x$
(d) $\int_{0}^{3}\left(1-\frac{1}{2} x\right) d x$
15. (a) $\int_{0}^{5} 2 d x$
(b) $\int_{0}^{\pi} \cos x d x$
(c) $\int_{-1}^{2}|2 x-3| d x$
(d) $\int_{-1}^{1} \sqrt{1-x^{2}} d x$
16. (a) $\int_{-10}^{-5} 6 d x$
(b) $\int_{-\pi / 3}^{\pi / 3} \sin x d x$
(c) $\int_{0}^{3}|x-2| d x$
(d) $\int_{0}^{2} \sqrt{4-x^{2}} d x$
17. In each part, evaluate the integral, given that

$$
f(x)= \begin{cases}|x-2|, & x \geq 0 \\ x+2, & x<0\end{cases}
$$

(a) $\int_{-2}^{0} f(x) d x$
(b) $\int_{-2}^{2} f(x) d x$
(c) $\int_{0}^{6} f(x) d x$
(d) $\int_{-4}^{6} f(x) d x$
18. In each part, evaluate the integral, given that

$$
f(x)= \begin{cases}2 x, & x \leq 1 \\ 2, & x>1\end{cases}
$$

(a) $\int_{0}^{1} f(x) d x$
(b) $\int_{-1}^{1} f(x) d x$
(c) $\int_{1}^{10} f(x) d x$
(d) $\int_{1 / 2}^{5} f(x) d x$

## FOCUS ON CONCEPTS

19-20 Use the areas shown in the figure to find
(a) $\int_{a}^{b} f(x) d x$
(b) $\int_{b}^{c} f(x) d x$
(c) $\int_{a}^{c} f(x) d x$
(d) $\int_{a}^{d} f(x) d x$.
19.

20.

21. Find $\int_{-1}^{2}[f(x)+2 g(x)] d x$ if

$$
\int_{-1}^{2} f(x) d x=5 \text { and } \int_{-1}^{2} g(x) d x=-3
$$

22. Find $\int_{1}^{4}[3 f(x)-g(x)] d x$ if

$$
\int_{1}^{4} f(x) d x=2 \text { and } \int_{1}^{4} g(x) d x=10
$$

23. Find $\int_{1}^{5} f(x) d x$ if

$$
\int_{0}^{1} f(x) d x=-2 \quad \text { and } \quad \int_{0}^{5} f(x) d x=1
$$

24. Find $\int_{3}^{-2} f(x) d x$ if

$$
\int_{-2}^{1} f(x) d x=2 \quad \text { and } \quad \int_{1}^{3} f(x) d x=-6
$$

25-28 Use Theorem 5.5.4 and appropriate formulas from geometry to evaluate the integrals.
25. $\int_{-1}^{3}(4-5 x) d x$
26. $\int_{-2}^{2}(1-3|x|) d x$
27. $\int_{0}^{1}\left(x+2 \sqrt{1-x^{2}}\right) d x$
28. $\int_{-3}^{0}\left(2+\sqrt{9-x^{2}}\right) d x$

29-32 True-False Determine whether the statement is true or false. Explain your answer.
29. If $f(x)$ is integrable on $[a, b]$, then $f(x)$ is continuous on $[a, b]$.
30. It is the case that

$$
0<\int_{-1}^{1} \frac{\cos x}{\sqrt{1+x^{2}}} d x
$$

31. If the integral of $f(x)$ over the interval $[a, b]$ is negative, then $f(x) \leq 0$ for $a \leq x \leq b$.
32. The function

$$
f(x)= \begin{cases}0, & x \leq 0 \\ x^{2}, & x>0\end{cases}
$$

is integrable over every closed interval $[a, b]$.
33-34 Use Theorem 5.5.6 to determine whether the value of the integral is positive or negative.
33. (a) $\int_{2}^{3} \frac{\sqrt{x}}{1-x} d x$
(b) $\int_{0}^{4} \frac{x^{2}}{3-\cos x} d x$
34. (a) $\int_{-3}^{-1} \frac{x^{4}}{\sqrt{3-x}} d x$
(b) $\int_{-2}^{2} \frac{x^{3}-9}{|x|+1} d x$
35. Prove that if $f$ is continuous and if $m \leq f(x) \leq M$ on $[a, b]$, then

$$
m(b-a) \leq \int_{a}^{b} f(x) d x \leq M(b-a)
$$

36. Find the maximum and minimum values of $\sqrt{x^{3}+2}$ for $0 \leq x \leq 3$. Use these values, and the inequalities in Exercise 35 , to find bounds on the value of the integral

$$
\int_{0}^{3} \sqrt{x^{3}+2} d x
$$

37-38 Evaluate the integrals by completing the square and applying appropriate formulas from geometry.
37. $\int_{0}^{10} \sqrt{10 x-x^{2}} d x$
38. $\int_{0}^{3} \sqrt{6 x-x^{2}} d x$

39-40 Evaluate the limit by expressing it as a definite integral over the interval $[a, b]$ and applying appropriate formulas from geometry.
39. $\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n}\left(3 x_{k}^{*}+1\right) \Delta x_{k} ; a=0, b=1$
40. $\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} \sqrt{4-\left(x_{k}^{*}\right)^{2}} \Delta x_{k} ; a=-2, b=2$

## FOCUS ON CONCEPTS

41. Let $f(x)=C$ be a constant function.
(a) Use a formula from geometry to show that

$$
\int_{a}^{b} f(x) d x=C(b-a)
$$

(b) Show that any Riemann sum for $f(x)$ over $[a, b]$ evaluates to $C(b-a)$. Use Definition 5.5.1 to show that

$$
\int_{a}^{b} f(x) d x=C(b-a)
$$

42. Define a function $f$ on $[0,1]$ by

$$
f(x)= \begin{cases}1, & 0<x \leq 1 \\ 0, & x=0\end{cases}
$$

Use Definition 5.5.1 to show that

$$
\int_{0}^{1} f(x) d x=1
$$

43. It can be shown that every interval contains both rational and irrational numbers. Accepting this to be so, do you believe that the function

$$
f(x)=\left\{\begin{array}{lll}
1 & \text { if } & x \text { is rational } \\
0 & \text { if } & x \text { is irrational }
\end{array}\right.
$$

is integrable on a closed interval $[a, b]$ ? Explain your reasoning.
44. Define the function $f$ by

$$
f(x)= \begin{cases}\frac{1}{x}, & x \neq 0 \\ 0, & x=0\end{cases}
$$

It follows from Theorem 5.5.8(b) that $f$ is not integrable on the interval $[0,1]$. Prove this to be the case by applying Definition 5.5.1. [Hint: Argue that no matter how small the mesh size is for a partition of [0, 1], there will always be a choice of $x_{1}^{*}$ that will make the Riemann sum in Definition 5.5.1 as large as we like.]
45. In each part, use Theorems 5.5.2 and 5.5.8 to determine whether the function $f$ is integrable on the interval $[-1,1]$.
(a) $f(x)=\cos x$
(b) $f(x)= \begin{cases}x /|x|, & x \neq 0 \\ 0, & x=0\end{cases}$
(c) $f(x)= \begin{cases}1 / x^{2}, & x \neq 0 \\ 0, & x=0\end{cases}$
(d) $f(x)= \begin{cases}\sin 1 / x, & x \neq 0 \\ 0, & x=0\end{cases}$
46. Writing Write a short paragraph that discusses the similarities and differences between indefinite integrals and definite integrals.
47. Writing Write a paragraph that explains informally what it means for a function to be "integrable."

## QUICK CHECK ANSWERS 5.5

1. (a) $n=4$ (b) $2,3,4.5,6.5,7$
(c) $1,1.5,2,0.5$ (d) 2
2. 3
3. 5
4. (a) -10
(b) 3
(c) 0 (d) -12

### 5.6 THE FUNDAMENTAL THEOREM OF CALCULUS

In this section we will establish two basic relationships between definite and indefinite integrals that together constitute a result called the "Fundamental Theorem of Calculus." One part of this theorem will relate the rectangle and antiderivative methods for calculating areas, and the second part will provide a powerful method for evaluating definite integrals using antiderivatives.


Figure 5.6.1


Figure 5.6.2

## THE FUNDAMENTAL THEOREM OF CALCULUS

As in earlier sections, let us begin by assuming that $f$ is nonnegative and continuous on an interval $[a, b]$, in which case the area $A$ under the graph of $f$ over the interval $[a, b]$ is represented by the definite integral

$$
\begin{equation*}
A=\int_{a}^{b} f(x) d x \tag{1}
\end{equation*}
$$

(Figure 5.6.1).
Recall that our discussion of the antiderivative method in Section 5.1 suggested that if $A(x)$ is the area under the graph of $f$ from $a$ to $x$ (Figure 5.6.2), then

- $A^{\prime}(x)=f(x)$
- $A(a)=0 \quad$ The area under the curve from $a$ to $a$ is the area above the single point $a$, and hence is zero.
- $A(b)=A \quad$ The area under the curve from $a$ to $b$ is $A$.

The formula $A^{\prime}(x)=f(x)$ states that $A(x)$ is an antiderivative of $f(x)$, which implies that every other antiderivative of $f(x)$ on $[a, b]$ can be obtained by adding a constant to $A(x)$. Accordingly, let

$$
F(x)=A(x)+C
$$

be any antiderivative of $f(x)$, and consider what happens when we subtract $F(a)$ from $F(b)$ :

$$
F(b)-F(a)=[A(b)+C]-[A(a)+C]=A(b)-A(a)=A-0=A
$$

Hence (1) can be expressed as

$$
\int_{a}^{b} f(x) d x=F(b)-F(a)
$$

In words, this equation states:

The definite integral can be evaluated by finding any antiderivative of the integrand and then subtracting the value of this antiderivative at the lower limit of integration from its value at the upper limit of integration.

Although our evidence for this result assumed that $f$ is nonnegative on $[a, b]$, this assumption is not essential.
5.6.1 THEOREM (The Fundamental Theorem of Calculus, Part 1) If $f$ is continuous on $[a, b]$ and $F$ is any antiderivative of $f$ on $[a, b]$, then

$$
\begin{equation*}
\int_{a}^{b} f(x) d x=F(b)-F(a) \tag{2}
\end{equation*}
$$

PROOF Let $x_{1}, x_{2}, \ldots, x_{n-1}$ be any points in $[a, b]$ such that

$$
a<x_{1}<x_{2}<\cdots<x_{n-1}<b
$$

These values divide $[a, b]$ into $n$ subintervals

$$
\begin{equation*}
\left[a, x_{1}\right],\left[x_{1}, x_{2}\right], \ldots,\left[x_{n-1}, b\right] \tag{3}
\end{equation*}
$$

whose lengths, as usual, we denote by

$$
\Delta x_{1}, \Delta x_{2}, \ldots, \Delta x_{n}
$$

(see Figure 5.6.3). By hypothesis, $F^{\prime}(x)=f(x)$ for all $x$ in $[a, b]$, so $F$ satisfies the hypotheses of the Mean-Value Theorem (4.8.2) on each subinterval in (3). Hence, we can find points $x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}$ in the respective subintervals in (3) such that

$$
\begin{aligned}
F\left(x_{1}\right)-F(a) & =F^{\prime}\left(x_{1}^{*}\right)\left(x_{1}-a\right)=f\left(x_{1}^{*}\right) \Delta x_{1} \\
F\left(x_{2}\right)-F\left(x_{1}\right) & =F^{\prime}\left(x_{2}^{*}\right)\left(x_{2}-x_{1}\right)=f\left(x_{2}^{*}\right) \Delta x_{2} \\
F\left(x_{3}\right)-F\left(x_{2}\right) & =F^{\prime}\left(x_{3}^{*}\right)\left(x_{3}-x_{2}\right)=f\left(x_{3}^{*}\right) \Delta x_{3} \\
& \vdots \\
F(b)-F\left(x_{n-1}\right) & =F^{\prime}\left(x_{n}^{*}\right)\left(b-x_{n-1}\right)=f\left(x_{n}^{*}\right) \Delta x_{n}
\end{aligned}
$$

Adding the preceding equations yields

$$
\begin{equation*}
F(b)-F(a)=\sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k} \tag{4}
\end{equation*}
$$

Let us now increase $n$ in such a way that max $\Delta x_{k} \rightarrow 0$. Since $f$ is assumed to be continuous, the right side of (4) approaches $\int_{a}^{b} f(x) d x$ by Theorem 5.5.2 and Definition 5.5.1. However,

Figure 5.6.3


The integral in Example 1 represents the area of a certain trapezoid. Sketch the trapezoid, and find its area using geometry.
the left side of (4) is independent of $n$; that is, the left side of (4) remains constant as $n$ increases. Thus,

$$
F(b)-F(a)=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k}=\int_{a}^{b} f(x) d x
$$

It is standard to denote the difference $F(b)-F(a)$ as

$$
F(x)]_{a}^{b}=F(b)-F(a) \quad \text { or } \quad[F(x)]_{a}^{b}=F(b)-F(a)
$$

For example, using the first of these notations we can express (2) as

$$
\begin{equation*}
\left.\int_{a}^{b} f(x) d x=F(x)\right]_{a}^{b} \tag{5}
\end{equation*}
$$

We will sometimes write

$$
F(x)]_{x=a}^{b}=F(b)-F(a)
$$

when it is important to emphasize that $a$ and $b$ are values for the variable $x$.
Example 1 Evaluate $\int_{1}^{2} x d x$.
Solution. The function $F(x)=\frac{1}{2} x^{2}$ is an antiderivative of $f(x)=x$; thus, from (2)

$$
\left.\int_{1}^{2} x d x=\frac{1}{2} x^{2}\right]_{1}^{2}=\frac{1}{2}(2)^{2}-\frac{1}{2}(1)^{2}=2-\frac{1}{2}=\frac{3}{2}
$$

- Example 2 In Example 5 of Section 5.4 we used the definition of area to show that the area under the graph of $y=9-x^{2}$ over the interval $[0,3]$ is 18 (square units). We can now solve that problem much more easily using the Fundamental Theorem of Calculus:

$$
A=\int_{0}^{3}\left(9-x^{2}\right) d x=\left[9 x-\frac{x^{3}}{3}\right]_{0}^{3}=\left(27-\frac{27}{3}\right)-0=18
$$

## - Example 3

(a) Find the area under the curve $y=\cos x$ over the interval $[0, \pi / 2]$ (Figure 5.6.4).
(b) Make a conjecture about the value of the integral

$$
\int_{0}^{\pi} \cos x d x
$$

and confirm your conjecture using the Fundamental Theorem of Calculus.
Solution (a). Since $\cos x \geq 0$ over the interval $[0, \pi / 2]$, the area $A$ under the curve is

$$
\left.A=\int_{0}^{\pi / 2} \cos x d x=\sin x\right]_{0}^{\pi / 2}=\sin \frac{\pi}{2}-\sin 0=1
$$

Solution (b). The given integral can be interpreted as the signed area between the graph of $y=\cos x$ and the interval $[0, \pi]$. The graph in Figure 5.6.4 suggests that over the interval $[0, \pi]$ the portion of area above the $x$-axis is the same as the portion of area below the $x$-axis,
so we conjecture that the signed area is zero; this implies that the value of the integral is zero. This is confirmed by the computations

$$
\left.\int_{0}^{\pi} \cos x d x=\sin x\right]_{0}^{\pi}=\sin \pi-\sin 0=0
$$

## THE RELATIONSHIP BETWEEN DEFINITE AND INDEFINITE INTEGRALS

Observe that in the preceding examples we did not include a constant of integration in the antiderivatives. In general, when applying the Fundamental Theorem of Calculus there is no need to include a constant of integration because it will drop out anyway. To see that this is so, let $F$ be any antiderivative of the integrand on $[a, b]$, and let $C$ be any constant; then

$$
\int_{a}^{b} f(x) d x=[F(x)+C]_{a}^{b}=[F(b)+C]-[F(a)+C]=F(b)-F(a)
$$

Thus, for purposes of evaluating a definite integral we can omit the constant of integration in

$$
\int_{a}^{b} f(x) d x=[F(x)+C]_{a}^{b}
$$

and express (5) as

$$
\begin{equation*}
\left.\int_{a}^{b} f(x) d x=\int f(x) d x\right]_{a}^{b} \tag{6}
\end{equation*}
$$

which relates the definite and indefinite integrals.

## Example 4

$$
\left.\left.\int_{1}^{9} \sqrt{x} d x=\int x^{1 / 2} d x\right]_{1}^{9}=\frac{2}{3} x^{3 / 2}\right]_{1}^{9}=\frac{2}{3}(27-1)=\frac{52}{3}
$$

- Example 5 Table 5.2 .1 will be helpful for the following computations.

$$
\begin{aligned}
& \left.\int_{4}^{9} x^{2} \sqrt{x} d x=\int_{4}^{9} x^{5 / 2} d x=\frac{2}{7} x^{7 / 2}\right]_{4}^{9}=\frac{2}{7}(2187-128)=\frac{4118}{7}=588 \frac{2}{7} \\
& \left.\int_{0}^{\pi / 2} \frac{\sin x}{5} d x=-\frac{1}{5} \cos x\right]_{0}^{\pi / 2}=-\frac{1}{5}\left[\cos \left(\frac{\pi}{2}\right)-\cos 0\right]=-\frac{1}{5}[0-1]=\frac{1}{5} \\
& \left.\int_{0}^{\pi / 3} \sec ^{2} x d x=\tan x\right]_{0}^{\pi / 3}=\tan \left(\frac{\pi}{3}\right)-\tan 0=\sqrt{3}-0=\sqrt{3} \\
& \left.\int_{0}^{\ln 3} 5 e^{x} d x=5 e^{x}\right]_{0}^{\ln 3}=5\left[e^{\ln 3}-e^{0}\right]=5[3-1]=10
\end{aligned}
$$

$$
\left.\int_{-e}^{-1} \frac{1}{x} d x=\ln |x|\right]_{-e}^{-1}=\ln |-1|-\ln |-e|=0-1=-1
$$

If you have a CAS, read the documentation on evaluating definite integrals and then check the results in Example 5.

$$
\left.\int_{-1 / 2}^{1 / 2} \frac{1}{\sqrt{1-x^{2}}} d x=\sin ^{-1} x\right]_{-1 / 2}^{1 / 2}=\sin ^{-1}\left(\frac{1}{2}\right)-\sin ^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)=\frac{\pi}{3}
$$



Figure 5.6.5

## WARNING

The requirements in the Fundamental Theorem of Calculus that $f$ be continuous on $[a, b]$ and that $F$ be an antiderivative for $f$ over the entire interval $[a, b]$ are important to keep in mind. Disregarding these assumptions will likely lead to incorrect results. For example, the function $f(x)=1 / x^{2}$ fails on two counts to be continuous at $x=0: f(x)$ is not defined at $x=0$ and $\lim _{x \rightarrow 0} f(x)$ does not exist. Thus, the Fundamental Theorem of Calculus should not be used to integrate $f$ on any interval that contains $x=0$. However, if we ignore this and mistakenly apply Formula (2) over the interval $[-1,1]$, we might incorrectly compute $\int_{-1}^{1}\left(1 / x^{2}\right) d x$ by evaluating an antiderivative, $-1 / x$, at the endpoints, arriving at the answer

$$
\left.-\frac{1}{x}\right]_{-1}^{1}=-[1-(-1)]=-2
$$

But $f(x)=1 / x^{2}$ is a nonnegative function, so clearly a negative value for the definite integral is impossible.

The Fundamental Theorem of Calculus can be applied without modification to definite integrals in which the lower limit of integration is greater than or equal to the upper limit of integration.

- Example 6

$$
\begin{aligned}
& \left.\int_{1}^{1} x^{2} d x=\frac{x^{3}}{3}\right]_{1}^{1}=\frac{1}{3}-\frac{1}{3}=0 \\
& \left.\int_{4}^{0} x d x=\frac{x^{2}}{2}\right]_{4}^{0}=\frac{0}{2}-\frac{16}{2}=-8
\end{aligned}
$$

The latter result is consistent with the result that would be obtained by first reversing the limits of integration in accordance with Definition 5.5.3(b):

$$
\left.\int_{4}^{0} x d x=-\int_{0}^{4} x d x=-\frac{x^{2}}{2}\right]_{0}^{4}=-\left[\frac{16}{2}-\frac{0}{2}\right]=-8
$$

To integrate a continuous function that is defined piecewise on an interval $[a, b]$, split this interval into subintervals at the breakpoints of the function, and integrate separately over each subinterval in accordance with Theorem 5.5.5.

Example 7 Evaluate $\int_{0}^{3} f(x) d x$ if

$$
f(x)= \begin{cases}x^{2}, & x<2 \\ 3 x-2, & x \geq 2\end{cases}
$$

Solution. See Figure 5.6.5. From Theorem 5.5 .5 we can integrate from 0 to 2 and from 2 to 3 separately and add the results. This yields

$$
\begin{aligned}
\int_{0}^{3} f(x) d x & =\int_{0}^{2} f(x) d x+\int_{2}^{3} f(x) d x=\int_{0}^{2} x^{2} d x+\int_{2}^{3}(3 x-2) d x \\
& \left.=\frac{x^{3}}{3}\right]_{0}^{2}+\left[\frac{3 x^{2}}{2}-2 x\right]_{2}^{3}=\left(\frac{8}{3}-0\right)+\left(\frac{15}{2}-2\right)=\frac{49}{6}
\end{aligned}
$$

If $f$ is a continuous function on the interval $[a, b]$, then we define the total area between the curve $y=f(x)$ and the interval $[a, b]$ to be

$$
\begin{equation*}
\text { total area }=\int_{a}^{b}|f(x)| d x \tag{7}
\end{equation*}
$$


(a)


$$
\text { Total area }=A_{I}+A_{I I}+A_{I I I}
$$

$\Delta$ Figure 5.6.6

$\Delta$ Figure 5.6.7
(Figure 5.6.6). To compute total area using Formula (7), begin by dividing the interval of integration into subintervals on which $f(x)$ does not change sign. On the subintervals for which $0 \leq f(x)$ replace $|f(x)|$ by $f(x)$, and on the subintervals for which $f(x) \leq 0$ replace $|f(x)|$ by $-f(x)$. Adding the resulting integrals then yields the total area.

- Example 8 Find the total area between the curve $y=1-x^{2}$ and the $x$-axis over the interval [0, 2] (Figure 5.6.7).

Solution. The area $A$ is given by

$$
\begin{aligned}
A=\int_{0}^{2}\left|1-x^{2}\right| d x & =\int_{0}^{1}\left(1-x^{2}\right) d x+\int_{1}^{2}-\left(1-x^{2}\right) d x \\
& =\left[x-\frac{x^{3}}{3}\right]_{0}^{1}-\left[x-\frac{x^{3}}{3}\right]_{1}^{2} \\
& =\frac{2}{3}-\left(-\frac{4}{3}\right)=2
\end{aligned}
$$

## DUMMY VARIABLES

To evaluate a definite integral using the Fundamental Theorem of Calculus, one needs to be able to find an antiderivative of the integrand; thus, it is important to know what kinds of functions have antiderivatives. It is our next objective to show that all continuous functions have antiderivatives, but to do this we will need some preliminary results.

Formula (6) shows that there is a close relationship between the integrals

$$
\int_{a}^{b} f(x) d x \text { and } \int f(x) d x
$$

However, the definite and indefinite integrals differ in some important ways. For one thing, the two integrals are different kinds of objects-the definite integral is a number (the net signed area between the graph of $y=f(x)$ and the interval $[a, b]$ ), whereas the indefinite integral is a function, or more accurately a family of functions [the antiderivatives of $f(x)$ ]. However, the two types of integrals also differ in the role played by the variable of integration. In an indefinite integral, the variable of integration is "passed through" to the antiderivative in the sense that integrating a function of $x$ produces a function of $x$, integrating a function of $t$ produces a function of $t$, and so forth. For example,

$$
\int x^{2} d x=\frac{x^{3}}{3}+C \quad \text { and } \quad \int t^{2} d t=\frac{t^{3}}{3}+C
$$

In contrast, the variable of integration in a definite integral is not passed through to the end result, since the end result is a number. Thus, integrating a function of $x$ over an interval and integrating the same function of $t$ over the same interval of integration produce the same value for the integral. For example,

$$
\left.\left.\int_{1}^{3} x^{2} d x=\frac{x^{3}}{3}\right]_{x=1}^{3}=\frac{27}{3}-\frac{1}{3}=\frac{26}{3} \quad \text { and } \quad \int_{1}^{3} t^{2} d t=\frac{t^{3}}{3}\right]_{t=1}^{3}=\frac{27}{3}-\frac{1}{3}=\frac{26}{3}
$$

However, this latter result should not be surprising, since the area under the graph of the curve $y=f(x)$ over an interval $[a, b]$ on the $x$-axis is the same as the area under the graph of the curve $y=f(t)$ over the interval $[a, b]$ on the $t$-axis (Figure 5.6.8).

$\Delta$ Figure 5.6.9


The area of the shaded rectangle is equal to the area of the shaded region in Figure 5.6.9.
$\Delta$ Figure 5.6.10

$>$ Figure 5.6.8

$$
A=\int_{a}^{b} f(x) d x=\int_{a}^{b} f(t) d t
$$

Because the variable of integration in a definite integral plays no role in the end result, it is often referred to as a dummy variable. In summary:

Whenever you find it convenient to change the letter used for the variable of integration in a definite integral, you can do so without changing the value of the integral.

## THE MEAN-VALUE THEOREM FOR INTEGRALS

To reach our goal of showing that continuous functions have antiderivatives, we will need to develop a basic property of definite integrals, known as the Mean-Value Theorem for Integrals. In Section 5.8 we will interpret this theorem using the concept of the "average value" of a continuous function over an interval. Here we will need it as a tool for developing other results.

Let $f$ be a continuous nonnegative function on $[a, b]$, and let $m$ and $M$ be the minimum and maximum values of $f(x)$ on this interval. Consider the rectangles of heights $m$ and $M$ over the interval $[a, b]$ (Figure 5.6.9). It is clear geometrically from this figure that the area

$$
A=\int_{a}^{b} f(x) d x
$$

under $y=f(x)$ is at least as large as the area of the rectangle of height $m$ and no larger than the area of the rectangle of height $M$. It seems reasonable, therefore, that there is a rectangle over the interval $[a, b]$ of some appropriate height $f\left(x^{*}\right)$ between $m$ and $M$ whose area is precisely $A$; that is,

$$
\int_{a}^{b} f(x) d x=f\left(x^{*}\right)(b-a)
$$

(Figure 5.6.10). This is a special case of the following result.
5.6.2 THEOREM (The Mean-Value Theorem for Integrals) If $f$ is continuous on a closed interval $[a, b]$, then there is at least one point $x^{*}$ in $[a, b]$ such that

$$
\begin{equation*}
\int_{a}^{b} f(x) d x=f\left(x^{*}\right)(b-a) \tag{8}
\end{equation*}
$$

Proof By the Extreme-Value Theorem (4.4.2), $f$ assumes a maximum value $M$ and a minimum value $m$ on $[a, b]$. Thus, for all $x$ in $[a, b]$,

$$
m \leq f(x) \leq M
$$

and from Theorem 5.5.6(b)

$$
\int_{a}^{b} m d x \leq \int_{a}^{b} f(x) d x \leq \int_{a}^{b} M d x
$$

or

$$
\begin{equation*}
m(b-a) \leq \int_{a}^{b} f(x) d x \leq M(b-a) \tag{9}
\end{equation*}
$$

or

$$
m \leq \frac{1}{b-a} \int_{a}^{b} f(x) d x \leq M
$$

This implies that

$$
\begin{equation*}
\frac{1}{b-a} \int_{a}^{b} f(x) d x \tag{10}
\end{equation*}
$$

is a number between $m$ and $M$, and since $f(x)$ assumes the values $m$ and $M$ on $[a, b]$, it follows from the Intermediate-Value Theorem (1.5.7) that $f(x)$ must assume the value (10) at some $x^{*}$ in $[a, b]$; that is,

$$
\frac{1}{b-a} \int_{a}^{b} f(x) d x=f\left(x^{*}\right) \quad \text { or } \quad \int_{a}^{b} f(x) d x=f\left(x^{*}\right)(b-a)
$$

Example 9 Since $f(x)=x^{2}$ is continuous on the interval [1,4], the Mean-Value Theorem for Integrals guarantees that there is a point $x^{*}$ in $[1,4]$ such that

$$
\int_{1}^{4} x^{2} d x=f\left(x^{*}\right)(4-1)=\left(x^{*}\right)^{2}(4-1)=3\left(x^{*}\right)^{2}
$$

But

$$
\left.\int_{1}^{4} x^{2} d x=\frac{x^{3}}{3}\right]_{1}^{4}=21
$$

so that

$$
3\left(x^{*}\right)^{2}=21 \quad \text { or } \quad\left(x^{*}\right)^{2}=7 \quad \text { or } \quad x^{*}= \pm \sqrt{7}
$$

Thus, $x^{*}=\sqrt{7} \approx 2.65$ is the point in the interval $[1,4]$ whose existence is guaranteed by the Mean-Value Theorem for Integrals.

## PART 2 OF THE FUNDAMENTAL THEOREM OF CALCULUS

In Section 5.1 we suggested that if $f$ is continuous and nonnegative on $[a, b]$, and if $A(x)$ is the area under the graph of $y=f(x)$ over the interval $[a, x]$ (Figure 5.6.2), then $A^{\prime}(x)=f(x)$. But $A(x)$ can be expressed as the definite integral

$$
A(x)=\int_{a}^{x} f(t) d t
$$

(where we have used $t$ rather than $x$ as the variable of integration to avoid confusion with the $x$ that appears as the upper limit of integration). Thus, the relationship $A^{\prime}(x)=f(x)$ can be expressed as

$$
\frac{d}{d x}\left[\int_{a}^{x} f(t) d t\right]=f(x)
$$

This is a special case of the following more general result, which applies even if $f$ has negative values.
5.6.3 THEOREM (The Fundamental Theorem of Calculus, Part 2) If $f$ is continuous on an interval, then $f$ has an antiderivative on that interval. In particular, if a is any point in the interval, then the function $F$ defined by

$$
F(x)=\int_{a}^{x} f(t) d t
$$

is an antiderivative of $f$; that is, $F^{\prime}(x)=f(x)$ for each $x$ in the interval, or in an alternative notation

$$
\begin{equation*}
\frac{d}{d x}\left[\int_{a}^{x} f(t) d t\right]=f(x) \tag{11}
\end{equation*}
$$

PROOF We will show first that $F(x)$ is defined at each $x$ in the interval. If $x>a$ and $x$ is in the interval, then Theorem 5.5.2 applied to the interval $[a, x]$ and the continuity of $f$ ensure that $F(x)$ is defined; and if $x$ is in the interval and $x \leq a$, then Definition 5.5.3 combined with Theorem 5.5.2 ensures that $F(x)$ is defined. Thus, $F(x)$ is defined for all $x$ in the interval.

Next we will show that $F^{\prime}(x)=f(x)$ for each $x$ in the interval. If $x$ is not an endpoint, then it follows from the definition of a derivative that

$$
\begin{align*}
F^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{F(x+h)-F(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\int_{a}^{x+h} f(t) d t-\int_{a}^{x} f(t) d t\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\int_{a}^{x+h} f(t) d t+\int_{x}^{a} f(t) d t\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h} \int_{x}^{x+h} f(t) d t \tag{12}
\end{align*}
$$

Applying the Mean-Value Theorem for Integrals (5.6.2) to the integral in (12) we obtain

$$
\begin{equation*}
\frac{1}{h} \int_{x}^{x+h} f(t) d t=\frac{1}{h}\left[f\left(t^{*}\right) \cdot h\right]=f\left(t^{*}\right) \tag{13}
\end{equation*}
$$

where $t^{*}$ is some number between $x$ and $x+h$. Because $t^{*}$ is trapped between $x$ and $x+h$, it follows that $t^{*} \rightarrow x$ as $h \rightarrow 0$. Thus, the continuity of $f$ at $x$ implies that $f\left(t^{*}\right) \rightarrow f(x)$ as $h \rightarrow 0$. Therefore, it follows from (12) and (13) that

$$
F^{\prime}(x)=\lim _{h \rightarrow 0}\left(\frac{1}{h} \int_{x}^{x+h} f(t) d t\right)=\lim _{h \rightarrow 0} f\left(t^{*}\right)=f(x)
$$

If $x$ is an endpoint of the interval, then the two-sided limits in the proof must be replaced by the appropriate one-sided limits, but otherwise the arguments are identical.

In words, Formula (11) states:

If a definite integral has a variable upper limit of integration, a constant lower limit of integration, and a continuous integrand, then the derivative of the integral with respect to its upper limit is equal to the integrand evaluated at the upper limit.

- Example 10 Find

$$
\frac{d}{d x}\left[\int_{1}^{x} t^{3} d t\right]
$$

by applying Part 2 of the Fundamental Theorem of Calculus, and then confirm the result by performing the integration and then differentiating.

Solution. The integrand is a continuous function, so from (11)

$$
\frac{d}{d x}\left[\int_{1}^{x} t^{3} d t\right]=x^{3}
$$

Alternatively, evaluating the integral and then differentiating yields

$$
\left.\int_{1}^{x} t^{3} d t=\frac{t^{4}}{4}\right]_{t=1}^{x}=\frac{x^{4}}{4}-\frac{1}{4}, \quad \frac{d}{d x}\left[\frac{x^{4}}{4}-\frac{1}{4}\right]=x^{3}
$$

so the two methods for differentiating the integral agree.

- Example 11 Since

$$
f(x)=\frac{\sin x}{x}
$$

is continuous on any interval that does not contain the origin, it follows from (11) that on the interval $(0,+\infty)$ we have

$$
\frac{d}{d x}\left[\int_{1}^{x} \frac{\sin t}{t} d t\right]=\frac{\sin x}{x}
$$

Unlike the preceding example, there is no way to evaluate the integral in terms of familiar functions, so Formula (11) provides the only simple method for finding the derivative.

## DIFFERENTIATION AND INTEGRATION ARE INVERSE PROCESSES

The two parts of the Fundamental Theorem of Calculus, when taken together, tell us that differentiation and integration are inverse processes in the sense that each undoes the effect of the other. To see why this is so, note that Part 1 of the Fundamental Theorem of Calculus (5.6.1) implies that

$$
\int_{a}^{x} f^{\prime}(t) d t=f(x)-f(a)
$$

which tells us that if the value of $f(a)$ is known, then the function $f$ can be recovered from its derivative $f^{\prime}$ by integrating. Conversely, Part 2 of the Fundamental Theorem of Calculus (5.6.3) states that

$$
\frac{d}{d x}\left[\int_{a}^{x} f(t) d t\right]=f(x)
$$

which tells us that the function $f$ can be recovered from its integral by differentiating. Thus, differentiation and integration can be viewed as inverse processes.

It is common to treat parts 1 and 2 of the Fundamental Theorem of Calculus as a single theorem and refer to it simply as the Fundamental Theorem of Calculus. This theorem ranks as one of the greatest discoveries in the history of science, and its formulation by Newton and Leibniz is generally regarded to be the "discovery of calculus."

INTEGRATING RATES OF CHANGE
The Fundamental Theorem of Calculus

$$
\begin{equation*}
\int_{a}^{b} f(x) d x=F(b)-F(a) \tag{14}
\end{equation*}
$$



Integrating the slope of $y=F(x)$ over the interval $[a, b]$ produces the change $F(b)-F(a)$ in the value of $F(x)$.

- Figure 5.6.11
has a useful interpretation that can be seen by rewriting it in a slightly different form. Since $F$ is an antiderivative of $f$ on the interval $[a, b]$, we can use the relationship $F^{\prime}(x)=f(x)$ to rewrite (14) as

$$
\begin{equation*}
\int_{a}^{b} F^{\prime}(x) d x=F(b)-F(a) \tag{15}
\end{equation*}
$$

In this formula we can view $F^{\prime}(x)$ as the rate of change of $F(x)$ with respect to $x$, and we can view $F(b)-F(a)$ as the change in the value of $F(x)$ as $x$ increases from $a$ to $b$ (Figure 5.6.11). Thus, we have the following useful principle.
5.6.4 INTEGRATING A RATE OF CHANGE Integrating the rate of change of $F(x)$ with respect to $x$ over an interval $[a, b]$ produces the change in the value of $F(x)$ that occurs as $x$ increases from $a$ to $b$.

Here are some examples of this idea:

- If $s(t)$ is the position of a particle in rectilinear motion, then $s^{\prime}(t)$ is the instantaneous velocity of the particle at time $t$, and

$$
\int_{t_{1}}^{t_{2}} s^{\prime}(t) d t=s\left(t_{2}\right)-s\left(t_{1}\right)
$$

is the displacement (or the change in the position) of the particle between the times $t_{1}$ and $t_{2}$.

- If $P(t)$ is a population (e.g., plants, animals, or people) at time $t$, then $P^{\prime}(t)$ is the rate at which the population is changing at time $t$, and

$$
\int_{t_{1}}^{t_{2}} P^{\prime}(t) d t=P\left(t_{2}\right)-P\left(t_{1}\right)
$$

is the change in the population between times $t_{1}$ and $t_{2}$.

- If $A(t)$ is the area of an oil spill at time $t$, then $A^{\prime}(t)$ is the rate at which the area of the spill is changing at time $t$, and

$$
\int_{t_{1}}^{t_{2}} A^{\prime}(t) d t=A\left(t_{2}\right)-A\left(t_{1}\right)
$$

is the change in the area of the spill between times $t_{1}$ and $t_{2}$.

- If $P^{\prime}(x)$ is the marginal profit that results from producing and selling $x$ units of a product (see Section 4.5), then

$$
\int_{x_{1}}^{x_{2}} P^{\prime}(x) d x=P\left(x_{2}\right)-P\left(x_{1}\right)
$$

is the change in the profit that results when the production level increases from $x_{1}$ units to $x_{2}$ units.

## QUICK CHECK EXERCISES 5.6 (See page 376 for answers.)

1. (a) If $F(x)$ is an antiderivative for $f(x)$, then

$$
\int_{a}^{b} f(x) d x=
$$

(b) $\int_{a}^{b} F^{\prime}(x) d x=$
(c) $\frac{d}{d x}\left[\int_{a}^{x} f(t) d t\right]=$
$\qquad$
$\qquad$
2. (a) $\int_{0}^{2}\left(3 x^{2}-2 x\right) d x=$ $\qquad$
(b) $\int_{-\pi}^{\pi} \cos x d x=$ $\qquad$
(c) $\int_{0}^{\frac{1}{2} \ln 5} e^{x} d x=$ $\qquad$
(d) $\int_{-1 / 2}^{1 / 2} \frac{1}{\sqrt{1-x^{2}}} d x=$
3. For the function $f(x)=3 x^{2}-2 x$ and an interval $[a, b]$, the point $x^{*}$ guaranteed by the Mean-Value Theorem for Integrals is $x^{*}=\frac{2}{3}$. It follows that

$$
\int_{a}^{b}\left(3 x^{2}-2 x\right) d x=
$$

4. The area of an oil spill is increasing at a rate of $25 t \mathrm{ft}^{2} / \mathrm{s}$ $t$ seconds after the spill. Between times $t=2$ and $t=4$ the area of the spill increases by $\qquad$ $\mathrm{ft}^{2}$.

## EXERCISE SET 5.6 $\square$ Graphing Utility © CAS

1. In each part, use a definite integral to find the area of the region, and check your answer using an appropriate formula from geometry.
(a)

(b)
(c)


2. In each part, use a definite integral to find the area under the curve $y=f(x)$ over the stated interval, and check your answer using an appropriate formula from geometry.
(a) $f(x)=x$; $[0,5]$
(b) $f(x)=5 ;[3,9]$
(c) $f(x)=x+3 ;[-1,2]$
3. In each part, sketch the analogue of Figure 5.6 .10 for the specified region. [Let $y=f(x)$ denote the upper boundary of the region. If $x^{*}$ is unique, label both it and $f\left(x^{*}\right)$ on your sketch. Otherwise, label $f\left(x^{*}\right)$ on your sketch, and determine all values of $x^{*}$ that satisfy Equation (8).]
(a) The region in part (a) of Exercise 1.
(b) The region in part (b) of Exercise 1.
(c) The region in part (c) of Exercise 1.
4. In each part, sketch the analogue of Figure 5.6.10 for the function and interval specified. [If $x^{*}$ is unique, label both it and $f\left(x^{*}\right)$ on your sketch. Otherwise, label $f\left(x^{*}\right)$ on your sketch, and determine all values of $x^{*}$ that satisfy Equation (8).]
(a) The function and interval in part (a) of Exercise 2.
(b) The function and interval in part (b) of Exercise 2.
(c) The function and interval in part (c) of Exercise 2.

5-10 Find the area under the curve $y=f(x)$ over the stated interval.
5. $f(x)=x^{3} ;[2,3]$
6. $f(x)=x^{4} ;[-1,1]$
7. $f(x)=3 \sqrt{x} ;[1,4]$
8. $f(x)=x^{-2 / 3} ;[1,27]$
9. $f(x)=e^{2 x} ;[0, \ln 2]$
10. $f(x)=\frac{1}{x} ;[1,5]$

11-12 Find all values of $x^{*}$ in the stated interval that satisfy Equation (8) in the Mean-Value Theorem for Integrals (5.6.2), and explain what these numbers represent.
11. (a) $f(x)=\sqrt{x} ;[0,3]$
(b) $f(x)=x^{2}+x ;[-12,0]$
12. (a) $f(x)=\sin x ;[-\pi, \pi]$
(b) $f(x)=1 / x^{2} ;[1,3]$

13-30 Evaluate the integrals using Part 1 of the Fundamental Theorem of Calculus.
13. $\int_{-2}^{1}\left(x^{2}-6 x+12\right) d x$
14. $\int_{-1}^{2} 4 x\left(1-x^{2}\right) d x$
15. $\int_{1}^{4} \frac{4}{x^{2}} d x$
16. $\int_{1}^{2} \frac{1}{x^{6}} d x$
17. $\int_{4}^{9} 2 x \sqrt{x} d x$
18. $\int_{1}^{4} \frac{1}{x \sqrt{x}} d x$
19. $\int_{-\pi / 2}^{\pi / 2} \sin \theta d \theta$
20. $\int_{0}^{\pi / 4} \sec ^{2} \theta d \theta$
21. $\int_{-\pi / 4}^{\pi / 4} \cos x d x$
22. $\int_{0}^{\pi / 3}(2 x-\sec x \tan x) d x$
23. $\int_{\ln 2}^{3} 5 e^{x} d x$
24. $\int_{1 / 2}^{1} \frac{1}{2 x} d x$
25. $\int_{0}^{1 / \sqrt{2}} \frac{d x}{\sqrt{1-x^{2}}}$
26. $\int_{-1}^{1} \frac{d x}{1+x^{2}}$
27. $\int_{\sqrt{2}}^{2} \frac{d x}{x \sqrt{x^{2}-1}}$
28. $\int_{-\sqrt{2}}^{-2 / \sqrt{3}} \frac{d x}{x \sqrt{x^{2}-1}}$
29. $\int_{1}^{4}\left(\frac{1}{\sqrt{t}}-3 \sqrt{t}\right) d t$
30. $\int_{\pi / 6}^{\pi / 2}\left(x+\frac{2}{\sin ^{2} x}\right) d x$

31-34 Use Theorem 5.5.5 to evaluate the given integrals.
31. (a) $\int_{-1}^{1}|2 x-1| d x$
(b) $\int_{0}^{3 \pi / 4}|\cos x| d x$
32. (a) $\int_{-1}^{2} \sqrt{2+|x|} d x$
(b) $\int_{0}^{\pi / 2}\left|\frac{1}{2}-\cos x\right| d x$
33. (a) $\int_{-1}^{1}\left|e^{x}-1\right| d x$
(b) $\int_{1}^{4} \frac{|2-x|}{x} d x$
34. (a) $\int_{-3}^{3}\left|x^{2}-1-\frac{15}{x^{2}+1}\right| d x$
(b) $\int_{0}^{\sqrt{3} / 2}\left|\frac{1}{\sqrt{1-x^{2}}}-\sqrt{2}\right| d x$

35-36 A function $f(x)$ is defined piecewise on an interval. In these exercises: (a) Use Theorem 5.5.5 to find the integral of $f(x)$ over the interval. (b) Find an antiderivative of $f(x)$ on the interval. (c) Use parts (a) and (b) to verify Part 1 of the Fundamental Theorem of Calculus.
35. $f(x)= \begin{cases}x, & 0 \leq x \leq 1 \\ x^{2}, & 1<x \leq 2\end{cases}$
36. $f(x)= \begin{cases}\sqrt{x}, & 0 \leq x<1 \\ 1 / x^{2}, & 1 \leq x \leq 4\end{cases}$

37-40 True-False Determine whether the statement is true or false. Explain your answer.
37. There does not exist a differentiable function $F(x)$ such that $F^{\prime}(x)=|x|$.
38. If $f(x)$ is continuous on the interval $[a, b]$, and if the definite integral of $f(x)$ over this interval has value 0 , then the equation $f(x)=0$ has at least one solution in the interval $[a, b]$.
39. If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, then

$$
\int_{a}^{b} f(x) d x=\int_{a}^{b} g(x) d x
$$

if and only if

$$
G(a)+F(b)=F(a)+G(b)
$$

40. If $f(x)$ is continuous everywhere and

$$
F(x)=\int_{0}^{x} f(t) d t
$$

then the equation $F(x)=0$ has at least one solution.
41-44 Use a calculating utility to find the midpoint approximation of the integral using $n=20$ subintervals, and then find the exact value of the integral using Part 1 of the Fundamental Theorem of Calculus.
41. $\int_{1}^{3} \frac{1}{x^{2}} d x$
42. $\int_{0}^{\pi / 2} \sin x d x$
43. $\int_{-1}^{1} \sec ^{2} x d x$
44. $\int_{1}^{3} \frac{1}{x} d x$

45-48 Sketch the region described and find its area.
45. The region under the curve $y=x^{2}+1$ and over the interval $[0,3]$.
46. The region below the curve $y=x-x^{2}$ and above the $x$ axis.
47. The region under the curve $y=3 \sin x$ and over the interval $[0,2 \pi / 3]$.
48. The region below the interval $[-2,-1]$ and above the curve $y=x^{3}$.

49-52 Sketch the curve and find the total area between the curve and the given interval on the $x$-axis.
49. $y=x^{2}-x ;[0,2]$
50. $y=\sin x ;[0,3 \pi / 2]$
51. $y=e^{x}-1 ;[-1,1]$
52. $y=\frac{x^{2}-1}{x^{2}} ;\left[\frac{1}{2}, 2\right]$
53. A student wants to find the area enclosed by the graphs of $y=1 / \sqrt{1-x^{2}}, y=0, x=0$, and $x=0.8$.
(a) Show that the exact area is $\sin ^{-1} 0.8$.
(b) The student uses a calculator to approximate the result in part (a) to two decimal places and obtains an incorrect answer of 53.13. What was the student's error? Find the correct approximation.

## FOCUS ON CONCEPTS

54. Explain why the Fundamental Theorem of Calculus may be applied without modification to definite integrals in which the lower limit of integration is greater than or equal to the upper limit of integration.
55. (a) If $h^{\prime}(t)$ is the rate of change of a child's height measured in inches per year, what does the integral $\int_{0}^{10} h^{\prime}(t) d t$ represent, and what are its units?
(b) If $r^{\prime}(t)$ is the rate of change of the radius of a spherical balloon measured in centimeters per second, what does the integral $\int_{1}^{2} r^{\prime}(t) d t$ represent, and what are its units?
(c) If $H(t)$ is the rate of change of the speed of sound with respect to temperature measured in $\mathrm{ft} / \mathrm{s}$ per ${ }^{\circ} \mathrm{F}$, what does the integral $\int_{32}^{100} H(t) d t$ represent, and what are its units?
(d) If $v(t)$ is the velocity of a particle in rectilinear motion, measured in $\mathrm{cm} / \mathrm{h}$, what does the integral $\int_{t_{1}}^{t_{2}} v(t) d t$ represent, and what are its units?
56. (a) Use a graphing utility to generate the graph of

$$
f(x)=\frac{1}{100}(x+2)(x+1)(x-3)(x-5)
$$

and use the graph to make a conjecture about the sign of the integral

$$
\int_{-2}^{5} f(x) d x
$$

(b) Check your conjecture by evaluating the integral.
57. Define $F(x)$ by

$$
F(x)=\int_{1}^{x}\left(3 t^{2}-3\right) d t
$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F^{\prime}(x)$.
(b) Check the result in part (a) by first integrating and then differentiating.
58. Define $F(x)$ by

$$
F(x)=\int_{\pi / 4}^{x} \cos 2 t d t
$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F^{\prime}(x)$.
(b) Check the result in part (a) by first integrating and then differentiating.

59-62 Use Part 2 of the Fundamental Theorem of Calculus to find the derivatives.
59. (a) $\frac{d}{d x} \int_{1}^{x} \sin \left(t^{2}\right) d t$
(b) $\frac{d}{d x} \int_{0}^{x} e^{\sqrt{t}} d t$
60. (a) $\frac{d}{d x} \int_{0}^{x} \frac{d t}{1+\sqrt{t}}$
(b) $\frac{d}{d x} \int_{1}^{x} \ln t d t$
61. $\frac{d}{d x} \int_{x}^{0} t \sec t d t$
[Hint: Use Definition 5.5.3(b).]
62. $\frac{d}{d u} \int_{0}^{u}|x| d x$
63. Let $F(x)=\int_{4}^{x} \sqrt{t^{2}+9} d t$. Find
(a) $F(4)$
(b) $F^{\prime}(4)$
(c) $F^{\prime \prime}(4)$.
64. Let $F(x)=\int_{\sqrt{3}}^{x} \tan ^{-1} t d t$. Find
(a) $F(\sqrt{3})$
(b) $F^{\prime}(\sqrt{3})$
(c) $F^{\prime \prime}(\sqrt{3})$.
65. Let $F(x)=\int_{0}^{x} \frac{t-3}{t^{2}+7} d t$ for $-\infty<x<+\infty$.
(a) Find the value of $x$ where $F$ attains its minimum value.
(b) Find intervals over which $F$ is only increasing or only decreasing.
(c) Find open intervals over which $F$ is only concave up or only concave down.
66. Use the plotting and numerical integration commands of a CAS to generate the graph of the function $F$ in Exercise 65 over the interval $-20 \leq x \leq 20$, and confirm that the graph is consistent with the results obtained in that exercise.
67. (a) Over what open interval does the formula

$$
F(x)=\int_{1}^{x} \frac{d t}{t}
$$

represent an antiderivative of $f(x)=1 / x$ ?
(b) Find a point where the graph of $F$ crosses the $x$-axis.
68. (a) Over what open interval does the formula

$$
F(x)=\int_{1}^{x} \frac{1}{t^{2}-9} d t
$$

represent an antiderivative of

$$
f(x)=\frac{1}{x^{2}-9} ?
$$

(b) Find a point where the graph of $F$ crosses the $x$-axis.
69. (a) Suppose that a reservoir supplies water to an industrial park at a constant rate of $r=4$ gallons per minute ( $\mathrm{gal} / \mathrm{min}$ ) between 8:30 A.m. and 9:00 A.m. How much water does the reservoir supply during that time period?
(b) Suppose that one of the industrial plants increases its water consumption between 9:00 A.M. and 10:00 A.m. and that the rate at which the reservoir supplies water increases linearly, as shown in the accompanying figure. How much water does the reservoir supply during that 1-hour time period?
(c) Suppose that from 10:00 A.m. to 12 noon the rate at which the reservoir supplies water is given by the formula $r(t)=10+\sqrt{t} \mathrm{gal} / \mathrm{min}$, where $t$ is the time (in minutes) since 10:00 A.m. How much water does the reservoir supply during that 2 -hour time period?

70. A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30 P.m. and 5:30 P.M. the rate $R(t)$ at which cars enter the highway is given by the formula $R(t)=100\left(1-0.0001 t^{2}\right)$ cars per minute, where $t$ is the time (in minutes) since 4:30 P.m.
(a) When does the peak traffic flow into the highway occur?
(b) Estimate the number of cars that enter the highway during the rush hour.

71-72 Evaluate each limit by interpreting it as a Riemann sum in which the given interval is divided into $n$ subintervals of equal width.
71. $\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} \frac{\pi}{4 n} \sec ^{2}\left(\frac{\pi k}{4 n}\right) ;\left[0, \frac{\pi}{4}\right]$
72. $\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} \frac{n}{n^{2}+k^{2}} ;[0,1]$
73. Prove the Mean-Value Theorem for Integrals (Theorem 5.6.2) by applying the Mean-Value Theorem (4.8.2) to an antiderivative $F$ for $f$.
74. Writing Write a short paragraph that describes the various ways in which integration and differentiation may be viewed as inverse processes. (Be sure to discuss both definite and indefinite integrals.)
75. Writing Let $f$ denote a function that is continuous on an interval $[a, b]$, and let $x^{*}$ denote the point guaranteed by the Mean-Value Theorem for Integrals. Explain geometrically why $f\left(x^{*}\right)$ may be interpreted as a "mean" or average value of $f(x)$ over $[a, b]$. (In Section 5.8 we will discuss the concept of "average value" in more detail.)

1. (a) $F(b)-F(a)$
(b) $F(b)-F(a)$
(c) $f(x)$
2. (a) 4 (b) 0
(c) $\sqrt{5}-1$
(d) $\pi / 3$
3. 0
4. $150 \mathrm{ft}^{2}$

### 5.7 RECTILINEAR MOTION REVISITED USING INTEGRATION

In Section 4.6 we used the derivative to define the notions of instantaneous velocity and acceleration for a particle in rectilinear motion. In this section we will resume the study of such motion using the tools of integration.

$\Delta$ Figure 5.7.1


There is a unique velocity function such that $v\left(t_{0}\right)=v_{0}$.
© Figure 5.7.2

## FINDING POSITION AND VELOCITY BY INTEGRATION

Recall from Formulas (1) and (3) of Section 4.6 that if a particle in rectilinear motion has position function $s(t)$, then its instantaneous velocity and acceleration are given by the formulas

$$
v(t)=s^{\prime}(t) \quad \text { and } \quad a(t)=v^{\prime}(t)
$$

It follows from these formulas that $s(t)$ is an antiderivative of $v(t)$ and $v(t)$ is an antiderivative of $a(t)$; that is,

$$
\begin{equation*}
s(t)=\int v(t) d t \quad \text { and } \quad v(t)=\int a(t) d t \tag{1-2}
\end{equation*}
$$

By Formula (1), if we know the velocity function $v(t)$ of a particle in rectilinear motion, then by integrating $v(t)$ we can produce a family of position functions with that velocity function. If, in addition, we know the position $s_{0}$ of the particle at any time $t_{0}$, then we have sufficient information to find the constant of integration and determine a unique position function (Figure 5.7.1). Similarly, if we know the acceleration function $a(t)$ of the particle, then by integrating $a(t)$ we can produce a family of velocity functions with that acceleration function. If, in addition, we know the velocity $v_{0}$ of the particle at any time $t_{0}$, then we have sufficient information to find the constant of integration and determine a unique velocity function (Figure 5.7.2).

- Example 1 Suppose that a particle moves with velocity $v(t)=\cos \pi t$ along a coordinate line. Assuming that the particle has coordinate $s=4$ at time $t=0$, find its position function.

Solution. The position function is

$$
s(t)=\int v(t) d t=\int \cos \pi t d t=\frac{1}{\pi} \sin \pi t+C
$$

Since $s=4$ when $t=0$, it follows that

$$
4=s(0)=\frac{1}{\pi} \sin 0+C=C
$$

Thus,

$$
s(t)=\frac{1}{\pi} \sin \pi t+4
$$

## COMPUTING DISPLACEMENT AND DISTANCE TRAVELED BY INTEGRATION

Recall that the displacement over a time interval of a particle in rectilinear motion is its final coordinate minus its initial coordinate. Thus, if the position function of the particle is $s(t)$,

Recall that Formula (3) is a special case of the formula

$$
\int_{a}^{b} F^{\prime}(x) d x=F(b)-F(a)
$$

for integrating a rate of change.


A Figure 5.7.3

In physical problems it is important to associate correct units with definite integrals. In general, the units for

$$
\int_{a}^{b} f(x) d x
$$

are units of $f(x)$ times units of $x$, since the integral is the limit of Riemann sums, each of whose terms has these units. For example, if $v(t)$ is in meters per second $(\mathrm{m} / \mathrm{s})$ and $t$ is in seconds (s), then

$$
\int_{a}^{b} v(t) d t
$$

is in meters since

$$
(\mathrm{m} / \mathrm{s}) \times \mathrm{s}=\mathrm{m}
$$

then its displacement (or change in position) over the time interval $\left[t_{0}, t_{1}\right]$ is $s\left(t_{1}\right)-s\left(t_{0}\right)$. This can be written in integral form as

$$
\left[\begin{array}{c}
\text { displacement }  \tag{3}\\
\text { over the time } \\
\text { interval }\left[t_{0}, t_{1}\right]
\end{array}\right]=\int_{t_{0}}^{t_{1}} v(t) d t=\int_{t_{0}}^{t_{1}} s^{\prime}(t) d t=s\left(t_{1}\right)-s\left(t_{0}\right)
$$

In contrast, to find the distance traveled by the particle over the time interval $\left[t_{0}, t_{1}\right]$ (distance traveled in the positive direction plus the distance traveled in the negative direction), we must integrate the absolute value of the velocity function; that is,

$$
\left[\begin{array}{c}
\text { distance traveled }  \tag{4}\\
\text { during time } \\
\text { interval }\left[t_{0}, t_{1}\right]
\end{array}\right]=\int_{t_{0}}^{t_{1}}|v(t)| d t
$$

Since the absolute value of velocity is speed, Formulas (3) and (4) can be summarized informally as follows:

Integrating velocity over a time interval produces displacement, and integrating speed over a time interval produces distance traveled.

- Example 2 Suppose that a particle moves on a coordinate line so that its velocity at time $t$ is $v(t)=t^{2}-2 t \mathrm{~m} / \mathrm{s}$ (Figure 5.7.3).
(a) Find the displacement of the particle during the time interval $0 \leq t \leq 3$.
(b) Find the distance traveled by the particle during the time interval $0 \leq t \leq 3$.

Solution (a). From (3) the displacement is

$$
\int_{0}^{3} v(t) d t=\int_{0}^{3}\left(t^{2}-2 t\right) d t=\left[\frac{t^{3}}{3}-t^{2}\right]_{0}^{3}=0
$$

Thus, the particle is at the same position at time $t=3$ as at $t=0$.
Solution (b). The velocity can be written as $v(t)=t^{2}-2 t=t(t-2)$, from which we see that $v(t) \leq 0$ for $0 \leq t \leq 2$ and $v(t) \geq 0$ for $2 \leq t \leq 3$. Thus, it follows from (4) that the distance traveled is

$$
\begin{aligned}
\int_{0}^{3}|v(t)| d t & =\int_{0}^{2}-v(t) d t+\int_{2}^{3} v(t) d t \\
& =\int_{0}^{2}-\left(t^{2}-2 t\right) d t+\int_{2}^{3}\left(t^{2}-2 t\right) d t \\
& =-\left[\frac{t^{3}}{3}-t^{2}\right]_{0}^{2}+\left[\frac{t^{3}}{3}-t^{2}\right]_{2}^{3}=\frac{4}{3}+\frac{4}{3}=\frac{8}{3} \mathrm{~m}
\end{aligned}
$$

## ANALYZING THE VELOCITY VERSUS TIME CURVE

In Section 4.6 we showed how to use the position versus time curve to obtain information about the behavior of a particle in rectilinear motion (Table 4.6.1). Similarly, there is valuable information that can be obtained from the velocity versus time curve. For example, the integral in (3) can be interpreted geometrically as the net signed area between the graph

$\Delta$ Figure 5.7.4
of $v(t)$ and the interval $\left[t_{0}, t_{1}\right]$, and the integral in (4) can be interpreted as the total area between the graph of $v(t)$ and the interval $\left[t_{0}, t_{1}\right]$. Thus we have the following result.

### 5.7.1 FINDING DISPLACEMENT AND DISTANCE TRAVELED FROM THE VELOCITY

 VERSUS TIME CURVE For a particle in rectilinear motion, the net signed area between the velocity versus time curve and the interval $\left[t_{0}, t_{1}\right]$ on the $t$-axis represents the displacement of the particle over that time interval, and the total area between the velocity versus time curve and the interval $\left[t_{0}, t_{1}\right]$ on the $t$-axis represents the distance traveled by the particle over that time interval (Figure 5.7.4).Example 3 Figure 5.7 .5 shows three velocity versus time curves for a particle in rectilinear motion along a horizontal line with the positive direction to the right. In each case find the displacement and the distance traveled over the time interval $0 \leq t \leq 4$, and explain what that information tells you about the motion of the particle.

Solution (a). In part (a) of the figure the area and the net signed area over the interval are both 2 . Thus, at the end of the time period the particle is 2 units to the right of its starting point and has traveled a distance of 2 units.

Solution (b). In part (b) of the figure the net signed area is -2 , and the total area is 2 . Thus, at the end of the time period the particle is 2 units to the left of its starting point and has traveled a distance of 2 units.

Solution (c). In part (c) of the figure the net signed area is 0 , and the total area is 2. Thus, at the end of the time period the particle is back at its starting point and has traveled a distance of 2 units. More specifically, it traveled 1 unit to the right over the time interval $0 \leq t \leq 1$ and then 1 unit to the left over the time interval $1 \leq t \leq 2$ (why?).

$\Delta$ Figure 5.7.5

## CONSTANT ACCELERATION

One of the most important cases of rectilinear motion occurs when a particle has constant acceleration. We will show that if a particle moves with constant acceleration along an $s$-axis, and if the position and velocity of the particle are known at some point in time, say when $t=0$, then it is possible to derive formulas for the position $s(t)$ and the velocity $v(t)$ at any time $t$. To see how this can be done, suppose that the particle has constant acceleration

$$
\begin{equation*}
a(t)=a \tag{5}
\end{equation*}
$$

and

$$
\begin{array}{lll}
s=s_{0} & \text { when } & t=0 \\
v=v_{0} & \text { when } & t=0 \tag{7}
\end{array}
$$

where $s_{0}$ and $v_{0}$ are known. We call (6) and (7) the initial conditions.

With (5) as a starting point, we can integrate $a(t)$ to obtain $v(t)$, and we can integrate $v(t)$ to obtain $s(t)$, using an initial condition in each case to determine the constant of integration. The computations are as follows:

$$
\begin{equation*}
v(t)=\int a(t) d t=\int a d t=a t+C_{1} \tag{8}
\end{equation*}
$$

To determine the constant of integration $C_{1}$ we apply initial condition (7) to this equation to obtain

$$
v_{0}=v(0)=a \cdot 0+C_{1}=C_{1}
$$

Substituting this in (8) and putting the constant term first yields

$$
v(t)=v_{0}+a t
$$

Since $v_{0}$ is constant, it follows that

$$
\begin{equation*}
s(t)=\int v(t) d t=\int\left(v_{0}+a t\right) d t=v_{0} t+\frac{1}{2} a t^{2}+C_{2} \tag{9}
\end{equation*}
$$

To determine the constant $C_{2}$ we apply initial condition (6) to this equation to obtain

$$
s_{0}=s(0)=v_{0} \cdot 0+\frac{1}{2} a \cdot 0+C_{2}=C_{2}
$$

Substituting this in (9) and putting the constant term first yields

$$
s(t)=s_{0}+v_{0} t+\frac{1}{2} a t^{2}
$$

In summary, we have the following result.
5.7.2 CONSTANT ACCELERATION If a particle moves with constant acceleration $a$ along an $s$-axis, and if the position and velocity at time $t=0$ are $s_{0}$ and $v_{0}$, respectively, then the position and velocity functions of the particle are

$$
\begin{gather*}
s(t)=s_{0}+v_{0} t+\frac{1}{2} a t^{2}  \tag{10}\\
v(t)=v_{0}+a t \tag{11}
\end{gather*}
$$

- Example 4 Suppose that an intergalactic spacecraft uses a sail and the "solar wind" to produce a constant acceleration of $0.032 \mathrm{~m} / \mathrm{s}^{2}$. Assuming that the spacecraft has a velocity of $10,000 \mathrm{~m} / \mathrm{s}$ when the sail is first raised, how far will the spacecraft travel in 1 hour, and what will its velocity be at the end of this hour?

Solution. In this problem the choice of a coordinate axis is at our discretion, so we will choose it to make the computations as simple as possible. Accordingly, let us introduce an $s$-axis whose positive direction is in the direction of motion, and let us take the origin to coincide with the position of the spacecraft at the time $t=0$ when the sail is raised. Thus, Formulas (10) and (11) apply with

$$
s_{0}=s(0)=0, \quad v_{0}=v(0)=10,000, \quad \text { and } \quad a=0.032
$$

Since 1 hour corresponds to $t=3600 \mathrm{~s}$, it follows from (10) that in 1 hour the spacecraft travels a distance of

$$
s(3600)=10,000(3600)+\frac{1}{2}(0.032)(3600)^{2} \approx 36,200,000 \mathrm{~m}
$$

and it follows from (11) that after 1 hour its velocity is

$$
v(3600)=10,000+(0.032)(3600) \approx 10,100 \mathrm{~m} / \mathrm{s}
$$


$\triangle$ Figure 5.7.6

- Example 5 A bus has stopped to pick up riders, and a woman is running at a constant velocity of $5 \mathrm{~m} / \mathrm{s}$ to catch it. When she is 11 m behind the front door the bus pulls away with a constant acceleration of $1 \mathrm{~m} / \mathrm{s}^{2}$. From that point in time, how long will it take for the woman to reach the front door of the bus if she keeps running with a velocity of $5 \mathrm{~m} / \mathrm{s}$ ?

Solution. As shown in Figure 5.7.6, choose the $s$-axis so that the bus and the woman are moving in the positive direction, and the front door of the bus is at the origin at the time $t=0$ when the bus begins to pull away. To catch the bus at some later time $t$, the woman will have to cover a distance $s_{w}(t)$ that is equal to 11 m plus the distance $s_{b}(t)$ traveled by the bus; that is, the woman will catch the bus when

$$
\begin{equation*}
s_{w}(t)=s_{b}(t)+11 \tag{12}
\end{equation*}
$$

Since the woman has a constant velocity of $5 \mathrm{~m} / \mathrm{s}$, the distance she travels in $t$ seconds is $s_{w}(t)=5 t$. Thus, (12) can be written as

$$
\begin{equation*}
s_{b}(t)=5 t-11 \tag{13}
\end{equation*}
$$

Since the bus has a constant acceleration of $a=1 \mathrm{~m} / \mathrm{s}^{2}$, and since $s_{0}=v_{0}=0$ at time $t=0$ (why?), it follows from (10) that

$$
s_{b}(t)=\frac{1}{2} t^{2}
$$

Substituting this equation into (13) and reorganizing the terms yields the quadratic equation

$$
\frac{1}{2} t^{2}-5 t+11=0 \quad \text { or } \quad t^{2}-10 t+22=0
$$

Solving this equation for $t$ using the quadratic formula yields two solutions:

$$
t=5-\sqrt{3} \approx 3.3 \quad \text { and } \quad t=5+\sqrt{3} \approx 6.7
$$

(verify). Thus, the woman can reach the door at two different times, $t=3.3 \mathrm{~s}$ and $t=6.7 \mathrm{~s}$. The reason that there are two solutions can be explained as follows: When the woman first reaches the door, she is running faster than the bus and can run past it if the driver does not see her. However, as the bus speeds up, it eventually catches up to her, and she has another chance to flag it down.

## FREE-FALL MODEL

Motion that occurs when an object near the Earth is imparted some initial velocity (up or down) and thereafter moves along a vertical line is called free-fall motion. In modeling free-fall motion we assume that the only force acting on the object is the Earth's gravity and that the object stays sufficiently close to the Earth that the gravitational force is constant. In particular, air resistance and the gravitational pull of other celestial bodies are neglected.

In our model we will ignore the physical size of the object by treating it as a particle, and we will assume that it moves along an $s$-axis whose origin is at the surface of the Earth and whose positive direction is up. With this convention, the $s$-coordinate of the particle is the height of the particle above the surface of the Earth (Figure 5.7.7).

It is a fact of physics that a particle with free-fall motion has constant acceleration. The magnitude of this constant, denoted by the letter $g$, is called the acceleration due to gravity and is approximately $9.8 \mathrm{~m} / \mathrm{s}^{2}$ or $32 \mathrm{ft} / \mathrm{s}^{2}$, depending on whether distance is measured in meters or feet. ${ }^{*}$

Recall that a particle is speeding up when its velocity and acceleration have the same sign and is slowing down when they have opposite signs. Thus, because we have chosen

[^1]How would Formulas (14), (15), and (16) change if we choose the direction of the positive $s$-axis to be down?


Corbis.Bettmann
Nolan Ryan's rookie baseball card.

In Example 6 the ball is moving up when the velocity is positive and is moving down when the velocity is negative, so it makes sense physically that the velocity is zero when the ball reaches its peak.

[^2]the positive direction to be up, it follows that the acceleration $a(t)$ of a particle in free fall is negative for all values of $t$. To see that this is so, observe that an upward-moving particle (positive velocity) is slowing down, so its acceleration must be negative; and a downward-moving particle (negative velocity) is speeding up, so its acceleration must also be negative. Thus, we conclude that
\[

$$
\begin{equation*}
a(t)=-g \tag{14}
\end{equation*}
$$

\]

It now follows from this and Formulas (10) and (11) that the position and velocity functions for a particle in free-fall motion are

$$
\begin{gather*}
s(t)=s_{0}+v_{0} t-\frac{1}{2} g t^{2}  \tag{15}\\
v(t)=v_{0}-g t \tag{16}
\end{gather*}
$$

- Example 6 Nolan Ryan, a member of the Baseball Hall of Fame and one of the fastest baseball pitchers of all time, was able to throw a baseball $150 \mathrm{ft} / \mathrm{s}$ (over $102 \mathrm{mi} / \mathrm{h}$ ). During his career, he had the opportunity to pitch in the Houston Astrodome, home to the Houston Astros Baseball Team from 1965 to 1999. The Astrodome was an indoor stadium with a ceiling 208 ft high. Could Nolan Ryan have hit the ceiling of the Astrodome if he were capable of giving a baseball an upward velocity of $100 \mathrm{ft} / \mathrm{s}$ from a height of 7 ft ?

Solution. Since distance is in feet, we take $g=32 \mathrm{ft} / \mathrm{s}^{2}$. Initially, we have $s_{0}=7 \mathrm{ft}$ and $v_{0}=100 \mathrm{ft} / \mathrm{s}$, so from (15) and (16) we have

$$
\begin{aligned}
& s(t)=7+100 t-16 t^{2} \\
& v(t)=100-32 t
\end{aligned}
$$

The ball will rise until $v(t)=0$, that is, until $100-32 t=0$. Solving this equation we see that the ball is at its maximum height at time $t=\frac{25}{8}$. To find the height of the ball at this instant we substitute this value of $t$ into the position function to obtain

$$
s\left(\frac{25}{8}\right)=7+100\left(\frac{25}{8}\right)-16\left(\frac{25}{8}\right)^{2}=163.25 \mathrm{ft}
$$

which is roughly 45 ft short of hitting the ceiling.

Example 7 A penny is released from rest near the top of the Empire State Building at a point that is 1250 ft above the ground (Figure 5.7.8). Assuming that the free-fall model applies, how long does it take for the penny to hit the ground, and what is its speed at the time of impact?

Solution. Since distance is in feet, we take $g=32 \mathrm{ft} / \mathrm{s}^{2}$. Initially, we have $s_{0}=1250$ and $v_{0}=0$, so from (15)

$$
\begin{equation*}
s(t)=1250-16 t^{2} \tag{17}
\end{equation*}
$$

Impact occurs when $s(t)=0$. Solving this equation for $t$, we obtain

$$
\begin{aligned}
& 1250-16 t^{2}=0 \\
& t^{2}=\frac{1250}{16}=\frac{625}{8} \\
& t= \pm \frac{25}{\sqrt{8}} \approx \pm 8.8 \mathrm{~s}
\end{aligned}
$$

Since $t \geq 0$, we can discard the negative solution and conclude that it takes $25 / \sqrt{8} \approx 8.8 \mathrm{~s}$
for the penny to hit the ground. To obtain the velocity at the time of impact, we substitute $t=25 / \sqrt{8}, v_{0}=0$, and $g=32$ in (16) to obtain

$$
v\left(\frac{25}{\sqrt{8}}\right)=0-32\left(\frac{25}{\sqrt{8}}\right)=-200 \sqrt{2} \approx-282.8 \mathrm{ft} / \mathrm{s}
$$

Thus, the speed at the time of impact is

$$
\left|v\left(\frac{25}{\sqrt{8}}\right)\right|=200 \sqrt{2} \approx 282.8 \mathrm{ft} / \mathrm{s}
$$

which is more than $192 \mathrm{mi} / \mathrm{h}$.

## QUICK CHECK EXERCISES 5.7 (See page 385 for answers.)

1. Suppose that a particle is moving along an $s$-axis with velocity $v(t)=2 t+1$. If at time $t=0$ the particle is at position $s=2$, the position function of the particle is $s(t)=$ $\qquad$ —.
2. Let $v(t)$ denote the velocity function of a particle that is moving along an $s$-axis with constant acceleration $a=-2$. If $v(1)=4$, then $v(t)=$ $\qquad$ -.
3. Let $v(t)$ denote the velocity function of a particle in rectilinear motion. Suppose that $v(0)=-1, v(3)=2$, and the
velocity versus time curve is a straight line. The displacement of the particle between times $t=0$ and $t=3$ is $\qquad$ , and the distance traveled by the particle over this period of time is $\qquad$ -.
4. Based on the free-fall model, from what height must a coin be dropped so that it strikes the ground with speed $48 \mathrm{ft} / \mathrm{s}$ ?

## EXERCISE SET 5.7 $\sim$ Graphing Utility c cAS

## FOCUS ON CONCEPTS

1. In each part, the velocity versus time curve is given for a particle moving along a line. Use the curve to find the displacement and the distance traveled by the particle over the time interval $0 \leq t \leq 3$.
(a)

(b)

(c)

(d)

2. Sketch a velocity versus time curve for a particle that travels a distance of 5 units along a coordinate line during the time interval $0 \leq t \leq 10$ and has a displacement of 0 units.
3. The accompanying figure shows the acceleration versus time curve for a particle moving along a coordinate line. If the initial velocity of the particle is $20 \mathrm{~m} / \mathrm{s}$, estimate
(a) the velocity at time $t=4 \mathrm{~s}$
(b) the velocity at time $t=6 \mathrm{~s}$.

< Figure Ex-3
4. The accompanying figure shows the velocity versus time curve over the time interval $1 \leq t \leq 5$ for a particle moving along a horizontal coordinate line.
(a) What can you say about the sign of the acceleration over the time interval?
(b) When is the particle speeding up? Slowing down?
(c) What can you say about the location of the particle at time $t=5$ relative to its location at time $t=1$ ? Explain your reasoning.

< Figure Ex-4

5-8 A particle moves along an $s$-axis. Use the given information to find the position function of the particle.
5. (a) $v(t)=3 t^{2}-2 t ; s(0)=1$
(b) $a(t)=3 \sin 3 t ; v(0)=3 ; s(0)=3$
6. (a) $v(t)=1+\sin t ; s(0)=-3$
(b) $a(t)=t^{2}-3 t+1 ; v(0)=0 ; s(0)=0$
7. (a) $v(t)=3 t+1 ; s(2)=4$
(b) $a(t)=t^{-2} ; v(1)=0 ; s(1)=2$
8. (a) $v(t)=t^{2 / 3} ; s(8)=0$
(b) $a(t)=\sqrt{t} ; v(4)=1 ; s(4)=-5$

9-12 A particle moves with a velocity of $v(t) \mathrm{m} / \mathrm{s}$ along an $s$-axis. Find the displacement and the distance traveled by the particle during the given time interval.
9. (a) $v(t)=\sin t ; 0 \leq t \leq \pi / 2$
(b) $v(t)=\cos t ; \pi / 2 \leq t \leq 2 \pi$
10. (a) $v(t)=3 t-2 ; 0 \leq t \leq 2$
(b) $v(t)=|1-2 t| ; 0 \leq t \leq 2$
11. (a) $v(t)=t^{3}-3 t^{2}+2 t ; 0 \leq t \leq 3$
(b) $v(t)=\sqrt{t}-2 ; 0 \leq t \leq 3$
12. (a) $v(t)=t-\sqrt{t} ; 0 \leq t \leq 4$
(b) $v(t)=\frac{1}{\sqrt{t+1}} ; 0 \leq t \leq 3$

13-16 A particle moves with acceleration $a(t) \mathrm{m} / \mathrm{s}^{2}$ along an $s$-axis and has velocity $v_{0} \mathrm{~m} / \mathrm{s}$ at time $t=0$. Find the displacement and the distance traveled by the particle during the given time interval.
13. $a(t)=3 ; v_{0}=-1 ; 0 \leq t \leq 2$
14. $a(t)=t-2 ; v_{0}=0 ; 1 \leq t \leq 5$
15. $a(t)=1 / \sqrt{3 t+1} ; \quad v_{0}=\frac{4}{3} ; \quad 1 \leq t \leq 5$
16. $a(t)=\sin t ; v_{0}=1 ; \pi / 4 \leq t \leq \pi / 2$
17. In each part, use the given information to find the position, velocity, speed, and acceleration at time $t=1$.
(a) $v=\sin \frac{1}{2} \pi t ; s=0$ when $t=0$
(b) $a=-3 t$; s $=1$ and $v=0$ when $t=0$
18. In each part, use the given information to find the position, velocity, speed, and acceleration at time $t=1$.
(a) $v=\cos \frac{1}{3} \pi t ; s=0$ when $t=\frac{3}{2}$
(b) $a=4 e^{2 t-2} ; s=1 / e^{2}$ and $v=\left(2 / e^{2}\right)-3$ when $t=0$
19. Suppose that a particle moves along a line so that its velocity $v$ at time $t$ is given by

$$
v(t)= \begin{cases}5 t, & 0 \leq t<1 \\ 6 \sqrt{t}-\frac{1}{t}, & 1 \leq t\end{cases}
$$

where $t$ is in seconds and $v$ is in centimeters per second $(\mathrm{cm} / \mathrm{s})$. Estimate the time(s) at which the particle is 4 cm from its starting position.
20. Suppose that a particle moves along a line so that its velocity $v$ at time $t$ is given by

$$
v(t)=\frac{3}{t^{2}+1}-0.5 t, \quad t \geq 0
$$

where $t$ is in seconds and $v$ is in centimeters per second $(\mathrm{cm} / \mathrm{s})$. Estimate the time(s) at which the particle is 2 cm from its starting position.
21. Suppose that the velocity function of a particle moving along an $s$-axis is $v(t)=20 t^{2}-110 t+120 \mathrm{ft} / \mathrm{s}$ and that the particle is at the origin at time $t=0$. Use a graphing utility to generate the graphs of $s(t), v(t)$, and $a(t)$ for the first 6 s of motion.
22. Suppose that the acceleration function of a particle moving along an $s$-axis is $a(t)=4 t-30 \mathrm{~m} / \mathrm{s}^{2}$ and that the position and velocity at time $t=0$ are $s_{0}=-5 \mathrm{~m}$ and $v_{0}=3 \mathrm{~m} / \mathrm{s}$. Use a graphing utility to generate the graphs of $s(t), v(t)$, and $a(t)$ for the first 25 s of motion.

23-26 True-False Determine whether the statement is true or false. Explain your answer. Each question refers to a particle in rectilinear motion.
23. If the particle has constant acceleration, the velocity versus time graph will be a straight line.
24. If the particle has constant nonzero acceleration, its position versus time curve will be a parabola.
25. If the total area between the velocity versus time curve and a time interval $[a, b]$ is positive, then the displacement of the particle over this time interval will be nonzero.
26. If $D(t)$ denotes the distance traveled by the particle over the time interval $[0, t]$, then $D(t)$ is an antiderivative for the speed of the particle.
[C 27-30 For the given velocity function $v(t)$ :
(a) Generate the velocity versus time curve, and use it to make a conjecture about the sign of the displacement over the given time interval.
(b) Use a CAS to find the displacement.
27. $v(t)=0.5-t \sin t ; 0 \leq t \leq 5$
28. $v(t)=0.5-t \cos \pi t ; 0 \leq t \leq 1$
29. $v(t)=0.5-t e^{-t} ; 0 \leq t \leq 5$
30. $v(t)=t \ln (t+0.1) ; 0 \leq t \leq 1$
31. Suppose that at time $t=0$ a particle is at the origin of an $x$-axis and has a velocity of $v_{0}=25 \mathrm{~cm} / \mathrm{s}$. For the first 4 s thereafter it has no acceleration, and then it is acted on by a retarding force that produces a constant negative acceleration of $a=-10 \mathrm{~cm} / \mathrm{s}^{2}$.
(a) Sketch the acceleration versus time curve over the interval $0 \leq t \leq 12$.
(b) Sketch the velocity versus time curve over the time interval $0 \leq t \leq 12$.
(c) Find the $x$-coordinate of the particle at times $t=8 \mathrm{~s}$ and $t=12 \mathrm{~s}$.
(d) What is the maximum $x$-coordinate of the particle over the time interval $0 \leq t \leq 12$ ?

32-36 In these exercises assume that the object is moving with constant acceleration in the positive direction of a coordinate line, and apply Formulas (10) and (11) as appropriate. In some of these problems you will need the fact that $88 \mathrm{ft} / \mathrm{s}=60 \mathrm{mi} / \mathrm{h}$.
32. A car traveling $60 \mathrm{mi} / \mathrm{h}$ along a straight road decelerates at a constant rate of $11 \mathrm{ft} / \mathrm{s}^{2}$.
(a) How long will it take until the speed is $45 \mathrm{mi} / \mathrm{h}$ ?
(b) How far will the car travel before coming to a stop?
33. Spotting a police car, you hit the brakes on your new Porsche to reduce your speed from $90 \mathrm{mi} / \mathrm{h}$ to $60 \mathrm{mi} / \mathrm{h}$ at a constant rate over a distance of 200 ft .
(a) Find the acceleration in $\mathrm{ft} / \mathrm{s}^{2}$.
(b) How long does it take for you to reduce your speed to $55 \mathrm{mi} / \mathrm{h}$ ?
(c) At the acceleration obtained in part (a), how long would it take for you to bring your Porsche to a complete stop from $90 \mathrm{mi} / \mathrm{h}$ ?
34. A particle moving along a straight line is accelerating at a constant rate of $5 \mathrm{~m} / \mathrm{s}^{2}$. Find the initial velocity if the particle moves 60 m in the first 4 s .
35. A car that has stopped at a toll booth leaves the booth with a constant acceleration of $4 \mathrm{ft} / \mathrm{s}^{2}$. At the time the car leaves the booth it is 2500 ft behind a truck traveling with a constant velocity of $50 \mathrm{ft} / \mathrm{s}$. How long will it take for the car to catch the truck, and how far will the car be from the toll booth at that time?
36. In the final sprint of a rowing race the challenger is rowing at a constant speed of $12 \mathrm{~m} / \mathrm{s}$. At the point where the leader is 100 m from the finish line and the challenger is 15 m behind, the leader is rowing at $8 \mathrm{~m} / \mathrm{s}$ but starts accelerating at a constant $0.5 \mathrm{~m} / \mathrm{s}^{2}$. Who wins?

37-46 Assume that a free-fall model applies. Solve these exercises by applying Formulas (15) and (16). In these exercises take $g=32 \mathrm{ft} / \mathrm{s}^{2}$ or $g=9.8 \mathrm{~m} / \mathrm{s}^{2}$, depending on the units.
37. A projectile is launched vertically upward from ground level with an initial velocity of $112 \mathrm{ft} / \mathrm{s}$.
(a) Find the velocity at $t=3 \mathrm{~s}$ and $t=5 \mathrm{~s}$.
(b) How high will the projectile rise?
(c) Find the speed of the projectile when it hits the ground.
38. A projectile fired downward from a height of 112 ft reaches the ground in 2 s . What is its initial velocity?
39. A projectile is fired vertically upward from ground level with an initial velocity of $16 \mathrm{ft} / \mathrm{s}$.
(a) How long will it take for the projectile to hit the ground?
(b) How long will the projectile be moving upward?
40. In 1939, Joe Sprinz of the San Francisco Seals Baseball Club attempted to catch a ball dropped from a blimp at a height of 800 ft (for the purpose of breaking the record for catching a
ball dropped from the greatest height set the preceding year by members of the Cleveland Indians).
(a) How long does it take for a ball to drop 800 ft ?
(b) What is the velocity of a ball in miles per hour after an 800 ft drop $(88 \mathrm{ft} / \mathrm{s}=60 \mathrm{mi} / \mathrm{h})$ ?
[Note: As a practical matter, it is unrealistic to ignore wind resistance in this problem; however, even with the slowing effect of wind resistance, the impact of the ball slammed Sprinz's glove hand into his face, fractured his upper jaw in 12 places, broke five teeth, and knocked him unconscious. He dropped the ball!]
41. A projectile is launched upward from ground level with an initial speed of $60 \mathrm{~m} / \mathrm{s}$.
(a) How long does it take for the projectile to reach its highest point?
(b) How high does the projectile go?
(c) How long does it take for the projectile to drop back to the ground from its highest point?
(d) What is the speed of the projectile when it hits the ground?
42. (a) Use the results in Exercise 41 to make a conjecture about the relationship between the initial and final speeds of a projectile that is launched upward from ground level and returns to ground level.
(b) Prove your conjecture.
43. A projectile is fired vertically upward with an initial velocity of $49 \mathrm{~m} / \mathrm{s}$ from a tower 150 m high.
(a) How long will it take for the projectile to reach its maximum height?
(b) What is the maximum height?
(c) How long will it take for the projectile to pass its starting point on the way down?
(d) What is the velocity when it passes the starting point on the way down?
(e) How long will it take for the projectile to hit the ground?
(f) What will be its speed at impact?
44. A man drops a stone from a bridge. What is the height of the bridge if
(a) the stone hits the water 4 s later
(b) the sound of the splash reaches the man 4 s later? [Take $1080 \mathrm{ft} / \mathrm{s}$ as the speed of sound.]
45. In Example 6, how fast would Nolan Ryan have to throw a ball upward from a height of 7 ft in order to hit the ceiling of the Astrodome?
46. A rock thrown downward with an unknown initial velocity from a height of 1000 ft reaches the ground in 5 s . Find the velocity of the rock when it hits the ground.
47. Writing Make a list of important features of a velocity versus time curve, and interpret each feature in terms of the motion.
48. Writing Use Riemann sums to argue informally that integrating speed over a time interval produces the distance traveled.

1. $t^{2}+t+2$
2. $6-2 t$
3. $\frac{3}{2} ; \frac{5}{2}$
4. 36 ft

### 5.8 AVERAGE VALUE OF A FUNCTION AND ITS APPLICATIONS

In this section we will define the notion of the "average value" of a function, and we will give various applications of this idea.

## AVERAGE VELOCITY REVISITED

Let $s=s(t)$ denote the position function of a particle in rectilinear motion. In Section 2.1 we defined the average velocity $v_{\text {ave }}$ of the particle over the time interval $\left[t_{0}, t_{1}\right]$ to be

$$
v_{\mathrm{ave}}=\frac{s\left(t_{1}\right)-s\left(t_{0}\right)}{t_{1}-t_{0}}
$$

Let $v(t)=s^{\prime}(t)$ denote the velocity function of the particle. We saw in Section 5.7 that integrating $s^{\prime}(t)$ over a time interval gives the displacement of the particle over that interval. Thus,

$$
\int_{t_{0}}^{t_{1}} v(t) d t=\int_{t_{0}}^{t_{1}} s^{\prime}(t) d t=s\left(t_{1}\right)-s\left(t_{0}\right)
$$

It follows that

$$
\begin{equation*}
v_{\mathrm{ave}}=\frac{s\left(t_{1}\right)-s\left(t_{0}\right)}{t_{1}-t_{0}}=\frac{1}{t_{1}-t_{0}} \int_{t_{0}}^{t_{1}} v(t) d t \tag{1}
\end{equation*}
$$

- Example 1 Suppose that a particle moves along a coordinate line so that its velocity at time $t$ is $v(t)=2+\cos t$. Find the average velocity of the particle during the time interval $0 \leq t \leq \pi$.

Solution. From (1) the average velocity is

$$
\frac{1}{\pi-0} \int_{0}^{\pi}(2+\cos t) d t=\frac{1}{\pi}[2 t+\sin t]_{0}^{\pi}=\frac{1}{\pi}(2 \pi)=2
$$

We will see that Formula (1) is a special case of a formula for what we will call the average value of a continuous function over a given interval.

## AVERAGE VALUE OF A CONTINUOUS FUNCTION

In scientific work, numerical information is often summarized by an average value or mean value of the observed data. There are various kinds of averages, but the most common is the arithmetic mean or arithmetic average, which is formed by adding the data and dividing by the number of data points. Thus, the arithmetic average $\bar{a}$ of $n$ numbers $a_{1}, a_{2}, \ldots, a_{n}$ is

$$
\bar{a}=\frac{1}{n}\left(a_{1}+a_{2}+\cdots+a_{n}\right)=\frac{1}{n} \sum_{k=1}^{n} a_{k}
$$

In the case where the $a_{k}$ 's are values of a function $f$, say,

$$
a_{1}=f\left(x_{1}\right), a_{2}=f\left(x_{2}\right), \ldots, a_{n}=f\left(x_{n}\right)
$$

then the arithmetic average $\bar{a}$ of these function values is

$$
\bar{a}=\frac{1}{n} \sum_{k=1}^{n} f\left(x_{k}\right)
$$

Note that the Mean-Value Theorem for Integrals, when expressed in form (3), ensures that there is always at least one point $x^{*}$ in $[a, b]$ at which the value of $f$ is equal to the average value of $f$ over the interval.

We will now show how to extend this concept so that we can compute not only the arithmetic average of finitely many function values but an average of all values of $f(x)$ as $x$ varies over a closed interval $[a, b]$. For this purpose recall the Mean-Value Theorem for Integrals (5.6.2), which states that if $f$ is continuous on the interval $[a, b]$, then there is at least one point $x^{*}$ in this interval such that

$$
\int_{a}^{b} f(x) d x=f\left(x^{*}\right)(b-a)
$$

The quantity

$$
f\left(x^{*}\right)=\frac{1}{b-a} \int_{a}^{b} f(x) d x
$$

will be our candidate for the average value of $f$ over the interval $[a, b]$. To explain what motivates this, divide the interval $[a, b]$ into $n$ subintervals of equal length

$$
\begin{equation*}
\Delta x=\frac{b-a}{n} \tag{2}
\end{equation*}
$$

and choose arbitrary points $x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}$ in successive subintervals. Then the arithmetic average of the values $f\left(x_{1}^{*}\right), f\left(x_{2}^{*}\right), \ldots, f\left(x_{n}^{*}\right)$ is

$$
\text { ave }=\frac{1}{n}\left[f\left(x_{1}^{*}\right)+f\left(x_{2}^{*}\right)+\cdots+f\left(x_{n}^{*}\right)\right]
$$

or from (2)

$$
\text { ave }=\frac{1}{b-a}\left[f\left(x_{1}^{*}\right) \Delta x+f\left(x_{2}^{*}\right) \Delta x+\cdots+f\left(x_{n}^{*}\right) \Delta x\right]=\frac{1}{b-a} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x
$$

Taking the limit as $n \rightarrow+\infty$ yields

$$
\lim _{n \rightarrow+\infty} \frac{1}{b-a} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x=\frac{1}{b-a} \int_{a}^{b} f(x) d x
$$

Since this equation describes what happens when we compute the average of "more and more" values of $f(x)$, we are led to the following definition.
5.8.1 DEFINITION If $f$ is continuous on $[a, b]$, then the average value (or mean value) of $f$ on $[a, b]$ is defined to be

$$
\begin{equation*}
f_{\mathrm{ave}}=\frac{1}{b-a} \int_{a}^{b} f(x) d x \tag{3}
\end{equation*}
$$

When $f$ is nonnegative on $[a, b]$, the quantity $f_{\text {ave }}$ has a simple geometric interpretation, which can be seen by writing (3) as

$$
f_{\mathrm{ave}} \cdot(b-a)=\int_{a}^{b} f(x) d x
$$

The left side of this equation is the area of a rectangle with a height of $f_{\text {ave }}$ and base of length $b-a$, and the right side is the area under $y=f(x)$ over $[a, b]$. Thus, $f_{\text {ave }}$ is the height of a rectangle constructed over the interval $[a, b]$, whose area is the same as the area under the graph of $f$ over that interval (Figure 5.8.1).

- Example 2 Find the average value of the function $f(x)=\sqrt{x}$ over the interval [1, 4], and find all points in the interval at which the value of $f$ is the same as the average.

$\Delta$ Figure 5.8.2

$\Delta$ Figure 5.8.3

In Example 3, the temperature $T$ of the lemonade rises from an initial temperature of $40^{\circ} \mathrm{F}$ toward the room temperature of $70^{\circ} \mathrm{F}$. Explain why the formula

$$
T=70-30 e^{-0.5 t}
$$

is a good model for this situation.

Solution.

$$
\begin{aligned}
f_{\mathrm{ave}}=\frac{1}{b-a} \int_{a}^{b} f(x) d x & =\frac{1}{4-1} \int_{1}^{4} \sqrt{x} d x=\frac{1}{3}\left[\frac{2 x^{3 / 2}}{3}\right]_{1}^{4} \\
& =\frac{1}{3}\left[\frac{16}{3}-\frac{2}{3}\right]=\frac{14}{9} \approx 1.6
\end{aligned}
$$

The $x$-values at which $f(x)=\sqrt{x}$ is the same as this average satisfy $\sqrt{x}=14 / 9$, from which we obtain $x=196 / 81 \approx 2.4$ (Figure 5.8.2).

- Example 3 A glass of lemonade with a temperature of $40^{\circ} \mathrm{F}$ is left to sit in a room whose temperature is a constant $70^{\circ} \mathrm{F}$. Using a principle of physics called Newton's Law of Cooling, one can show that if the temperature of the lemonade reaches $52^{\circ} \mathrm{F}$ in 1 hour, then the temperature $T$ of the lemonade as a function of the elapsed time $t$ is modeled by the equation

$$
T=70-30 e^{-0.5 t}
$$

where $T$ is in degrees Fahrenheit and $t$ is in hours. The graph of this equation, shown in Figure 5.8.3, conforms to our everyday experience that the temperature of the lemonade gradually approaches the temperature of the room. Find the average temperature $T_{\text {ave }}$ of the lemonade over the first 5 hours.

Solution. From Definition 5.8.1 the average value of $T$ over the time interval [0,5] is

$$
\begin{equation*}
T_{\mathrm{ave}}=\frac{1}{5} \int_{0}^{5}\left(70-30 e^{-0.5 t}\right) d t \tag{4}
\end{equation*}
$$

To evaluate the definite integral, we first find the indefinite integral

$$
\int\left(70-30 e^{-0.5 t}\right) d t
$$

by making the substitution

$$
u=-0.5 t \quad \text { so that } \quad d u=-0.5 d t \quad(\text { or } d t=-2 d u)
$$

Thus,

$$
\begin{aligned}
\int\left(70-30 e^{-0.5 t}\right) d t & =\int\left(70-30 e^{u}\right)(-2) d u=-2\left(70 u-30 e^{u}\right)+C \\
& =-2\left[70(-0.5 t)-30 e^{-0.5 t}\right]+C=70 t+60 e^{-0.5 t}+C
\end{aligned}
$$

and (4) can be expressed as

$$
\begin{aligned}
T_{\mathrm{ave}} & =\frac{1}{5}\left[70 t+60 e^{-0.5 t}\right]_{0}^{5}=\frac{1}{5}\left[\left(350+60 e^{-2.5}\right)-60\right] \\
& =58+12 e^{-2.5} \approx 59^{\circ} \mathrm{F}
\end{aligned}
$$

## AVERAGE VALUE AND AVERAGE VELOCITY

We now have two ways to calculate the average velocity of a particle in rectilinear motion, since

$$
\begin{equation*}
\frac{s\left(t_{1}\right)-s\left(t_{0}\right)}{t_{1}-t_{0}}=\frac{1}{t_{1}-t_{0}} \int_{t_{0}}^{t_{1}} v(t) d t \tag{5}
\end{equation*}
$$

and both of these expressions are equal to the average velocity. The left side of (5) gives the average rate of change of $s$ over $\left[t_{0}, t_{1}\right]$, while the right side gives the average value of

The result of Example 4 can be generalized to show that the average velocity of a particle with constant acceleration during a time interval $[a, b]$ is the velocity at time $t=(a+b) / 2$. (See Exercise 18.)
$v=s^{\prime}$ over the interval $\left[t_{0}, t_{1}\right]$. That is, the average velocity of the particle over the time interval $\left[t_{0}, t_{1}\right]$ is the same as the average value of the velocity function over that interval.

Since velocity functions are generally continuous, it follows from the marginal note associated with Definition 5.8.1 that a particle's average velocity over a time interval matches the particle's velocity at some time in the interval.

- Example 4 Show that if a body released from rest (initial velocity zero) is in free fall, then its average velocity over a time interval $[0, T]$ during its fall is its velocity at time $t=T / 2$.

Solution. It follows from Formula (16) of Section 5.7 with $v_{0}=0$ that the velocity function of the body is $v(t)=-g t$. Thus, its average velocity over a time interval $[0, T]$ is

$$
\begin{aligned}
v_{\mathrm{ave}} & =\frac{1}{T-0} \int_{0}^{T} v(t) d t \\
& =\frac{1}{T} \int_{0}^{T}-g t d t \\
& =-\frac{g}{T}\left[\frac{1}{2} t^{2}\right]_{0}^{T}=-g \cdot \frac{T}{2}=v\left(\frac{T}{2}\right)
\end{aligned}
$$

QUICK CHECK EXERCISES 5.8 (See page 390 for answers.)

1. The arithmetic average of $n$ numbers, $a_{1}, a_{2}, \ldots, a_{n}$ is
$\qquad$
2. If $f$ is continuous on $[a, b]$, then the average value of $f$ on $[a, b]$ is $\qquad$
3. If $f$ is continuous on $[a, b]$, then the Mean-Value Theorem for Integrals guarantees that for at least one point $x^{*}$ in $[a, b]$
$\qquad$ equals the average value of $f$ on $[a, b]$.
4. The average value of $f(x)=4 x^{3}$ on $[1,3]$ is $\qquad$ .

## EXERCISE SET 5.8 <br> CAS

1. (a) Find $f_{\text {ave }}$ of $f(x)=2 x$ over $[0,4]$.
(b) Find a point $x^{*}$ in $[0,4]$ such that $f\left(x^{*}\right)=f_{\text {ave }}$.
(c) Sketch a graph of $f(x)=2 x$ over [0,4], and construct a rectangle over the interval whose area is the same as the area under the graph of $f$ over the interval.
2. (a) Find $f_{\text {ave }}$ of $f(x)=x^{2}$ over $[0,2]$.
(b) Find a point $x^{*}$ in $[0,2]$ such that $f\left(x^{*}\right)=f_{\text {ave }}$.
(c) Sketch a graph of $f(x)=x^{2}$ over [ 0,2 ], and construct a rectangle over the interval whose area is the same as the area under the graph of $f$ over the interval.

3-12 Find the average value of the function over the given interval.
3. $f(x)=3 x$; $[1,3]$
4. $f(x)=\sqrt[3]{x} ;[-1,8]$
5. $f(x)=\sin x ;[0, \pi]$
6. $f(x)=\sec x \tan x ;[0, \pi / 3]$
7. $f(x)=1 / x ;[1, e]$
8. $f(x)=e^{x} ;[-1, \ln 5]$
9. $f(x)=\frac{1}{1+x^{2}} ;[1, \sqrt{3}]$
10. $f(x)=\frac{1}{\sqrt{1-x^{2}}} ;\left[-\frac{1}{2}, 0\right]$
11. $f(x)=e^{-2 x} ;[0,4]$
12. $f(x)=\sec ^{2} x ;[-\pi / 4, \pi / 4]$

## FOCUS ON CONCEPTS

13. Let $f(x)=3 x^{2}$.
(a) Find the arithmetic average of the values $f(0.4)$, $f(0.8), f(1.2), f(1.6)$, and $f(2.0)$.
(b) Find the arithmetic average of the values $f(0.1)$, $f(0.2), f(0.3), \ldots, f(2.0)$.
(c) Find the average value of $f$ on $[0,2]$.
(d) Explain why the answer to part (c) is less than the answers to parts (a) and (b).
14. In parts (a)-(d), let $f(x)=1+(1 / x)$.
(a) Find the arithmetic average of the values $f\left(\frac{6}{5}\right)$, $f\left(\frac{7}{5}\right), f\left(\frac{8}{5}\right), f\left(\frac{9}{5}\right)$, and $f(2)$.
(b) Find the arithmetic average of the values $f(1.1)$, $f(1.2), f(1.3), \ldots, f(2)$.
(c) Find the average value of $f$ on $[1,2]$.
(d) Explain why the answer to part (c) is greater than the answers to parts (a) and (b).
15. In each part, the velocity versus time curve is given for a particle moving along a line. Use the curve to find the average velocity of the particle over the time interval $0 \leq t \leq 3$.

> (a)
(b)


16. Suppose that a particle moving along a line starts from rest and has an average velocity of $2 \mathrm{ft} / \mathrm{s}$ over the time interval $0 \leq t \leq 5$. Sketch a velocity versus time curve for the particle assuming that the particle is also at rest at time $t=5$. Explain how your curve satisfies the required properties.
17. Suppose that $f$ is a linear function. Using the graph of $f$, explain why the average value of $f$ on $[a, b]$ is

$$
f\left(\frac{a+b}{2}\right)
$$

18. Suppose that a particle moves along a coordinate line with constant acceleration. Show that the average velocity of the particle during a time interval $[a, b]$ matches the velocity of the particle at the midpoint of the interval.

19-22 True-False Determine whether the statement is true or false. Explain your answer. (Assume that $f$ and $g$ denote continuous functions on an interval $[a, b]$ and that $f_{\text {ave }}$ and $g_{\text {ave }}$ denote the respective average values of $f$ and $g$ on $[a, b]$.)
19. If $g_{\text {ave }}<f_{\text {ave }}$, then $g(x) \leq f(x)$ on $[a, b]$.
20. The average value of a constant multiple of $f$ is the same multiple of $f_{\text {ave }}$; that is, if $c$ is any constant,

$$
(c \cdot f)_{\mathrm{ave}}=c \cdot f_{\mathrm{ave}}
$$

21. The average of the sum of two functions on an interval is the sum of the average values of the two functions on the interval; that is,

$$
(f+g)_{\text {ave }}=f_{\text {ave }}+g_{\text {ave }}
$$

22. The average of the product of two functions on an interval is the product of the average values of the two functions on the interval; that is

$$
(f \cdot g)_{\mathrm{ave}}=f_{\mathrm{ave}} \cdot g_{\mathrm{ave}}
$$

23. (a) Suppose that the velocity function of a particle moving along a coordinate line is $v(t)=3 t^{3}+2$. Find the average velocity of the particle over the time interval $1 \leq t \leq 4$ by integrating.
(b) Suppose that the position function of a particle moving along a coordinate line is $s(t)=6 t^{2}+t$. Find the average velocity of the particle over the time interval $1 \leq t \leq 4$ algebraically.
24. (a) Suppose that the acceleration function of a particle moving along a coordinate line is $a(t)=t+1$. Find the average acceleration of the particle over the time interval $0 \leq t \leq 5$ by integrating.
(b) Suppose that the velocity function of a particle moving along a coordinate line is $v(t)=\cos t$. Find the average acceleration of the particle over the time interval $0 \leq t \leq \pi / 4$ algebraically.
25. Water is run at a constant rate of $1 \mathrm{ft}^{3} / \mathrm{min}$ to fill a cylindrical tank of radius 3 ft and height 5 ft . Assuming that the tank is initially empty, make a conjecture about the average weight of the water in the tank over the time period required to fill it, and then check your conjecture by integrating. [Take the weight density of water to be $62.4 \mathrm{lb} / \mathrm{ft}^{3}$.]
26. (a) The temperature of a 10 m long metal bar is $15^{\circ} \mathrm{C}$ at one end and $30^{\circ} \mathrm{C}$ at the other end. Assuming that the temperature increases linearly from the cooler end to the hotter end, what is the average temperature of the bar?
(b) Explain why there must be a point on the bar where the temperature is the same as the average, and find it.
27. A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30 P.m. and 5:30 p.м. the rate $R(t)$ at which cars enter the highway is given by the formula $R(t)=100\left(1-0.0001 t^{2}\right)$ cars per minute, where $t$ is the time (in minutes) since 4:30 p.m. Find the average rate, in cars per minute, at which cars enter the highway during the first half-hour of rush hour.
28. Suppose that the value of a yacht in dollars after $t$ years of use is $V(t)=275,000 e^{-0.17 t}$. What is the average value of the yacht over its first 10 years of use?
29. A large juice glass containing 60 ml of orange juice is replenished by a server. The accompanying figure shows the rate at which orange juice is poured into the glass in milliliters per second $(\mathrm{ml} / \mathrm{s})$. Show that the average rate of change of the volume of juice in the glass during these 5 s is equal to the average value of the rate of flow of juice into the glass.


C 30. The function $J_{0}$ defined by

$$
J_{0}(x)=\frac{1}{\pi} \int_{0}^{\pi} \cos (x \sin t) d t
$$

is called the Bessel function of order zero.
(a) Find a function $f$ and an interval $[a, b]$ for which $J_{0}(1)$ is the average value of $f$ over $[a, b]$.
(b) Estimate $J_{0}(1)$.
(c) Use a CAS to graph the equation $y=J_{0}(x)$ over the interval $0 \leq x \leq 8$.
(d) Estimate the smallest positive zero of $J_{0}$.
31. Find a positive value of $k$ such that the average value of $f(x)=\sqrt{3 x}$ over the interval $[0, k]$ is 6.
32. Suppose that a tumor grows at the rate of $r(t)=k t$ grams per week for some positive constant $k$, where $t$ is the num-
ber of weeks since the tumor appeared. When, during the second 26 weeks of growth, is the mass of the tumor the same as its average mass during that period?
33. Writing Consider the following statement: The average value of the rate of change of a function over an interval is equal to the average rate of change of the function over that interval. Write a short paragraph that explains why this statement may be interpreted as a rewording of Part 1 of the Fundamental Theorem of Calculus.
34. Writing If an automobile gets an average of 25 miles per gallon of gasoline, then it is also the case that on average the automobile expends $1 / 25$ gallon of gasoline per mile. Interpret this statement using the concept of the average value of a function over an interval.

QUICK CHECK ANSWERS 5.8

1. $\frac{1}{n} \sum_{k=1}^{n} a_{k}$
2. $\frac{1}{b-a} \int_{a}^{b} f(x) d x$
3. $f\left(x^{*}\right)$
4. 40

### 5.9 EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION

In this section we will discuss two methods for evaluating definite integrals in which a substitution is required.

## TWO METHODS FOR MAKING SUBSTITUTIONS IN DEFINITE INTEGRALS

Recall from Section 5.3 that indefinite integrals of the form

$$
\int f(g(x)) g^{\prime}(x) d x
$$

can sometimes be evaluated by making the $u$-substitution

$$
\begin{equation*}
u=g(x), \quad d u=g^{\prime}(x) d x \tag{1}
\end{equation*}
$$

which converts the integral to the form

$$
\int f(u) d u
$$

To apply this method to a definite integral of the form

$$
\int_{a}^{b} f(g(x)) g^{\prime}(x) d x
$$

we need to account for the effect that the substitution has on the $x$-limits of integration. There are two ways of doing this.

## Method 1.

First evaluate the indefinite integral

$$
\int f(g(x)) g^{\prime}(x) d x
$$

by substitution, and then use the relationship

$$
\int_{a}^{b} f(g(x)) g^{\prime}(x) d x=\left[\int f(g(x)) g^{\prime}(x) d x\right]_{a}^{b}
$$

to evaluate the definite integral. This procedure does not require any modification of the $x$-limits of integration.

## Method 2.

Make the substitution (1) directly in the definite integral, and then use the relationship $u=g(x)$ to replace the $x$-limits, $x=a$ and $x=b$, by corresponding $u$-limits, $u=g(a)$ and $u=g(b)$. This produces a new definite integral

$$
\int_{g(a)}^{g(b)} f(u) d u
$$

that is expressed entirely in terms of $u$.
$\overline{\text { Example } 1}$ Use the two methods above to evaluate $\int_{0}^{2} x\left(x^{2}+1\right)^{3} d x$.
Solution by Method 1. If we let

$$
\begin{equation*}
u=x^{2}+1 \quad \text { so that } \quad d u=2 x d x \tag{2}
\end{equation*}
$$

then we obtain

$$
\int x\left(x^{2}+1\right)^{3} d x=\frac{1}{2} \int u^{3} d u=\frac{u^{4}}{8}+C=\frac{\left(x^{2}+1\right)^{4}}{8}+C
$$

Thus,

$$
\begin{aligned}
\int_{0}^{2} x\left(x^{2}+1\right)^{3} d x & =\left[\int x\left(x^{2}+1\right)^{3} d x\right]_{x=0}^{2} \\
& \left.=\frac{\left(x^{2}+1\right)^{4}}{8}\right]_{x=0}^{2}=\frac{625}{8}-\frac{1}{8}=78
\end{aligned}
$$

Solution by Method 2. If we make the substitution $u=x^{2}+1$ in (2), then

$$
\begin{array}{lll}
\text { if } & x=0, & u=1 \\
\text { if } & x=2, & u=5
\end{array}
$$

Thus,

$$
\begin{aligned}
\int_{0}^{2} x\left(x^{2}+1\right)^{3} d x & =\frac{1}{2} \int_{1}^{5} u^{3} d u \\
& \left.=\frac{u^{4}}{8}\right]_{u=1}^{5}=\frac{625}{8}-\frac{1}{8}=78
\end{aligned}
$$

which agrees with the result obtained by Method 1.

The following theorem states precise conditions under which Method 2 can be used.
5.9.1 THEOREM If $g^{\prime}$ is continuous on $[a, b]$ and $f$ is continuous on an interval containing the values of $g(x)$ for $a \leq x \leq b$, then

$$
\int_{a}^{b} f(g(x)) g^{\prime}(x) d x=\int_{g(a)}^{g(b)} f(u) d u
$$

PROOF Since $f$ is continuous on an interval containing the values of $g(x)$ for $a \leq x \leq b$, it follows that $f$ has an antiderivative $F$ on that interval. If we let $u=g(x)$, then the chain rule implies that

$$
\frac{d}{d x} F(g(x))=\frac{d}{d x} F(u)=\frac{d F}{d u} \frac{d u}{d x}=f(u) \frac{d u}{d x}=f(g(x)) g^{\prime}(x)
$$

for each $x$ in $[a, b]$. Thus, $F(g(x))$ is an antiderivative of $f(g(x)) g^{\prime}(x)$ on $[a, b]$. Therefore, by Part 1 of the Fundamental Theorem of Calculus (5.6.1)

$$
\left.\int_{a}^{b} f(g(x)) g^{\prime}(x) d x=F(g(x))\right]_{a}^{b}=F(g(b))-F(g(a))=\int_{g(a)}^{g(b)} f(u) d u
$$

The choice of methods for evaluating definite integrals by substitution is generally a matter of taste, but in the following examples we will use the second method, since the idea is new.

## - Example 2 Evaluate

(a) $\int_{0}^{\pi / 8} \sin ^{5} 2 x \cos 2 x d x$
(b) $\int_{2}^{5}(2 x-5)(x-3)^{9} d x$

Solution (a). Let

$$
u=\sin 2 x \quad \text { so that } \quad d u=2 \cos 2 x d x \quad\left(\text { or } \frac{1}{2} d u=\cos 2 x d x\right)
$$

With this substitution,

$$
\begin{array}{ll}
\text { if } & x=0, \quad u=\sin (0)=0 \\
\text { if } & x=\pi / 8, \quad u=\sin (\pi / 4)=1 / \sqrt{2}
\end{array}
$$

so

$$
\begin{aligned}
\int_{0}^{\pi / 8} \sin ^{5} 2 x \cos 2 x d x & =\frac{1}{2} \int_{0}^{1 / \sqrt{2}} u^{5} d u \\
& \left.=\frac{1}{2} \cdot \frac{u^{6}}{6}\right]_{u=0}^{1 / \sqrt{2}}=\frac{1}{2}\left[\frac{1}{6(\sqrt{2})^{6}}-0\right]=\frac{1}{96}
\end{aligned}
$$

Solution (b). Let

$$
u=x-3 \text { so that } d u=d x
$$

This leaves a factor of $2 x-5$ unresolved in the integrand. However,

$$
x=u+3, \quad \text { so } \quad 2 x-5=2(u+3)-5=2 u+1
$$

With this substitution,

$$
\begin{array}{ll}
\text { if } \quad x=2, & u=2-3=-1 \\
\text { if } & x=5, \\
\quad u=5-3=2
\end{array}
$$

so

$$
\begin{aligned}
\int_{2}^{5}(2 x-5)(x-3)^{9} d x & =\int_{-1}^{2}(2 u+1) u^{9} d u=\int_{-1}^{2}\left(2 u^{10}+u^{9}\right) d u \\
& =\left[\frac{2 u^{11}}{11}+\frac{u^{10}}{10}\right]_{u=-1}^{2}=\left(\frac{2^{12}}{11}+\frac{2^{10}}{10}\right)-\left(-\frac{2}{11}+\frac{1}{10}\right) \\
& =\frac{52,233}{110} \approx 474.8
\end{aligned}
$$

The $u$-substitution in Example 3(a) produces an integral in which the upper $u$ limit is smaller than the lower $u$-limit. Use Definition 5.5.3(b) to convert this integral to one whose lower limit is smaller than the upper limit and verify that it produces an integral with the same value as that in the example.

## Example 3 Evaluate

(a) $\int_{0}^{3 / 4} \frac{d x}{1-x}$
(b) $\int_{0}^{\ln 3} e^{x}\left(1+e^{x}\right)^{1 / 2} d x$

Solution (a). Let

$$
u=1-x \quad \text { so that } \quad d u=-d x
$$

With this substitution,

$$
\begin{array}{lll}
\text { if } & x=0, & u=1 \\
\text { if } & x=\frac{3}{4}, & u=\frac{1}{4}
\end{array}
$$

Thus,

$$
\begin{aligned}
\int_{0}^{3 / 4} \frac{d x}{1-x} & =-\int_{1}^{1 / 4} \frac{d u}{u} \\
& =-\ln |u|]_{u=1}^{1 / 4}=-\left[\ln \left(\frac{1}{4}\right)-\ln (1)\right]=\ln 4
\end{aligned}
$$

Solution (b). Make the $u$-substitution

$$
u=1+e^{x}, \quad d u=e^{x} d x
$$

and change the $x$-limits of integration $(x=0, x=\ln 3)$ to the $u$-limits

$$
u=1+e^{0}=2, \quad u=1+e^{\ln 3}=1+3=4
$$

This yields

$$
\begin{aligned}
\int_{0}^{\ln 3} e^{x}\left(1+e^{x}\right)^{1 / 2} d x & =\int_{2}^{4} u^{1 / 2} d u \\
& \left.=\frac{2}{3} u^{3 / 2}\right]_{u=2}^{4}=\frac{2}{3}\left[4^{3 / 2}-2^{3 / 2}\right]=\frac{16-4 \sqrt{2}}{3}
\end{aligned}
$$

## QUICK CHECK EXERCISES 5.9 (See page 396 for answers.)

1. Assume that $g^{\prime}$ is continuous on $[a, b]$ and that $f$ is continuous on an interval containing the values of $g(x)$ for $a \leq x \leq b$. If $F$ is an antiderivative for $f$, then

$$
\int_{a}^{b} f(g(x)) g^{\prime}(x) d x=
$$

2. In each part, use the substitution to replace the given integral with an integral involving the variable $u$. (Do not evaluate the integral.)
(a) $\int_{0}^{2} 3 x^{2}\left(1+x^{3}\right)^{3} d x ; u=1+x^{3}$
(b) $\int_{0}^{2} \frac{x}{\sqrt{5-x^{2}}} d x ; u=5-x^{2}$
(c) $\int_{0}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}} d x ; u=\sqrt{x}$
3. Evaluate the integral by making an appropriate substitution.
(a) $\int_{-\pi}^{0} \sin (3 x-\pi) d x=$ $\qquad$
(b) $\int_{2}^{3} \frac{x}{x^{2}-2} d x=$ $\qquad$
(c) $\int_{0}^{\pi / 2} \sqrt[3]{\sin x} \cos x d x=$ $\qquad$

1-4 Express the integral in terms of the variable $u$, but do not evaluate it.

1. (a) $\int_{1}^{3}(2 x-1)^{3} d x ; u=2 x-1$
(b) $\int_{0}^{4} 3 x \sqrt{25-x^{2}} d x ; u=25-x^{2}$
(c) $\int_{-1 / 2}^{1 / 2} \cos (\pi \theta) d \theta ; u=\pi \theta$
(d) $\int_{0}^{1}(x+2)(x+1)^{5} d x ; u=x+1$
2. (a) $\int_{-1}^{4}(5-2 x)^{8} d x ; u=5-2 x$
(b) $\int_{-\pi / 3}^{2 \pi / 3} \frac{\sin x}{\sqrt{2+\cos x}} d x ; u=2+\cos x$
(c) $\int_{0}^{\pi / 4} \tan ^{2} x \sec ^{2} x d x ; u=\tan x$
(d) $\int_{0}^{1} x^{3} \sqrt{x^{2}+3} d x ; u=x^{2}+3$
3. (a) $\int_{0}^{1} e^{2 x-1} d x ; u=2 x-1$
(b) $\int_{e}^{e^{2}} \frac{\ln x}{x} d x ; u=\ln x$
4. (a) $\int_{1}^{\sqrt{3}} \frac{\sqrt{\tan ^{-1} x}}{1+x^{2}} d x ; u=\tan ^{-1} x$
(b) $\int_{1}^{\sqrt{e}} \frac{d x}{x \sqrt{1-(\ln x)^{2}}} ; u=\ln x$

5-18 Evaluate the definite integral two ways: first by a $u$ substitution in the definite integral and then by a $u$-substitution in the corresponding indefinite integral.
5. $\int_{0}^{1}(2 x+1)^{3} d x$
6. $\int_{1}^{2}(4 x-2)^{3} d x$
7. $\int_{0}^{1}(2 x-1)^{3} d x$
8. $\int_{1}^{2}(4-3 x)^{8} d x$
9. $\int_{0}^{8} x \sqrt{1+x} d x$
10. $\int_{-3}^{0} x \sqrt{1-x} d x$
11. $\int_{0}^{\pi / 2} 4 \sin (x / 2) d x$
12. $\int_{0}^{\pi / 6} 2 \cos 3 x d x$
13. $\int_{-2}^{-1} \frac{x}{\left(x^{2}+2\right)^{3}} d x$
14. $\int_{1-\pi}^{1+\pi} \sec ^{2}\left(\frac{1}{4} x-\frac{1}{4}\right) d x$
15. $\int_{-\ln 3}^{\ln 3} \frac{e^{x}}{e^{x}+4} d x$
16. $\int_{0}^{\ln 5} e^{x}\left(3-4 e^{x}\right) d x$
17. $\int_{1}^{3} \frac{d x}{\sqrt{x}(x+1)}$
18. $\int_{\ln 2}^{\ln (2 / \sqrt{3})} \frac{e^{-x} d x}{\sqrt{1-e^{-2 x}}}$

19-22 Evaluate the definite integral by expressing it in terms of $u$ and evaluating the resulting integral using a formula from geometry.
19. $\int_{-5 / 3}^{5 / 3} \sqrt{25-9 x^{2}} d x ; u=3 x$
20. $\int_{0}^{2} x \sqrt{16-x^{4}} d x$; u $=x^{2}$
21. $\int_{\pi / 3}^{\pi / 2} \sin \theta \sqrt{1-4 \cos ^{2} \theta} d \theta$; u=2 $\cos \theta$
22. $\int_{e^{-3}}^{e^{3}} \frac{\sqrt{9-(\ln x)^{2}}}{x} d x ; u=\ln x$
23. A particle moves with a velocity of $v(t)=\sin \pi t \mathrm{~m} / \mathrm{s}$ along an $s$-axis. Find the distance traveled by the particle over the time interval $0 \leq t \leq 1$.
24. A particle moves with a velocity of $v(t)=3 \cos 2 t \mathrm{~m} / \mathrm{s}$ along an $s$-axis. Find the distance traveled by the particle over the time interval $0 \leq t \leq \pi / 8$.
25. Find the area under the curve $y=9 /(x+2)^{2}$ over the interval $[-1,1]$.
26. Find the area under the curve $y=1 /(3 x+1)^{2}$ over the interval $[0,1]$.
27. Find the area of the region enclosed by the graphs of $y=1 / \sqrt{1-9 x^{2}}, y=0, x=0$, and $x=\frac{1}{6}$.
28. Find the area of the region enclosed by the graphs of $y=\sin ^{-1} x, x=0$, and $y=\pi / 2$.

29-48 Evaluate the integrals by any method.
29. $\int_{1}^{5} \frac{d x}{\sqrt{2 x-1}}$
30. $\int_{1}^{2} \sqrt{5 x-1} d x$
31. $\int_{-1}^{1} \frac{x^{2} d x}{\sqrt{x^{3}+9}}$
32. $\int_{\pi / 2}^{\pi} 6 \sin x(\cos x+1)^{5} d x$
33. $\int_{1}^{3} \frac{x+2}{\sqrt{x^{2}+4 x+7}} d x$
34. $\int_{1}^{2} \frac{d x}{x^{2}-6 x+9}$
35. $\int_{0}^{\pi / 4} 4 \sin x \cos x d x$
36. $\int_{0}^{\pi / 4} \sqrt{\tan x} \sec ^{2} x d x$
37. $\int_{0}^{\sqrt{\pi}} 5 x \cos \left(x^{2}\right) d x$
38. $\int_{\pi^{2}}^{4 \pi^{2}} \frac{1}{\sqrt{x}} \sin \sqrt{x} d x$
39. $\int_{\pi / 12}^{\pi / 9} \sec ^{2} 3 \theta d \theta$
40. $\int_{0}^{\pi / 6} \tan 2 \theta d \theta$
41. $\int_{0}^{1} \frac{y^{2} d y}{\sqrt{4-3 y}}$
42. $\int_{-1}^{4} \frac{x d x}{\sqrt{5+x}}$
43. $\int_{0}^{e} \frac{d x}{2 x+e}$
44. $\int_{1}^{\sqrt{2}} x e^{-x^{2}} d x$
45. $\int_{0}^{1} \frac{x}{\sqrt{4-3 x^{4}}} d x$
46. $\int_{1}^{2} \frac{1}{\sqrt{x} \sqrt{4-x}} d x$
47. $\int_{0}^{1 / \sqrt{3}} \frac{1}{1+9 x^{2}} d x$
48. $\int_{1}^{\sqrt{2}} \frac{x}{3+x^{4}} d x$
49. (a) Use a CAS to find the exact value of the integral

$$
\int_{0}^{\pi / 6} \sin ^{4} x \cos ^{3} x d x
$$

(b) Confirm the exact value by hand calculation. [Hint: Use the identity $\cos ^{2} x=1-\sin ^{2} x$.]
C 50. (a) Use a CAS to find the exact value of the integral

$$
\int_{-\pi / 4}^{\pi / 4} \tan ^{4} x d x
$$

(b) Confirm the exact value by hand calculation.
[Hint: Use the identity $1+\tan ^{2} x=\sec ^{2} x$.]
51. (a) Find $\int_{0}^{1} f(3 x+1) d x$ if $\int_{1}^{4} f(x) d x=5$.
(b) Find $\int_{0}^{3} f(3 x) d x$ if $\int_{0}^{9} f(x) d x=5$.
(c) Find $\int_{-2}^{0} x f\left(x^{2}\right) d x$ if $\int_{0}^{4} f(x) d x=1$.
52. Given that $m$ and $n$ are positive integers, show that

$$
\int_{0}^{1} x^{m}(1-x)^{n} d x=\int_{0}^{1} x^{n}(1-x)^{m} d x
$$

by making a substitution. Do not attempt to evaluate the integrals.
53. Given that $n$ is a positive integer, show that

$$
\int_{0}^{\pi / 2} \sin ^{n} x d x=\int_{0}^{\pi / 2} \cos ^{n} x d x
$$

by using a trigonometric identity and making a substitution. Do not attempt to evaluate the integrals.
54. Given that $n$ is a positive integer, evaluate the integral

$$
\int_{0}^{1} x(1-x)^{n} d x
$$

55. Suppose that at time $t=0$ there are 750 bacteria in a growth medium and the bacteria population $y(t)$ grows at the rate $y^{\prime}(t)=802.137 e^{1.528 t}$ bacteria per hour. How many bacteria will there be in 12 hours?
56. Suppose that a particle moving along a coordinate line has velocity $v(t)=25+10 e^{-0.05 t} \mathrm{ft} / \mathrm{s}$.
(a) What is the distance traveled by the particle from time $t=0$ to time $t=10$ ?
(b) Does the term $10 e^{-0.05 t}$ have much effect on the distance traveled by the particle over that time interval? Explain your reasoning.
57. (a) The accompanying table shows the fraction of the Moon that is illuminated (as seen from Earth) at midnight (Eastern Standard Time) for the first week of 2005. Find the average fraction of the Moon illuminated during the first week of 2005.
Source: Data from the U.S Naval Observatory Astronomical Applications Department.
(b) The function $f(x)=0.5+0.5 \sin (0.213 x+2.481)$ models data for illumination of the Moon for the first 60 days of 2005. Find the average value of this illumination function over the interval $[0,7]$.

| DAY | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ILLUMINATION | 0.74 | 0.65 | 0.56 | 0.45 | 0.35 | 0.25 | 0.16 |

A Table Ex-57
58. Electricity is supplied to homes in the form of alternating current, which means that the voltage has a sinusoidal waveform described by an equation of the form

$$
V=V_{p} \sin (2 \pi f t)
$$

(see the accompanying figure). In this equation, $V_{p}$ is called the peak voltage or amplitude of the current, $f$ is called its frequency, and $1 / f$ is called its period. The voltages $V$ and $V_{p}$ are measured in volts $(\mathrm{V})$, the time $t$ is measured in seconds (s), and the frequency is measured in hertz (Hz). ( $1 \mathrm{~Hz}=1$ cycle per second; a cycle is the electrical term for one period of the waveform.) Most alternating-current voltmeters read what is called the rms or root-mean-square value of $V$. By definition, this is the square root of the average value of $V^{2}$ over one period.
(a) Show that

$$
V_{\mathrm{rms}}=\frac{V_{p}}{\sqrt{2}}
$$

[Hint: Compute the average over the cycle from $t=0$ to $t=1 / f$, and use the identity $\sin ^{2} \theta=\frac{1}{2}(1-\cos 2 \theta)$ to help evaluate the integral.]
(b) In the United States, electrical outlets supply alternating current with an rms voltage of 120 V at a frequency of 60 Hz . What is the peak voltage at such an outlet?


## < Figure Ex-58

59. Find a positive value of $k$ such that the area under the graph of $y=e^{2 x}$ over the interval $[0, k]$ is 3 square units.
60. Use a graphing utility to estimate the value of $k(k>0)$ so that the region enclosed by $y=1 /\left(1+k x^{2}\right), y=0, x=0$, and $x=2$ has an area of 0.6 square unit.
61. (a) Find the limit

$$
\lim _{n \rightarrow+\infty} \sum_{k=1}^{n} \frac{\sin (k \pi / n)}{n}
$$

by evaluating an appropriate definite integral over the interval $[0,1]$.
(b) Check your answer to part (a) by evaluating the limit directly with a CAS.

## FOCUS ON CONCEPTS

62. Let

$$
I=\int_{-1}^{1} \frac{1}{1+x^{2}} d x
$$

(a) Explain why $I>0$.
(b) Show that the substitution $x=1 / u$ results in

$$
I=-\int_{-1}^{1} \frac{1}{1+x^{2}} d x=-I
$$

Thus, $2 I=0$, which implies that $I=0$. But this contradicts part (a). What is the error?
63. (a) Prove that if $f$ is an odd function, then

$$
\int_{-a}^{a} f(x) d x=0
$$

and give a geometric explanation of this result.
[Hint: One way to prove that a quantity $q$ is zero is to show that $q=-q$.]
(b) Prove that if $f$ is an even function, then

$$
\int_{-a}^{a} f(x) d x=2 \int_{0}^{a} f(x) d x
$$

and give a geometric explanation of this result. [Hint: Split the interval of integration from $-a$ to $a$ into two parts at 0.]
64. Show that if $f$ and $g$ are continuous functions, then

$$
\int_{0}^{t} f(t-x) g(x) d x=\int_{0}^{t} f(x) g(t-x) d x
$$

65. (a) Let

$$
I=\int_{0}^{a} \frac{f(x)}{f(x)+f(a-x)} d x
$$

Show that $I=a / 2$.
[Hint: Let $u=a-x$, and then note the difference between the resulting integrand and 1.]
(b) Use the result of part (a) to find

$$
\int_{0}^{3} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}} d x
$$

(c) Use the result of part (a) to find

$$
\int_{0}^{\pi / 2} \frac{\sin x}{\sin x+\cos x} d x
$$

66. Evaluate
(a) $\int_{-1}^{1} x \sqrt{\cos \left(x^{2}\right)} d x$
(b) $\int_{0}^{\pi} \sin ^{8} x \cos ^{5} x d x$.
[Hint: Use the substitution $u=x-(\pi / 2)$.]
67. Writing The two substitution methods discussed in this section yield the same result when used to evaluate a definite integral. Write a short paragraph that carefully explains why this is the case.
68. Writing In some cases, the second method for the evaluation of definite integrals has distinct advantages over the first. Provide some illustrations, and write a short paragraph that discusses the advantages of the second method in each case. [Hint: To get started, consider the results in Exercises $52-54,63$, and 65.]

## QUICK CHECK ANSWERS 5.9

1. $F(g(b))-F(g(a))$
2. (a) $\int_{1}^{9} u^{3} d u$
(b) $\int_{1}^{5} \frac{1}{2 \sqrt{u}} d u$
(c) $\int_{0}^{1} 2 e^{u} d u$
3. (a) $\frac{2}{3}$
(b) $\frac{1}{2} \ln \left(\frac{7}{2}\right)$
(c) $\frac{3}{4}$

### 5.10 LOGARITHMIC AND OTHER FUNCTIONS DEFINED BY INTEGRALS

In Section 0.5 we defined the natural logarithm function $\ln x$ to be the inverse of $e^{x}$. Although this was convenient and enabled us to deduce many properties of $\ln x$, the mathematical foundation was shaky in that we accepted the continuity of $e^{x}$ and of all exponential functions without proof. In this section we will show that $\ln x$ can be defined as a certain integral, and we will use this new definition to prove that exponential functions are continuous. This integral definition is also important in applications because it provides a way of recognizing when integrals that appear in solutions of problems can be expressed as natural logarithms.


## - Figure 5.10.1

Review Theorem 5.5.8 and then explain why $x$ is required to be positive in Definition 5.10.1.

None of the properties of $\ln x$ obtained in this section should be new, but now, for the first time, we give them a sound mathematical footing.

## THE CONNECTION BETWEEN NATURAL LOGARITHMS AND INTEGRALS

The connection between natural logarithms and integrals was made in the middle of the seventeenth century in the course of investigating areas under the curve $y=1 / t$. The problem being considered was to find values of $t_{1}, t_{2}, t_{3}, \ldots, t_{n}, \ldots$ for which the areas $A_{1}, A_{2}, A_{3}, \ldots, A_{n}, \ldots$ in Figure 5.10.1 $a$ would be equal. Through the combined work of Isaac Newton, the Belgian Jesuit priest Gregory of St. Vincent (1584-1667), and Gregory's student Alfons A. de Sarasa (1618-1667), it was shown that by taking the points to be

$$
t_{1}=e, \quad t_{2}=e^{2}, \quad t_{3}=e^{3}, \ldots, \quad t_{n}=e^{n}, \ldots
$$

each of the areas would be 1 (Figure 5.10.1b). Thus, in modern integral notation

$$
\int_{1}^{e^{n}} \frac{1}{t} d t=n
$$

which can be expressed as

$$
\int_{1}^{e^{n}} \frac{1}{t} d t=\ln \left(e^{n}\right)
$$

By comparing the upper limit of the integral and the expression inside the logarithm, it is a natural leap to the more general result

$$
\int_{1}^{x} \frac{1}{t} d t=\ln x
$$

which today we take as the formal definition of the natural logarithm.
5.10.1 Definition The natural logarithm of $x$ is denoted by $\ln x$ and is defined by the integral

$$
\begin{equation*}
\ln x=\int_{1}^{x} \frac{1}{t} d t, \quad x>0 \tag{1}
\end{equation*}
$$

Our strategy for putting the study of logarithmic and exponential functions on a sound mathematical footing is to use (1) as a starting point and then define $e^{x}$ as the inverse of $\ln x$. This is the exact opposite of our previous approach in which we defined $\ln x$ to be the inverse of $e^{x}$. However, whereas previously we had to assume that $e^{x}$ is continuous, the continuity of $e^{x}$ will now follow from our definitions as a theorem. Our first challenge is to demonstrate that the properties of $\ln x$ resulting from Definition 5.10.1 are consistent with those obtained earlier. To start, observe that Part 2 of the Fundamental Theorem of Calculus (5.6.3) implies that $\ln x$ is differentiable and

$$
\begin{equation*}
\frac{d}{d x}[\ln x]=\frac{d}{d x}\left[\int_{1}^{x} \frac{1}{t} d t\right]=\frac{1}{x} \quad(x>0) \tag{2}
\end{equation*}
$$

This is consistent with the derivative formula for $\ln x$ that we obtained previously. Moreover, because differentiability implies continuity, it follows that $\ln x$ is a continuous function on the interval $(0,+\infty)$.

Other properties of $\ln x$ can be obtained by interpreting the integral in (1) geometrically: In the case where $x>1$, this integral represents the area under the curve $y=1 / t$ from $t=1$ to $t=x$ (Figure 5.10.2a); in the case where $0<x<1$, the integral represents the negative of the area under the curve $y=1 / t$ from $t=x$ to $t=1$ (Figure 5.10.2b); and in the case where $x=1$, the integral has value 0 because its upper and lower limits of integration are the same. These geometric observations imply that

$$
\begin{array}{lll}
\ln x>0 & \text { if } & x>1 \\
\ln x<0 & \text { if } & 0<x<1 \\
\ln x=0 & \text { if } & x=1
\end{array}
$$



$$
\ln x=\int_{1}^{x} \frac{1}{t} d t=A
$$

(a)

$\ln x=\int_{1}^{x} \frac{1}{t} d t=-\int_{x}^{1} \frac{1}{t} d t=-A$
(b)

Also, since $1 / x$ is positive for $x>0$, it follows from (2) that $\ln x$ is an increasing function on the interval $(0,+\infty)$. This is all consistent with the graph of $\ln x$ in Figure 5.10.3.

## ALGEBRAIC PROPERTIES OF $\ln x$

We can use (1) to show that Definition 5.10.1 produces the standard algebraic properties of logarithms.
5.10.2 THEOREM For any positive numbers $a$ and $c$ and any rational number $r$ :
(a) $\ln a c=\ln a+\ln c$
(b) $\ln \frac{1}{c}=-\ln c$
(c) $\ln \frac{a}{c}=\ln a-\ln c$
(d) $\ln a^{r}=r \ln a$

PROOF (a) Treating $a$ as a constant, consider the function $f(x)=\ln (a x)$. Then

$$
f^{\prime}(x)=\frac{1}{a x} \cdot \frac{d}{d x}(a x)=\frac{1}{a x} \cdot a=\frac{1}{x}
$$

Thus, $\ln a x$ and $\ln x$ have the same derivative on $(0,+\infty)$, so these functions must differ by a constant on this interval. That is, there is a constant $k$ such that

$$
\begin{equation*}
\ln a x-\ln x=k \tag{3}
\end{equation*}
$$

on $(0,+\infty)$. Substituting $x=1$ into this equation we conclude that $\ln a=k$ (verify). Thus, (3) can be written as

$$
\ln a x-\ln x=\ln a
$$

Setting $x=c$ establishes that

$$
\ln a c-\ln c=\ln a \quad \text { or } \quad \ln a c=\ln a+\ln c
$$

PROOFS (b) AND (c) Part (b) follows immediately from part (a) by substituting $1 / c$ for $a$ (verify). Then

$$
\ln \frac{a}{c}=\ln \left(a \cdot \frac{1}{c}\right)=\ln a+\ln \frac{1}{c}=\ln a-\ln c
$$

Proof $(d)$ First, we will argue that part $(d)$ is satisfied if $r$ is any nonnegative integer. If $r=1$, then $(d)$ is clearly satisfied; if $r=0$, then $(d)$ follows from the fact that $\ln 1=0$. Suppose that we know $(d)$ is satisfied for $r$ equal to some integer $n$. It then follows from part (a) that

$$
\ln a^{n+1}=\ln \left[a \cdot a^{n}\right]=\ln a+\ln a^{n}=\ln a+n \ln a=(n+1) \ln a
$$

How is the proof of Theorem 5.10.2(d) for the case where $r$ is a nonnegative integer analogous to a row of falling dominos? (This "domino" argument uses an informal version of a property of the integers known as the principle of mathematical induction.)

Table 5.10.1

| $n=10$ |  |  |
| :---: | :---: | :---: |
| $\Delta t=(b-a) / n=(2-1) / 10=0.1$ |  |  |
| $k$ | $t_{k}^{*}$ | $1 / t_{k}^{*}$ |
| 1 | 1.05 | 0.952381 |
| 2 | 1.15 | 0.869565 |
| 3 | 1.25 | 0.800000 |
| 4 | 1.35 | 0.740741 |
| 5 | 1.45 | 0.689655 |
| 6 | 1.55 | 0.645161 |
| 7 | 1.65 | 0.606061 |
| 8 | 1.75 | 0.571429 |
| 9 | 1.85 | 0.540541 |
| 10 | 1.95 | 0.512821 |
|  |  | 6.928355 |

That is, if $(d)$ is valid for $r$ equal to some integer $n$, then it is also valid for $r=n+1$. However, since we know $(d)$ is satisfied if $r=1$, it follows that $(d)$ is valid for $r=2$. But this implies that $(d)$ is satisfied for $r=3$, which in turn implies that $(d)$ is valid for $r=4$, and so forth. We conclude that $(d)$ is satisfied if $r$ is any nonnegative integer.

Next, suppose that $r=-m$ is a negative integer. Then

$$
\begin{array}{rlr}
\ln a^{r}=\ln a^{-m}=\ln \frac{1}{a^{m}} & =-\ln a^{m} & \\
& =-m \ln a & \text { By part }(b) \\
& =r \ln a &
\end{array}
$$

which shows that $(d)$ is valid for any negative integer $r$. Combining this result with our previous conclusion that $(d)$ is satisfied for a nonnegative integer $r$ shows that $(d)$ is valid if $r$ is any integer.

Finally, suppose that $r=m / n$ is any rational number, where $m \neq 0$ and $n \neq 0$ are integers. Then

$$
\begin{array}{rlrl}
\ln a^{r}=\frac{n \ln a^{r}}{n} & =\frac{\ln \left[\left(a^{r}\right)^{n}\right]}{n} & & \text { Part }(d) \text { is valid for integer powers. } \\
& =\frac{\ln a^{r n}}{n} & & \text { Property of exponents } \\
& =\frac{\ln a^{m}}{n} & & \text { Definition of } r \\
& =\frac{m \ln a}{n} & & \text { Part }(d) \text { is valid for integer powers. } \\
& =\frac{m}{n} \ln a=r \ln a
\end{array}
$$

which shows that $(d)$ is valid for any rational number $r$.

## APPROXIMATING In $x$ NUMERICALLY

For specific values of $x$, the value of $\ln x$ can be approximated numerically by approximating the definite integral in (1), say by using the midpoint approximation that was discussed in Section 5.4.

## - Example 1 Approximate $\ln 2$ using the midpoint approximation with $n=10$.

Solution. From (1), the exact value of $\ln 2$ is represented by the integral

$$
\ln 2=\int_{1}^{2} \frac{1}{t} d t
$$

The midpoint rule is given in Formulas (5) and (6) of Section 5.4. Expressed in terms of $t$, the latter formula is

$$
\int_{a}^{b} f(t) d t \approx \Delta t \sum_{k=1}^{n} f\left(t_{k}^{*}\right)
$$

where $\Delta t$ is the common width of the subintervals and $t_{1}^{*}, t_{2}^{*}, \ldots, t_{n}^{*}$ are the midpoints. In this case we have 10 subintervals, so $\Delta t=(2-1) / 10=0.1$. The computations to six decimal places are shown in Table 5.10.1. By comparison, a calculator set to display six decimal places gives $\ln 2 \approx 0.693147$, so the magnitude of the error in the midpoint approximation is about 0.000311 . Greater accuracy in the midpoint approximation can be obtained by increasing $n$. For example, the midpoint approximation with $n=100$ yields $\ln 2 \approx 0.693144$, which is correct to five decimal places.

## DOMAIN, RANGE, AND END BEHAVIOR OF $\ln x$

5.10.3 THEOREM
(a) The domain of $\ln x$ is $(0,+\infty)$.
(b) $\lim _{x \rightarrow 0^{+}} \ln x=-\infty$ and $\lim _{x \rightarrow+\infty} \ln x=+\infty$
(c) The range of $\ln x$ is $(-\infty,+\infty)$.

Proofs (a) and (b) We have already shown that $\ln x$ is defined and increasing on the interval $(0,+\infty)$. To prove that $\ln x \rightarrow+\infty$ as $x \rightarrow+\infty$, we must show that given any number $M>0$, the value of $\ln x$ exceeds $M$ for sufficiently large values of $x$. To do this, let $N$ be any integer. If $x>2^{N}$, then

$$
\begin{equation*}
\ln x>\ln 2^{N}=N \ln 2 \tag{4}
\end{equation*}
$$

by Theorem 5.10.2(d). Since

$$
\ln 2=\int_{1}^{2} \frac{1}{t} d t>0
$$

it follows that $N \ln 2$ can be made arbitrarily large by choosing $N$ sufficiently large. In particular, we can choose $N$ so that $N \ln 2>M$. It now follows from (4) that if $x>2^{N}$, then $\ln x>M$, and this proves that

$$
\lim _{x \rightarrow+\infty} \ln x=+\infty
$$

Furthermore, by observing that $v=1 / x \rightarrow+\infty$ as $x \rightarrow 0^{+}$, we can use the preceding limit and Theorem 5.10.2(b) to conclude that

$$
\lim _{x \rightarrow 0^{+}} \ln x=\lim _{v \rightarrow+\infty} \ln \frac{1}{v}=\lim _{v \rightarrow+\infty}(-\ln v)=-\infty
$$

PROOF (c) It follows from part (a), the continuity of $\ln x$, and the Intermediate-Value Theorem (1.5.7) that $\ln x$ assumes every real value as $x$ varies over the interval $(0,+\infty)$ (why?).

## DEFINITION OF $e^{x}$

In Chapter 0 we defined $\ln x$ to be the inverse of the natural exponential function $e^{x}$. Now that we have a formal definition of $\ln x$ in terms of an integral, we will define the natural exponential function to be the inverse of $\ln x$.

Since $\ln x$ is increasing and continuous on $(0,+\infty)$ with range $(-\infty,+\infty)$, there is exactly one (positive) solution to the equation $\ln x=1$. We define $e$ to be the unique solution to $\ln x=1$, so

$$
\begin{equation*}
\ln e=1 \tag{5}
\end{equation*}
$$

Furthermore, if $x$ is any real number, there is a unique positive solution $y$ to $\ln y=x$, so for irrational values of $x$ we define $e^{x}$ to be this solution. That is, when $x$ is irrational, $e^{x}$ is defined by

$$
\begin{equation*}
\ln e^{x}=x \tag{6}
\end{equation*}
$$

Note that for rational values of $x$, we also have $\ln e^{x}=x \ln e=x$ from Theorem 5.10.2(d). Moreover, it follows immediately that $e^{\ln x}=x$ for any $x>0$. Thus, (6) defines the exponential function for all real values of $x$ as the inverse of the natural logarithm function.
5.10.4 DEFINITION The inverse of the natural logarithm function $\ln x$ is denoted by $e^{x}$ and is called the natural exponential function.

We can now establish the differentiability of $e^{x}$ and confirm that

$$
\frac{d}{d x}\left[e^{x}\right]=e^{x}
$$

5.10.5 THEOREM The natural exponential function $e^{x}$ is differentiable, and hence continuous, on $(-\infty,+\infty)$, and its derivative is

$$
\frac{d}{d x}\left[e^{x}\right]=e^{x}
$$

Proof Because $\ln x$ is differentiable and

$$
\frac{d}{d x}[\ln x]=\frac{1}{x}>0
$$

for all $x$ in $(0,+\infty)$, it follows from Theorem 3.3.1, with $f(x)=\ln x$ and $f^{-1}(x)=e^{x}$, that $e^{x}$ is differentiable on $(-\infty,+\infty)$ and its derivative is

$$
\frac{d}{d x} \underbrace{\left[e^{x}\right]}_{f^{-1}(x)}=\underbrace{\frac{1}{1 / e^{x}}}_{f^{\prime}\left(f^{-1}(x)\right)}=e^{x}
$$

## IRRATIONAL EXPONENTS

Recall from Theorem 5.10.2(d) that if $a>0$ and $r$ is a rational number, then $\ln a^{r}=r \ln a$. Then $a^{r}=e^{\ln a^{r}}=e^{r \ln a}$ for any positive value of $a$ and any rational number $r$. But the expression $e^{r \ln a}$ makes sense for any real number $r$, whether rational or irrational, so it is a good candidate to give meaning to $a^{r}$ for any real number $r$.
5.10.6 DEFINITION If $a>0$ and $r$ is a real number, $a^{r}$ is defined by

$$
\begin{equation*}
a^{r}=e^{r \ln a} \tag{7}
\end{equation*}
$$

With this definition it can be shown that the standard algebraic properties of exponents, such as

$$
a^{p} a^{q}=a^{p+q}, \quad \frac{a^{p}}{a^{q}}=a^{p-q}, \quad\left(a^{p}\right)^{q}=a^{p q}, \quad\left(a^{p}\right)\left(b^{p}\right)=(a b)^{p}
$$

hold for any real values of $a, b, p$, and $q$, where $a$ and $b$ are positive. In addition, using (7) for a real exponent $r$, we can define the power function $x^{r}$ whose domain consists of all positive real numbers, and for a positive base $b$ we can define the base $\boldsymbol{b}$ exponential function $\boldsymbol{b}^{\boldsymbol{x}}$ whose domain consists of all real numbers.

### 5.10.7 THEOREM

(a) For any real number $r$, the power function $x^{r}$ is differentiable on $(0,+\infty)$ and its derivative is

$$
\frac{d}{d x}\left[x^{r}\right]=r x^{r-1}
$$

(b) For $b>0$ and $b \neq 1$, the base $b$ exponential function $b^{x}$ is differentiable on $(-\infty,+\infty)$ and its derivative is

$$
\frac{d}{d x}\left[b^{x}\right]=b^{x} \ln b
$$

PROOF The differentiability of $x^{r}=e^{r \ln x}$ and $b^{x}=e^{x \ln b}$ on their domains follows from the differentiability of $\ln x$ on $(0,+\infty)$ and of $e^{x}$ on $(-\infty,+\infty)$ :

$$
\begin{aligned}
\frac{d}{d x}\left[x^{r}\right] & =\frac{d}{d x}\left[e^{r \ln x}\right]=e^{r \ln x} \cdot \frac{d}{d x}[r \ln x]=x^{r} \cdot \frac{r}{x}=r x^{r-1} \\
\frac{d}{d x}\left[b^{x}\right] & =\frac{d}{d x}\left[e^{x \ln b}\right]=e^{x \ln b} \cdot \frac{d}{d x}[x \ln b]=b^{x} \ln b
\end{aligned}
$$

We expressed $e$ as the value of a limit in Formulas (7) and (8) of Section 1.3 and in Formula (1) of Section 3.2. We now have the mathematical tools necessary to prove the existence of these limits.

### 5.10.8 THEOREM

(a) $\lim _{x \rightarrow 0}(1+x)^{1 / x}=e \quad$ (b) $\quad \lim _{x \rightarrow+\infty}\left(1+\frac{1}{x}\right)^{x}=e$
(c) $\lim _{x \rightarrow-\infty}\left(1+\frac{1}{x}\right)^{x}=e$

PROOF We will prove part ( $a$; the proofs of parts (b) and (c) follow from this limit and are left as exercises. We first observe that

$$
\left.\frac{d}{d x}[\ln (x+1)]\right|_{x=0}=\left.\frac{1}{x+1} \cdot 1\right|_{x=0}=1
$$

However, using the definition of the derivative, we obtain

$$
\begin{aligned}
1=\left.\frac{d}{d x}[\ln (x+1)]\right|_{x=0} & =\lim _{h \rightarrow 0} \frac{\ln (0+h+1)-\ln (0+1)}{h} \\
& =\lim _{h \rightarrow 0}\left[\frac{1}{h} \cdot \ln (1+h)\right]
\end{aligned}
$$

or, equivalently,

$$
\begin{equation*}
\lim _{x \rightarrow 0} \frac{1}{x} \cdot \ln (1+x)=1 \tag{8}
\end{equation*}
$$

Now

$$
\begin{array}{rlrl}
\lim _{x \rightarrow 0}(1+x)^{1 / x} & =\lim _{x \rightarrow 0} e^{(\ln (1+x)) / x} & & \text { Definition 5.1 } \\
& =e^{\lim _{x \rightarrow 0}[(\ln (1+x)) / x]} & & \text { Theorem 1.5.5 } \\
& =e^{1} & & \text { Equation (8) } \\
& =e &
\end{array}
$$

## GENERAL LOGARITHMS

We note that for $b>0$ and $b \neq 1$, the function $b^{x}$ is one-to-one and so has an inverse function. Using the definition of $b^{x}$, we can solve $y=b^{x}$ for $x$ as a function of $y$ :

$$
\begin{aligned}
& y=b^{x}=e^{x \ln b} \\
& \ln y=\ln \left(e^{x \ln b}\right)=x \ln b \\
& \frac{\ln y}{\ln b}=x
\end{aligned}
$$

Thus, the inverse function for $b^{x}$ is $(\ln x) /(\ln b)$.
5.10.9 definition For $b>0$ and $b \neq 1$, the base $\boldsymbol{b}$ logarithm function, denoted $\log _{b} x$, is defined by

$$
\begin{equation*}
\log _{b} x=\frac{\ln x}{\ln b} \tag{9}
\end{equation*}
$$

It follows immediately from this definition that $\log _{b} x$ is the inverse function for $b^{x}$ and satisfies the properties in Table 0.5.3. Furthermore, $\log _{b} x$ is differentiable, and hence continuous, on $(0,+\infty)$, and its derivative is

$$
\frac{d}{d x}\left[\log _{b} x\right]=\frac{1}{x \ln b}
$$

As a final note of consistency, we observe that $\log _{e} x=\ln x$.

## FUNCTIONS DEFINED BY INTEGRALS

The functions we have dealt with thus far in this text are called elementary functions; they include polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric functions, and all other functions that can be obtained from these by addition, subtraction, multiplication, division, root extraction, and composition.

However, there are many important functions that do not fall into this category. Such functions occur in many ways, but they commonly arise in the course of solving initial-value problems of the form

$$
\begin{equation*}
\frac{d y}{d x}=f(x), \quad y\left(x_{0}\right)=y_{0} \tag{10}
\end{equation*}
$$

Recall from Example 6 of Section 5.2 and the discussion preceding it that the basic method for solving (10) is to integrate $f(x)$, and then use the initial condition to determine the constant of integration. It can be proved that if $f$ is continuous, then (10) has a unique solution and that this procedure produces it. However, there is another approach: Instead of solving each initial-value problem individually, we can find a general formula for the solution of (10), and then apply that formula to solve specific problems. We will now show that

$$
\begin{equation*}
y(x)=y_{0}+\int_{x_{0}}^{x} f(t) d t \tag{11}
\end{equation*}
$$

is a formula for the solution of (10). To confirm this we must show that $d y / d x=f(x)$ and that $y\left(x_{0}\right)=y_{0}$. The computations are as follows:

$$
\begin{aligned}
& \frac{d y}{d x}=\frac{d}{d x}\left[y_{0}+\int_{x_{0}}^{x} f(t) d t\right]=0+f(x)=f(x) \\
& y\left(x_{0}\right)=y_{0}+\int_{x_{0}}^{x_{0}} f(t) d t=y_{0}+0=y_{0}
\end{aligned}
$$

- Example 2 In Example 6 of Section 5.2 we showed that the solution of the initial-value problem

$$
\frac{d y}{d x}=\cos x, \quad y(0)=1
$$

is $y(x)=1+\sin x$. This initial-value problem can also be solved by applying Formula (11) with $f(x)=\cos x, x_{0}=0$, and $y_{0}=1$. This yields

$$
y(x)=1+\int_{0}^{x} \cos t d t=1+[\sin t]_{t=0}^{x}=1+\sin x
$$

In the last example we were able to perform the integration in Formula (11) and express the solution of the initial-value problem as an elementary function. However, sometimes this will not be possible, in which case the solution of the initial-value problem must be left in terms of an "unevaluated" integral. For example, from (11), the solution of the
initial-value problem

$$
\frac{d y}{d x}=e^{-x^{2}}, \quad y(0)=1
$$

is

$$
y(x)=1+\int_{0}^{x} e^{-t^{2}} d t
$$

However, it can be shown that there is no way to express the integral in this solution as an elementary function. Thus, we have encountered a new function, which we regard to be defined by the integral. A close relative of this function, known as the error function, plays an important role in probability and statistics; it is denoted by $\operatorname{erf}(x)$ and is defined as

$$
\begin{equation*}
\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^{2}} d t \tag{12}
\end{equation*}
$$

Indeed, many of the most important functions in science and engineering are defined as integrals that have special names and notations associated with them. For example, the functions defined by

$$
\begin{equation*}
S(x)=\int_{0}^{x} \sin \left(\frac{\pi t^{2}}{2}\right) d t \quad \text { and } \quad C(x)=\int_{0}^{x} \cos \left(\frac{\pi t^{2}}{2}\right) d t \tag{13-14}
\end{equation*}
$$

are called the Fresnel sine and cosine functions, respectively, in honor of the French physicist Augustin Fresnel (1788-1827), who first encountered them in his study of diffraction of light waves.

## EVALUATING AND GRAPHING FUNCTIONS DEFINED BY INTEGRALS

The following values of $S(1)$ and $C(1)$ were produced by a CAS that has a built-in algorithm for approximating definite integrals:

$$
S(1)=\int_{0}^{1} \sin \left(\frac{\pi t^{2}}{2}\right) d t \approx 0.438259, \quad C(1)=\int_{0}^{1} \cos \left(\frac{\pi t^{2}}{2}\right) d t \approx 0.779893
$$

To generate graphs of functions defined by integrals, computer programs choose a set of $x$-values in the domain, approximate the integral for each of those values, and then plot the resulting points. Thus, there is a lot of computation involved in generating such graphs, since each plotted point requires the approximation of an integral. The graphs of the Fresnel functions in Figure 5.10 .4 were generated in this way using a CAS.


Fresnel sine function


Fresnel cosine function

Although it required a considerable amount of computation to generate the graphs of the Fresnel functions, the derivatives of $S(x)$ and $C(x)$ are easy to obtain using Part 2 of the Fundamental Theorem of Calculus (5.6.3); they are

$$
\begin{equation*}
S^{\prime}(x)=\sin \left(\frac{\pi x^{2}}{2}\right) \text { and } C^{\prime}(x)=\cos \left(\frac{\pi x^{2}}{2}\right) \tag{15-16}
\end{equation*}
$$

These derivatives can be used to determine the locations of the relative extrema and inflection points and to investigate other properties of $S(x)$ and $C(x)$.

## INTEGRALS WITH FUNCTIONS AS LIMITS OF INTEGRATION

Various applications can lead to integrals in which at least one of the limits of integration is a function of $x$. Some examples are

$$
\int_{x}^{1} \sqrt{\sin t} d t, \quad \int_{x^{2}}^{\sin x} \sqrt{t^{3}+1} d t, \quad \int_{\ln x}^{\pi} \frac{d t}{t^{7}-8}
$$

We will complete this section by showing how to differentiate integrals of the form

$$
\begin{equation*}
\int_{a}^{g(x)} f(t) d t \tag{17}
\end{equation*}
$$

where $a$ is constant. Derivatives of other kinds of integrals with functions as limits of integration will be discussed in the exercises.

To differentiate (17) we can view the integral as a composition $F(g(x))$, where

$$
F(x)=\int_{a}^{x} f(t) d t
$$

If we now apply the chain rule, we obtain

$$
\frac{d}{d x}\left[\int_{a}^{g(x)} f(t) d t\right]=\frac{d}{d x}[F(g(x))]=F^{\prime}(g(x)) g^{\prime}(x)=f(g(x)) g^{\prime}(x)
$$

Thus,

$$
\begin{equation*}
\frac{d}{d x}\left[\int_{a}^{g(x)} f(t) d t\right]=f(g(x)) g^{\prime}(x) \tag{18}
\end{equation*}
$$

In words:

To differentiate an integral with a constant lower limit and a function as the upper limit, substitute the upper limit into the integrand, and multiply by the derivative of the upper limit.

## - Example 3

$$
\frac{d}{d x}\left[\int_{1}^{\sin x}\left(1-t^{2}\right) d t\right]=\left(1-\sin ^{2} x\right) \cos x=\cos ^{3} x
$$

1. $\int_{1}^{1 / e} \frac{1}{t} d t=$ $\qquad$
2. Estimate $\ln 2$ using Definition 5.10.1 and
(a) a left endpoint approximation with $n=2$
(b) a right endpoint approximation with $n=2$.
3. $\pi^{1 /(\ln \pi)}=$ $\qquad$
4. A solution to the initial-value problem

$$
\frac{d y}{d x}=\cos x^{3}, \quad y(0)=2
$$

that is defined by an integral is $y=$ $\qquad$
5. $\frac{d}{d x}\left[\int_{0}^{e^{-x}} \frac{1}{1+t^{4}} d t\right]=$

1. Sketch the curve $y=1 / t$, and shade a region under the curve whose area is
(a) $\ln 2$
(b) $-\ln 0.5$
(c) 2.
2. Sketch the curve $y=1 / t$, and shade two different regions under the curve whose areas are $\ln 1.5$.
3. Given that $\ln a=2$ and $\ln c=5$, find
(a) $\int_{1}^{a c} \frac{1}{t} d t$
(b) $\int_{1}^{1 / c} \frac{1}{t} d t$
(c) $\int_{1}^{a / c} \frac{1}{t} d t$
(d) $\int_{1}^{a^{3}} \frac{1}{t} d t$.
4. Given that $\ln a=9$, find
(a) $\int_{1}^{\sqrt{a}} \frac{1}{t} d t$
(b) $\int_{1}^{2 a} \frac{1}{t} d t$
(c) $\int_{1}^{2 / a} \frac{1}{t} d t$
(d) $\int_{2}^{a} \frac{1}{t} d t$.
5. Approximate $\ln 5$ using the midpoint rule with $n=10$, and estimate the magnitude of the error by comparing your answer to that produced directly by a calculating utility.
6. Approximate $\ln 3$ using the midpoint rule with $n=20$, and estimate the magnitude of the error by comparing your answer to that produced directly by a calculating utility.
7. Simplify the expression and state the values of $x$ for which your simplification is valid.
(a) $e^{-\ln x}$
(b) $e^{\ln x^{2}}$
(c) $\ln \left(e^{-x^{2}}\right)$
(d) $\ln \left(1 / e^{x}\right)$
(e) $\exp (3 \ln x)$
(f) $\ln \left(x e^{x}\right)$
(g) $\ln \left(e^{x-\sqrt[3]{x}}\right)$
(h) $e^{x-\ln x}$
8. (a) Let $f(x)=e^{-2 x}$. Find the simplest exact value of the function $f(\ln 3)$.
(b) Let $f(x)=e^{x}+3 e^{-x}$. Find the simplest exact value of the function $f(\ln 2)$.

9-10 Express the given quantity as a power of $e$.
9. (a) $3^{\pi}$
(b) $2^{\sqrt{2}}$
10. (a) $\pi^{-x}$
(b) $x^{2 x}, \quad x>0$

11-12 Find the limits by making appropriate substitutions in the limits given in Theorem 5.10.8.
11. (a) $\lim _{x \rightarrow+\infty}\left(1+\frac{1}{2 x}\right)^{x}$
(b) $\lim _{x \rightarrow 0}(1+2 x)^{1 / x}$
12. (a) $\lim _{x \rightarrow+\infty}\left(1+\frac{3}{x}\right)^{x}$
(b) $\lim _{x \rightarrow 0}(1+x)^{1 /(3 x)}$

13-14 Find $g^{\prime}(x)$ using Part 2 of the Fundamental Theorem of Calculus, and check your answer by evaluating the integral and then differentiating.
13. $g(x)=\int_{1}^{x}\left(t^{2}-t\right) d t$
14. $g(x)=\int_{\pi}^{x}(1-\cos t) d t$

15-16 Find the derivative using Formula (18), and check your answer by evaluating the integral and then differentiating the result.
15. (a) $\frac{d}{d x} \int_{1}^{x^{3}} \frac{1}{t} d t$
(b) $\frac{d}{d x} \int_{1}^{\ln x} e^{t} d t$
16. (a) $\frac{d}{d x} \int_{-1}^{x^{2}} \sqrt{t+1} d t$
(b) $\frac{d}{d x} \int_{\pi}^{1 / x} \sin t d t$
17. Let $F(x)=\int_{0}^{x} \frac{\sin t}{t^{2}+1} d t$. Find
(a) $F(0)$
(b) $F^{\prime}(0)$
(c) $F^{\prime \prime}(0)$.
18. Let $F(x)=\int_{2}^{x} \sqrt{3 t^{2}+1} d t$. Find
(a) $F(2)$
(b) $F^{\prime}(2)$
(c) $F^{\prime \prime}(2)$.

19-22 True-False Determine whether the equation is true or false. Explain your answer.
19. $\int_{1}^{1 / a} \frac{1}{t} d t=-\int_{1}^{a} \frac{1}{t} d t, \quad$ for $0<a$
20. $\int_{1}^{\sqrt{a}} \frac{1}{t} d t=\frac{1}{2} \int_{1}^{a} \frac{1}{t} d t, \quad$ for $0<a$
21. $\int_{-1}^{e} \frac{1}{t} d t=1$
22. $\int \frac{2 x}{1+x^{2}} d x=\int_{1}^{1+x^{2}} \frac{1}{t} d t+C$
c 23. (a) Use Formula (18) to find

$$
\frac{d}{d x} \int_{1}^{x^{2}} t \sqrt{1+t} d t
$$

(b) Use a CAS to evaluate the integral and differentiate the resulting function.
(c) Use the simplification command of the CAS, if necessary, to confirm that the answers in parts (a) and (b) are the same.
24. Show that
(a) $\frac{d}{d x}\left[\int_{x}^{a} f(t) d t\right]=-f(x)$
(b) $\frac{d}{d x}\left[\int_{g(x)}^{a} f(t) d t\right]=-f(g(x)) g^{\prime}(x)$.

25-26 Use the results in Exercise 24 to find the derivative.
25. (a) $\frac{d}{d x} \int_{x}^{\pi} \cos \left(t^{3}\right) d t$
(b) $\frac{d}{d x} \int_{\tan x}^{3} \frac{t^{2}}{1+t^{2}} d t$
26. (a) $\frac{d}{d x} \int_{x}^{0} \frac{1}{\left(t^{2}+1\right)^{2}} d t$
(b) $\frac{d}{d x} \int_{1 / x}^{\pi} \cos ^{3} t d t$
27. Find

$$
\frac{d}{d x}\left[\int_{3 x}^{x^{2}} \frac{t-1}{t^{2}+1} d t\right]
$$

by writing

$$
\int_{3 x}^{x^{2}} \frac{t-1}{t^{2}+1} d t=\int_{3 x}^{0} \frac{t-1}{t^{2}+1} d t+\int_{0}^{x^{2}} \frac{t-1}{t^{2}+1} d t
$$

28. Use Exercise 24(b) and the idea in Exercise 27 to show that

$$
\frac{d}{d x} \int_{h(x)}^{g(x)} f(t) d t=f(g(x)) g^{\prime}(x)-f(h(x)) h^{\prime}(x)
$$

29. Use the result obtained in Exercise 28 to perform the following differentiations:
(a) $\frac{d}{d x} \int_{x^{2}}^{x^{3}} \sin ^{2} t d t$
(b) $\frac{d}{d x} \int_{-x}^{x} \frac{1}{1+t} d t$.
30. Prove that the function

$$
F(x)=\int_{x}^{5 x} \frac{1}{t} d t
$$

is constant on the interval $(0,+\infty)$ by using Exercise 28 to find $F^{\prime}(x)$. What is that constant?

## FOCUS ON CONCEPTS

31. Let $F(x)=\int_{0}^{x} f(t) d t$, where $f$ is the function whose graph is shown in the accompanying figure.
(a) Find $F(0), F(3), F(5), F(7)$, and $F(10)$.
(b) On what subintervals of the interval $[0,10]$ is $F$ increasing? Decreasing?
(c) Where does $F$ have its maximum value? Its minimum value?
(d) Sketch the graph of $F$.


## < Figure Ex-31

32. Determine the inflection point(s) for the graph of $F$ in Exercise 31.

33-34 Express $F(x)$ in a piecewise form that does not involve an integral.
33. $F(x)=\int_{-1}^{x}|t| d t$
34. $F(x)=\int_{0}^{x} f(t) d t$, where $f(x)= \begin{cases}x, & 0 \leq x \leq 2 \\ 2, & x>2\end{cases}$

35-38 Use Formula (11) to solve the initial-value problem.
35. $\frac{d y}{d x}=\frac{2 x^{2}+1}{x}, y(1)=2 \quad$ 36. $\frac{d y}{d x}=\frac{x+1}{\sqrt{x}}, y(1)=0$
37. $\frac{d y}{d x}=\sec ^{2} x-\sin x, y(\pi / 4)=1$
38. $\frac{d y}{d x}=\frac{1}{x \ln x}, y(e)=1$
39. Suppose that at time $t=0$ there are $P_{0}$ individuals who have disease X, and suppose that a certain model for the spread
of the disease predicts that the disease will spread at the rate of $r(t)$ individuals per day. Write a formula for the number of individuals who will have disease X after $x$ days.
40. Suppose that $v(t)$ is the velocity function of a particle moving along an $s$-axis. Write a formula for the coordinate of the particle at time $T$ if the particle is at $s_{1}$ at time $t=1$.

## FOCUS ON CONCEPTS

41. The accompanying figure shows the graphs of $y=f(x)$ and $y=\int_{0}^{x} f(t) d t$. Determine which graph is which, and explain your reasoning.

< Figure Ex-41
42. (a) Make a conjecture about the value of the limit

$$
\lim _{k \rightarrow 0} \int_{1}^{b} t^{k-1} d t \quad(b>0)
$$

(b) Check your conjecture by evaluating the integral and finding the limit. [Hint: Interpret the limit as the definition of the derivative of an exponential function.]
43. Let $F(x)=\int_{0}^{x} f(t) d t$, where $f$ is the function graphed in the accompanying figure.
(a) Where do the relative minima of $F$ occur?
(b) Where do the relative maxima of $F$ occur?
(c) Where does the absolute maximum of $F$ on the interval $[0,5]$ occur?
(d) Where does the absolute minimum of $F$ on the interval [0,5] occur?
(e) Where is $F$ concave up? Concave down?
(f) Sketch the graph of $F$.

44. CAS programs have commands for working with most of the important nonelementary functions. Check your CAS documentation for information about the error function $\operatorname{erf}(x)$ [see Formula (12)], and then complete the following.
(a) Generate the graph of $\operatorname{erf}(x)$.
(b) Use the graph to make a conjecture about the existence and location of any relative maxima and minima of $\operatorname{erf}(x)$.
(c) Check your conjecture in part (b) using the derivative of $\operatorname{erf}(x)$.
(d) Use the graph to make a conjecture about the existence and location of any inflection points of $\operatorname{erf}(x)$.
(e) Check your conjecture in part (d) using the second derivative of $\operatorname{erf}(x)$.
(f) Use the graph to make a conjecture about the existence of horizontal asymptotes of $\operatorname{erf}(x)$.
(g) Check your conjecture in part (f) by using the CAS to find the limits of $\operatorname{erf}(x)$ as $x \rightarrow \pm \infty$.
45. The Fresnel sine and cosine functions $S(x)$ and $C(x)$ were defined in Formulas (13) and (14) and graphed in Figure 5.10.4. Their derivatives were given in Formulas (15) and (16).
(a) At what points does $C(x)$ have relative minima? Relative maxima?
(b) Where do the inflection points of $C(x)$ occur?
(c) Confirm that your answers in parts (a) and (b) are consistent with the graph of $C(x)$.
46. Find the limit

$$
\lim _{h \rightarrow 0} \frac{1}{h} \int_{x}^{x+h} \ln t d t
$$

47. Find a function $f$ and a number $a$ such that

$$
4+\int_{a}^{x} f(t) d t=e^{2 x}
$$

48. (a) Give a geometric argument to show that

$$
\frac{1}{x+1}<\int_{x}^{x+1} \frac{1}{t} d t<\frac{1}{x}, \quad x>0
$$

(b) Use the result in part (a) to prove that

$$
\frac{1}{x+1}<\ln \left(1+\frac{1}{x}\right)<\frac{1}{x}, \quad x>0
$$

(c) Use the result in part (b) to prove that

$$
e^{x /(x+1)}<\left(1+\frac{1}{x}\right)^{x}<e, \quad x>0
$$

and hence that

$$
\lim _{x \rightarrow+\infty}\left(1+\frac{1}{x}\right)^{x}=e
$$

(d) Use the result in part (b) to prove that

$$
\left(1+\frac{1}{x}\right)^{x}<e<\left(1+\frac{1}{x}\right)^{x+1}, \quad x>0
$$49. Use a graphing utility to generate the graph of

$$
y=\left(1+\frac{1}{x}\right)^{x+1}-\left(1+\frac{1}{x}\right)^{x}
$$

in the window $[0,100] \times[0,0.2]$, and use that graph and part (d) of Exercise 48 to make a rough estimate of the error in the approximation

$$
e \approx\left(1+\frac{1}{50}\right)^{50}
$$

50. Prove: If $f$ is continuous on an open interval and $a$ is any point in that interval, then

$$
F(x)=\int_{a}^{x} f(t) d t
$$

is continuous on the interval.
51. Writing A student objects that it is circular reasoning to make the definition

$$
\ln x=\int_{1}^{x} \frac{1}{t} d t
$$

since to evaluate the integral we need to know the value of $\ln x$. Write a short paragraph that answers this student's objection.
52. Writing Write a short paragraph that compares Definition 5.10.1 with the definition of the natural logarithm function given in Chapter 0. Be sure to discuss the issues surrounding continuity and differentiability.

## QUICK CHECK ANSWERS 5.10

1. -1
2. (a) $\frac{5}{6}$
(b) $\frac{7}{12}$
3. $e$
4. $y=2+\int_{0}^{x} \cos t^{3} d t$
5. $-\frac{e^{-x}}{1+e^{-4 x}}$

## CHAPTER 5 REVIEW EXERCISES $\sim$ Graphing Utility c CAS

1-8 Evaluate the integrals.

1. $\int\left[\frac{1}{2 x^{3}}+4 \sqrt{x}\right] d x$
2. $\int\left[u^{3}-2 u+7\right] d u$
3. $\int[4 \sin x+2 \cos x] d x$
4. $\int \sec x(\tan x+\cos x) d x$
5. $\int\left[x^{-2 / 3}-5 e^{x}\right] d x$
6. $\int\left[\frac{3}{4 x}-\sec ^{2} x\right] d x$
7. $\int\left[\frac{1}{1+x^{2}}+\frac{2}{\sqrt{1-x^{2}}}\right] d x$
8. $\int\left[\frac{12}{x \sqrt{x^{2}-1}}+\frac{1-x^{4}}{1+x^{2}}\right] d x$
9. Solve the initial-value problems.
(a) $\frac{d y}{d x}=\frac{1-x}{\sqrt{x}}, y(1)=0$
(b) $\frac{d y}{d x}=\cos x-5 e^{x}, y(0)=0$
(c) $\frac{d y}{d x}=\sqrt[3]{x}, y(1)=2$
(d) $\frac{d y}{d x}=x e^{x^{2}}, y(0)=0$
10. The accompanying figure shows the slope field for a differential equation $d y / d x=f(x)$. Which of the following functions is most likely to be $f(x)$ ?

$$
\sqrt{x}, \quad \sin x, \quad x^{4}, \quad x
$$

Explain your reasoning.


4Figure Ex-10
11. (a) Show that the substitutions $u=\sec x$ and $u=\tan x$ produce different values for the integral

$$
\int \sec ^{2} x \tan x d x
$$

(b) Explain why both are correct.
12. Use the two substitutions in Exercise 11 to evaluate the definite integral

$$
\int_{0}^{\pi / 4} \sec ^{2} x \tan x d x
$$

and confirm that they produce the same result.
13. Evaluate the integral

$$
\int \frac{x}{\left(x^{2}-1\right) \sqrt{x^{4}-2 x^{2}}} d x
$$

by making the substitution $u=x^{2}-1$.
14. Evaluate the integral

$$
\int \sqrt{1+x^{-2 / 3}} d x
$$

by making the substitution $u=1+x^{2 / 3}$.
C 15-18 Evaluate the integrals by hand, and check your answers with a CAS if you have one.
15. $\int \frac{\cos 3 x}{\sqrt{5+2 \sin 3 x}} d x$
16. $\int \frac{\sqrt{3+\sqrt{x}}}{\sqrt{x}} d x$
17. $\int \frac{x^{2}}{\left(a x^{3}+b\right)^{2}} d x$
18. $\int x \sec ^{2}\left(a x^{2}\right) d x$
19. Express

$$
\sum_{k=4}^{18} k(k-3)
$$

in sigma notation with
(a) $k=0$ as the lower limit of summation
(b) $k=5$ as the lower limit of summation.
20. (a) Fill in the blank:

$$
1+3+5+\cdots+(2 n-1)=\sum_{k=1}^{n}
$$

(b) Use part (a) to prove that the sum of the first $n$ consecutive odd integers is a perfect square.
21. Find the area under the graph of $f(x)=4 x-x^{2}$ over the interval $[0,4]$ using Definition 5.4.3 with $x_{k}^{*}$ as the right endpoint of each subinterval.
22. Find the area under the graph of $f(x)=5 x-x^{2}$ over the interval [0,5] using Definition 5.4.3 with $x_{k}^{*}$ as the left endpoint of each subinterval.

23-24 Use a calculating utility to find the left endpoint, right endpoint, and midpoint approximations to the area under the curve $y=f(x)$ over the stated interval using $n=10$ subintervals.
23. $y=\ln x$; $[1,2]$
24. $y=e^{x} ;[0,1]$
25. The definite integral of $f$ over the interval $[a, b]$ is defined as the limit

$$
\int_{a}^{b} f(x) d x=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k}
$$

Explain what the various symbols on the right side of this equation mean.
26. Use a geometric argument to evaluate

$$
\int_{0}^{1}|2 x-1| d x
$$

27. Suppose that

$$
\begin{aligned}
& \int_{0}^{1} f(x) d x=\frac{1}{2}, \quad \int_{1}^{2} f(x) d x=\frac{1}{4} \\
& \int_{0}^{3} f(x) d x=-1, \quad \int_{0}^{1} g(x) d x=2
\end{aligned}
$$

In each part, use this information to evaluate the given integral, if possible. If there is not enough information to evaluate the integral, then say so.
(a) $\int_{0}^{2} f(x) d x$
(b) $\int_{1}^{3} f(x) d x$
(c) $\int_{2}^{3} 5 f(x) d x$
(d) $\int_{1}^{0} g(x) d x$
(e) $\int_{0}^{1} g(2 x) d x$
(f) $\int_{0}^{1}[g(x)]^{2} d x$
28. In parts (a)-(d), use the information in Exercise 27 to evaluate the given integral. If there is not enough information to evaluate the integral, then say so.
(a) $\int_{0}^{1}[f(x)+g(x)] d x$
(b) $\int_{0}^{1} f(x) g(x) d x$
(c) $\int_{0}^{1} \frac{f(x)}{g(x)} d x$
(d) $\int_{0}^{1}[4 g(x)-3 f(x)] d x$
29. In each part, evaluate the integral. Where appropriate, you may use a geometric formula.
(a) $\int_{-1}^{1}\left(1+\sqrt{1-x^{2}}\right) d x$
(b) $\int_{0}^{3}\left(x \sqrt{x^{2}+1}-\sqrt{9-x^{2}}\right) d x$
(c) $\int_{0}^{1} x \sqrt{1-x^{4}} d x$
30. In each part, find the limit by interpreting it as a limit of Riemann sums in which the interval $[0,1]$ is divided into $n$ subintervals of equal length.
(a) $\lim _{n \rightarrow+\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n^{3 / 2}}$
(b) $\lim _{n \rightarrow+\infty} \frac{1^{4}+2^{4}+3^{4}+\cdots+n^{4}}{n^{5}}$
(c) $\lim _{n \rightarrow+\infty} \frac{e^{1 / n}+e^{2 / n}+e^{3 / n}+\cdots+e^{n / n}}{n}$

31-38 Evaluate the integrals using the Fundamental Theorem of Calculus and (if necessary) properties of the definite integral.
31. $\int_{-3}^{0}\left(x^{2}-4 x+7\right) d x$
32. $\int_{-1}^{2} x\left(1+x^{3}\right) d x$
33. $\int_{1}^{3} \frac{1}{x^{2}} d x$
34. $\int_{1}^{8}\left(5 x^{2 / 3}-4 x^{-2}\right) d x$
35. $\int_{0}^{1}(x-\sec x \tan x) d x$
36. $\int_{1}^{4}\left(\frac{3}{\sqrt{t}}-5 \sqrt{t}-t^{-3 / 2}\right) d t$
37. $\int_{0}^{2}|2 x-3| d x$
38. $\int_{0}^{\pi / 2}\left|\frac{1}{2}-\sin x\right| d x$

39-42 Find the area under the curve $y=f(x)$ over the stated interval.
39. $f(x)=\sqrt{x} ;[1,9]$
40. $f(x)=x^{-3 / 5} ;[1,4]$
41. $f(x)=e^{x} ;[1,3]$
42. $f(x)=\frac{1}{x} ;\left[1, e^{3}\right]$
43. Find the area that is above the $x$-axis but below the curve $y=(1-x)(x-2)$. Make a sketch of the region.
44. Use a CAS to find the area of the region in the first quadrant that lies below the curve $y=x+x^{2}-x^{3}$ and above the $x$-axis.

45-46 Sketch the curve and find the total area between the curve and the given interval on the $x$-axis.
45. $y=x^{2}-1 ;[0,3]$
46. $y=\sqrt{x+1}-1 ;[-1,1]$
47. Define $F(x)$ by

$$
F(x)=\int_{1}^{x}\left(t^{3}+1\right) d t
$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F^{\prime}(x)$.
(b) Check the result in part (a) by first integrating and then differentiating.
48. Define $F(x)$ by

$$
F(x)=\int_{4}^{x} \frac{1}{\sqrt{t}} d t
$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F^{\prime}(x)$.
(b) Check the result in part (a) by first integrating and then differentiating.

49-54 Use Part 2 of the Fundamental Theorem of Calculus and (where necessary) Formula (18) of Section 5.10 to find the derivatives.
49. $\frac{d}{d x}\left[\int_{0}^{x} e^{t^{2}} d t\right]$
50. $\frac{d}{d x}\left[\int_{0}^{x} \frac{t}{\cos t^{2}} d t\right]$
51. $\frac{d}{d x}\left[\int_{0}^{x}|t-1| d t\right]$
52. $\frac{d}{d x}\left[\int_{\pi}^{x} \cos \sqrt{t} d t\right]$
53. $\frac{d}{d x}\left[\int_{2}^{\sin x} \frac{1}{1+t^{3}} d t\right]$
54. $\frac{d}{d x}\left[\int_{e}^{\sqrt{x}}(\ln t)^{2} d t\right]$
55. State the two parts of the Fundamental Theorem of Calculus, and explain what is meant by the statement "Differentiation and integration are inverse processes."
56. Let $F(x)=\int_{0}^{x} \frac{t^{2}-3}{t^{4}+7} d t$.
(a) Find the intervals on which $F$ is increasing and those on which $F$ is decreasing.
(b) Find the open intervals on which $F$ is concave up and those on which $F$ is concave down.
(c) Find the $x$-values, if any, at which the function $F$ has absolute extrema.
(d) Use a CAS to graph $F$, and confirm that the results in parts (a), (b), and (c) are consistent with the graph.
57. (a) Use differentiation to prove that the function

$$
F(x)=\int_{0}^{x} \frac{1}{1+t^{2}} d t+\int_{0}^{1 / x} \frac{1}{1+t^{2}} d t
$$

is constant on the interval $(0,+\infty)$.
(b) Determine the constant value of the function in part (a) and then interpret (a) as an identity involving the inverse tangent function.
58. What is the natural domain of the function

$$
F(x)=\int_{1}^{x} \frac{1}{t^{2}-9} d t ?
$$

Explain your reasoning.
59. In each part, determine the values of $x$ for which $F(x)$ is positive, negative, or zero without performing the integration; explain your reasoning.
(a) $F(x)=\int_{1}^{x} \frac{t^{4}}{t^{2}+3} d t$
(b) $F(x)=\int_{-1}^{x} \sqrt{4-t^{2}} d t$
60. Use a CAS to approximate the largest and smallest values of the integral

$$
\int_{-1}^{x} \frac{t}{\sqrt{2+t^{3}}} d t
$$

for $1 \leq x \leq 3$.
61. Find all values of $x^{*}$ in the stated interval that are guaranteed to exist by the Mean-Value Theorem for Integrals, and explain what these numbers represent.
(a) $f(x)=\sqrt{x} ;[0,3]$
(b) $f(x)=1 / x ;[1, e]$
62. A 10 -gram tumor is discovered in a laboratory rat on March 1. The tumor is growing at a rate of $r(t)=t / 7$ grams per week, where $t$ denotes the number of weeks since March 1. What will be the mass of the tumor on June 7?
63. Use the graph of $f$ shown in the accompanying figure to find the average value of $f$ on the interval $[0,10]$.

< Figure Ex-63
64. Find the average value of $f(x)=e^{x}+e^{-x}$ over the interval $\left[\ln \frac{1}{2}, \ln 2\right]$.
65. Derive the formulas for the position and velocity functions of a particle that moves with constant acceleration along a coordinate line.
66. The velocity of a particle moving along an $s$-axis is measured at 5 s intervals for 40 s , and the velocity function is modeled by a smooth curve. (The curve and the data points are shown in the accompanying figure.) Use this model in each part.
(a) Does the particle have constant acceleration? Explain your reasoning.
(b) Is there any 15 s time interval during which the acceleration is constant? Explain your reasoning.
(c) Estimate the distance traveled by the particle from time $t=0$ to time $t=40$.
(d) Estimate the average velocity of the particle over the 40 s time period.
(e) Is the particle ever slowing down during the 40 s time period? Explain your reasoning.
(f) Is there sufficient information for you to determine the $s$-coordinate of the particle at time $t=10$ ? If so,
find it. If not, explain what additional information you need.


67-70 A particle moves along an $s$-axis. Use the given information to find the position function of the particle.
67. $v(t)=t^{3}-2 t^{2}+1 ; s(0)=1$
68. $a(t)=4 \cos 2 t ; v(0)=-1, s(0)=-3$
69. $v(t)=2 t-3 ; s(1)=5$
70. $a(t)=\cos t-2 t ; v(0)=0, s(0)=0$

71-74 A particle moves with a velocity of $v(t) \mathrm{m} / \mathrm{s}$ along an $s$-axis. Find the displacement and the distance traveled by the particle during the given time interval.
71. $v(t)=2 t-4 ; 0 \leq t \leq 6$
72. $v(t)=|t-3| ; 0 \leq t \leq 5$
73. $v(t)=\frac{1}{2}-\frac{1}{t^{2}} ; 1 \leq t \leq 3$
74. $v(t)=\frac{3}{\sqrt{t}} ; 4 \leq t \leq 9$

75-76 A particle moves with acceleration $a(t) \mathrm{m} / \mathrm{s}^{2}$ along an $s$-axis and has velocity $v_{0} \mathrm{~m} / \mathrm{s}$ at time $t=0$. Find the displacement and the distance traveled by the particle during the given time interval.
75. $a(t)=-2 ; \quad v_{0}=3 ; 1 \leq t \leq 4$
76. $a(t)=\frac{1}{\sqrt{5 t+1}} ; \quad v_{0}=2 ; 0 \leq t \leq 3$
77. A car traveling $60 \mathrm{mi} / \mathrm{h}(=88 \mathrm{ft} / \mathrm{s})$ along a straight road decelerates at a constant rate of $10 \mathrm{ft} / \mathrm{s}^{2}$.
(a) How long will it take until the speed is $45 \mathrm{mi} / \mathrm{h}$ ?
(b) How far will the car travel before coming to a stop?
78. Suppose that the velocity function of a particle moving along an $s$-axis is $v(t)=20 t^{2}-100 t+50 \mathrm{ft} / \mathrm{s}$ and that the particle is at the origin at time $t=0$. Use a graphing utility to generate the graphs of $s(t), v(t)$, and $a(t)$ for the first 6 s of motion.
79. A ball is thrown vertically upward from a height of $s_{0} \mathrm{ft}$ with an initial velocity of $v_{0} \mathrm{ft} / \mathrm{s}$. If the ball is caught at height $s_{0}$, determine its average speed through the air using the free-fall model.
80. A rock, dropped from an unknown height, strikes the ground with a speed of $24 \mathrm{~m} / \mathrm{s}$. Find the height from which the rock was dropped.

81-88 Evaluate the integrals by making an appropriate substitution.
81. $\int_{0}^{1}(2 x+1)^{4} d x$
82. $\int_{-5}^{0} x \sqrt{4-x} d x$
83. $\int_{0}^{1} \frac{d x}{\sqrt{3 x+1}}$
84. $\int_{0}^{\sqrt{\pi}} x \sin x^{2} d x$
85. $\int_{0}^{1} \sin ^{2}(\pi x) \cos (\pi x) d x$ 86. $\int_{e}^{e^{2}} \frac{d x}{x \ln x}$
87. $\int_{0}^{1} \frac{d x}{\sqrt{e^{x}}}$
88. $\int_{0}^{2 / \sqrt{3}} \frac{1}{4+9 x^{2}} d x$
89. Evaluate the limits.
(a) $\lim _{x \rightarrow+\infty}\left(1+\frac{1}{x}\right)^{2 x}$
(b) $\lim _{x \rightarrow+\infty}\left(1+\frac{1}{3 x}\right)^{x}$
90. Find a function $f$ and a number $a$ such that

$$
2+\int_{a}^{x} f(t) d t=e^{3 x}
$$

## CHAPTER 5 MAKING CONNECTIONS

1. Consider a Riemann sum

$$
\sum_{k=1}^{n} 2 x_{k}^{*} \Delta x_{k}
$$

for the integral of $f(x)=2 x$ over an interval $[a, b]$.
(a) Show that if $x_{k}^{*}$ is the midpoint of the $k$ th subinterval, the Riemann sum is a telescoping sum. (See Exercises 57-60 of Section 5.4 for other examples of telescoping sums.)
(b) Use part (a), Definition 5.5.1, and Theorem 5.5.2 to evaluate the definite integral of $f(x)=2 x$ over $[a, b]$.
2. The function $f(x)=\sqrt{x}$ is continuous on $[0,4]$ and therefore integrable on this interval. Evaluate

$$
\int_{0}^{4} \sqrt{x} d x
$$

by using Definition 5.5.1. Use subintervals of unequal length given by the partition

$$
0<4(1)^{2} / n^{2}<4(2)^{2} / n^{2}<\cdots<4(n-1)^{2} / n^{2}<4
$$

and let $x_{k}^{*}$ be the right endpoint of the $k$ th subinterval.
3. Make appropriate modifications and repeat Exercise 2 for

$$
\int_{0}^{8} \sqrt[3]{x} d x
$$

4. Given a continuous function $f$ and a positive real number $m$, let $g$ denote the function defined by the composition $g(x)=f(m x)$.
(a) Suppose that

$$
\sum_{k=1}^{n} g\left(x_{k}^{*}\right) \Delta x_{k}
$$

is any Riemann sum for the integral of $g$ over $[0,1]$. Use the correspondence $u_{k}=m x_{k}, u_{k}^{*}=m x_{k}^{*}$ to create a Riemann sum for the integral of $f$ over $[0, m]$. How are the values of the two Riemann sums related?
(b) Use part (a), Definition 5.5.1, and Theorem 5.5.2 to find an equation that relates the integral of $g$ over $[0,1]$ with the integral of $f$ over $[0, m]$.
(c) How is your answer to part (b) related to Theorem 5.9.1?
5. Given a continuous function $f$, let $g$ denote the function defined by $g(x)=2 x f\left(x^{2}\right)$.
(a) Suppose that

$$
\sum_{k=1}^{n} g\left(x_{k}^{*}\right) \Delta x_{k}
$$

is any Riemann sum for the integral of $g$ over [2,3], with $x_{k}^{*}=\left(x_{k}+x_{k-1}\right) / 2$ the midpoint of the $k$ th subinterval. Use the correspondence $u_{k}=x_{k}^{2}, u_{k}^{*}=\left(x_{k}^{*}\right)^{2}$ to create a Riemann sum for the integral of $f$ over $[4,9]$. How are the values of the two Riemann sums related?
(b) Use part (a), Definition 5.5.1, and Theorem 5.5.2 to find an equation that relates the integral of $g$ over $[2,3]$ with the integral of $f$ over $[4,9]$.
(c) How is your answer to part (b) related to Theorem 5.9.1?


Courtesy NASA

Calculus is essential for the computations required to land an astronaut on the moon.

## APPLICATIONS OF THE DEFINITE INTEGRAL IN GEOMETRY, SCIENCE, AND ENGINEERING

In the last chapter we introduced the definite integral as the limit of Riemann sums in the context of finding areas. However, Riemann sums and definite integrals have applications that extend far beyond the area problem. In this chapter we will show how Riemann sums and definite integrals arise in such problems as finding the volume and surface area of a solid, finding the length of a plane curve, calculating the work done by a force, finding the center of gravity of a planar region, finding the pressure and force exerted by a fluid on a submerged object, and finding properties of suspended cables.

Although these problems are diverse, the required calculations can all be approached by the same procedure that we used to find areas-breaking the required calculation into "small parts," making an approximation for each part, adding the approximations from the parts to produce a Riemann sum that approximates the entire quantity to be calculated, and then taking the limit of the Riemann sums to produce an exact result.

### 6.1 AREA BETWEEN TWO CURVES


$\Delta$ Figure 6.1.1

In the last chapter we showed how to find the area between a curve $y=f(x)$ and an interval on the $x$-axis. Here we will show how to find the area between two curves.

## A REVIEW OF RIEMANN SUMS

Before we consider the problem of finding the area between two curves it will be helpful to review the basic principle that underlies the calculation of area as a definite integral. Recall that if $f$ is continuous and nonnegative on $[a, b]$, then the definite integral for the area $A$ under $y=f(x)$ over the interval $[a, b]$ is obtained in four steps (Figure 6.1.1):

- Divide the interval $[a, b]$ into $n$ subintervals, and use those subintervals to divide the region under the curve $y=f(x)$ into $n$ strips.
- Assuming that the width of the $k$ th strip is $\Delta x_{k}$, approximate the area of that strip by the area $f\left(x_{k}^{*}\right) \Delta x_{k}$ of a rectangle of width $\Delta x_{k}$ and height $f\left(x_{k}^{*}\right)$, where $x_{k}^{*}$ is a point in the $k$ th subinterval.
- Add the approximate areas of the strips to approximate the entire area $A$ by the Riemann sum:

$$
A \approx \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k}
$$



Effect of the limit process on the Riemann sum

- Take the limit of the Riemann sums as the number of subintervals increases and all their widths approach zero. This causes the error in the approximations to approach zero and produces the following definite integral for the exact area $A$ :

$$
A=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} f\left(x_{k}^{*}\right) \Delta x_{k}=\int_{a}^{b} f(x) d x
$$

Figure 6.1.2 illustrates the effect that the limit process has on the various parts of the Riemann sum:

- The quantity $x_{k}^{*}$ in the Riemann sum becomes the variable $x$ in the definite integral.
- The interval width $\Delta x_{k}$ in the Riemann sum becomes the $d x$ in the definite integral.
- The interval $[a, b]$, which is the union of the subintervals with widths $\Delta x_{1}, \Delta x_{2}, \ldots$, $\Delta x_{n}$, does not appear explicitly in the Riemann sum but is represented by the upper and lower limits of integration in the definite integral.

AREA BETWEEN $y=f(x)$ AND $y=g(x)$
We will now consider the following extension of the area problem.
6.1.1 FIRST AREA PROBLEM Suppose that $f$ and $g$ are continuous functions on an interval $[a, b]$ and

$$
f(x) \geq g(x) \quad \text { for } \quad a \leq x \leq b
$$

[This means that the curve $y=f(x)$ lies above the curve $y=g(x)$ and that the two can touch but not cross.] Find the area $A$ of the region bounded above by $y=f(x)$, below by $y=g(x)$, and on the sides by the lines $x=a$ and $x=b$ (Figure 6.1.3a).

$>$ Figure 6.1.3

(b)

To solve this problem we divide the interval $[a, b]$ into $n$ subintervals, which has the effect of subdividing the region into $n$ strips (Figure 6.1.3b). If we assume that the width of the $k$ th strip is $\Delta x_{k}$, then the area of the strip can be approximated by the area of a rectangle of width $\Delta x_{k}$ and height $f\left(x_{k}^{*}\right)-g\left(x_{k}^{*}\right)$, where $x_{k}^{*}$ is a point in the $k$ th subinterval. Adding these approximations yields the following Riemann sum that approximates the area $A$ :

$$
A \approx \sum_{k=1}^{n}\left[f\left(x_{k}^{*}\right)-g\left(x_{k}^{*}\right)\right] \Delta x_{k}
$$

Taking the limit as $n$ increases and the widths of all the subintervals approach zero yields the following definite integral for the area $A$ between the curves:

$$
A=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n}\left[f\left(x_{k}^{*}\right)-g\left(x_{k}^{*}\right)\right] \Delta x_{k}=\int_{a}^{b}[f(x)-g(x)] d x
$$



## A Figure 6.1.4

What does the integral in (1) represent if the graphs of $f$ and $g$ cross each other over the interval $[a, b]$ ? How would you find the area between the curves in this case?

In summary, we have the following result.
6.1.2 AREA FORMULA If $f$ and $g$ are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all $x$ in $[a, b]$, then the area of the region bounded above by $y=f(x)$, below by $y=g(x)$, on the left by the line $x=a$, and on the right by the line $x=b$ is

$$
\begin{equation*}
A=\int_{a}^{b}[f(x)-g(x)] d x \tag{1}
\end{equation*}
$$

Example 1 Find the area of the region bounded above by $y=x+6$, bounded below by $y=x^{2}$, and bounded on the sides by the lines $x=0$ and $x=2$.

Solution. The region and a cross section are shown in Figure 6.1.4. The cross section extends from $g(x)=x^{2}$ on the bottom to $f(x)=x+6$ on the top. If the cross section is moved through the region, then its leftmost position will be $x=0$ and its rightmost position will be $x=2$. Thus, from (1)

$$
A=\int_{0}^{2}\left[(x+6)-x^{2}\right] d x=\left[\frac{x^{2}}{2}+6 x-\frac{x^{3}}{3}\right]_{0}^{2}=\frac{34}{3}-0=\frac{34}{3}
$$

It is possible that the upper and lower boundaries of a region may intersect at one or both endpoints, in which case the sides of the region will be points, rather than vertical line segments (Figure 6.1.5). When that occurs you will have to determine the points of intersection to obtain the limits of integration.
$>$ Figure 6.1.5


The left-hand boundary reduces to a point.


Both side boundaries reduce to points.

$\Delta$ Figure 6.1.6

- Example 2 Find the area of the region that is enclosed between the curves $y=x^{2}$ and $y=x+6$.

Solution. A sketch of the region (Figure 6.1.6) shows that the lower boundary is $y=x^{2}$ and the upper boundary is $y=x+6$. At the endpoints of the region, the upper and lower boundaries have the same $y$-coordinates; thus, to find the endpoints we equate

$$
\begin{equation*}
y=x^{2} \quad \text { and } \quad y=x+6 \tag{2}
\end{equation*}
$$

This yields

$$
x^{2}=x+6 \quad \text { or } \quad x^{2}-x-6=0 \quad \text { or } \quad(x+2)(x-3)=0
$$

from which we obtain

$$
x=-2 \quad \text { and } \quad x=3
$$

Although the $y$-coordinates of the endpoints are not essential to our solution, they may be obtained from (2) by substituting $x=-2$ and $x=3$ in either equation. This yields $y=4$ and $y=9$, so the upper and lower boundaries intersect at $(-2,4)$ and $(3,9)$.

From (1) with $f(x)=x+6, g(x)=x^{2}, a=-2$, and $b=3$, we obtain the area

$$
A=\int_{-2}^{3}\left[(x+6)-x^{2}\right] d x=\left[\frac{x^{2}}{2}+6 x-\frac{x^{3}}{3}\right]_{-2}^{3}=\frac{27}{2}-\left(-\frac{22}{3}\right)=\frac{125}{6}
$$

In the case where $f$ and $g$ are nonnegative on the interval $[a, b]$, the formula

$$
A=\int_{a}^{b}[f(x)-g(x)] d x=\int_{a}^{b} f(x) d x-\int_{a}^{b} g(x) d x
$$

states that the area $A$ between the curves can be obtained by subtracting the area under $y=g(x)$ from the area under $y=f(x)$ (Figure 6.1.7).

$\triangle$ Figure 6.1.7

$\triangle$ Figure 6.1.8

It is not necessary to make an extremely accurate sketch in Step 1; the only purpose of the sketch is to determine which curve is the upper boundary and which is the lower boundary.

- Example 3 Figure 6.1 .8 shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same time. Give a physical interpretation of the area $A$ between the curves over the interval $0 \leq t \leq T$.

Solution. From (1)

$$
A=\int_{0}^{T}\left[v_{2}(t)-v_{1}(t)\right] d t=\int_{0}^{T} v_{2}(t) d t-\int_{0}^{T} v_{1}(t) d t
$$

Since $v_{1}$ and $v_{2}$ are nonnegative functions on [0, T], it follows from Formula (4) of Section 5.7 that the integral of $v_{1}$ over $[0, T]$ is the distance traveled by car 1 during the time interval $0 \leq t \leq T$, and the integral of $v_{2}$ over $[0, T]$ is the distance traveled by car 2 during the same time interval. Since $v_{1}(t) \leq v_{2}(t)$ on [ $0, T$ ], car 2 travels farther than car 1 does over the time interval $0 \leq t \leq T$, and the area $A$ represents the distance by which car 2 is ahead of car 1 at time $T$.

Some regions may require careful thought to determine the integrand and limits of integration in (1). Here is a systematic procedure that you can follow to set up this formula.

## Finding the Limits of Integration for the Area Between Two Curves

Step 1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point $x$ on the $x$-axis, connecting the top and bottom boundaries (Figure 6.1.9a).

Step 2. The $y$-coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x)-g(x)$. This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x=a$ and the rightmost is $x=b$ (Figures 6.1.9b and 6.1.9c).

$\Delta$ Figure 6.1.9
There is a useful way of thinking about this procedure:

If you view the vertical line segment as the "cross section" of the region at the point $x$, then Formula (1) states that the area between the curves is obtained by integrating the length of the cross section over the interval $[a, b]$.

It is possible for the upper or lower boundary of a region to consist of two or more different curves, in which case it will be convenient to subdivide the region into smaller pieces in order to apply Formula (1). This is illustrated in the next example.

- Example 4 Find the area of the region enclosed by $x=y^{2}$ and $y=x-2$.

Solution. To determine the appropriate boundaries of the region, we need to know where the curves $x=y^{2}$ and $y=x-2$ intersect. In Example 2 we found intersections by equating the expressions for $y$. Here it is easier to rewrite the latter equation as $x=y+2$ and equate the expressions for $x$, namely,

$$
\begin{equation*}
x=y^{2} \quad \text { and } \quad x=y+2 \tag{3}
\end{equation*}
$$


(a)

(b)

Figure 6.1.10
This yields

$$
y^{2}=y+2 \quad \text { or } \quad y^{2}-y-2=0 \quad \text { or } \quad(y+1)(y-2)=0
$$

from which we obtain $y=-1, y=2$. Substituting these values in either equation in (3) we see that the corresponding $x$-values are $x=1$ and $x=4$, respectively, so the points of intersection are $(1,-1)$ and $(4,2)$ (Figure 6.1.10a).

To apply Formula (1), the equations of the boundaries must be written so that $y$ is expressed explicitly as a function of $x$. The upper boundary can be written as $y=\sqrt{x}$ (rewrite $x=y^{2}$ as $y= \pm \sqrt{x}$ and choose the + for the upper portion of the curve). The lower boundary consists of two parts:

$$
y=-\sqrt{x} \quad \text { for } \quad 0 \leq x \leq 1 \quad \text { and } \quad y=x-2 \text { for } \quad 1 \leq x \leq 4
$$

(Figure 6.1.10b). Because of this change in the formula for the lower boundary, it is necessary to divide the region into two parts and find the area of each part separately.

From (1) with $f(x)=\sqrt{x}, g(x)=-\sqrt{x}, a=0$, and $b=1$, we obtain

$$
A_{1}=\int_{0}^{1}[\sqrt{x}-(-\sqrt{x})] d x=2 \int_{0}^{1} \sqrt{x} d x=2\left[\frac{2}{3} x^{3 / 2}\right]_{0}^{1}=\frac{4}{3}-0=\frac{4}{3}
$$

From (1) with $f(x)=\sqrt{x}, g(x)=x-2, a=1$, and $b=4$, we obtain

$$
\begin{aligned}
A_{2} & =\int_{1}^{4}[\sqrt{x}-(x-2)] d x=\int_{1}^{4}(\sqrt{x}-x+2) d x \\
& =\left[\frac{2}{3} x^{3 / 2}-\frac{1}{2} x^{2}+2 x\right]_{1}^{4}=\left(\frac{16}{3}-8+8\right)-\left(\frac{2}{3}-\frac{1}{2}+2\right)=\frac{19}{6}
\end{aligned}
$$


$\Delta$ Figure 6.1.11

$\triangle$ Figure 6.1.12

The choice between Formulas (1) and (4) is usually dictated by the shape of the region and which formula requires the least amount of splitting. However, sometimes one might choose the formula that requires more splitting because it is easier to evaluate the resulting integrals.

Thus, the area of the entire region is

$$
A=A_{1}+A_{2}=\frac{4}{3}+\frac{19}{6}=\frac{9}{2}
$$

## REVERSING THE ROLES OF $x$ AND $y$

Sometimes it is much easier to find the area of a region by integrating with respect to $y$ rather than $x$. We will now show how this can be done.
6.1.3 SECOND AREA PROBLEM Suppose that $w$ and $v$ are continuous functions of $y$ on an interval $[c, d]$ and that

$$
w(y) \geq v(y) \quad \text { for } \quad c \leq y \leq d
$$

[This means that the curve $x=w(y)$ lies to the right of the curve $x=v(y)$ and that the two can touch but not cross.] Find the area $A$ of the region bounded on the left by $x=v(y)$, on the right by $x=w(y)$, and above and below by the lines $y=d$ and $y=c$ (Figure 6.1.11).

Proceeding as in the derivation of (1), but with the roles of $x$ and $y$ reversed, leads to the following analog of 6.1.2.
6.1.4 AREA FORMULA If $w$ and $v$ are continuous functions and if $w(y) \geq v(y)$ for all $y$ in $[c, d]$, then the area of the region bounded on the left by $x=v(y)$, on the right by $x=w(y)$, below by $y=c$, and above by $y=d$ is

$$
\begin{equation*}
A=\int_{c}^{d}[w(y)-v(y)] d y \tag{4}
\end{equation*}
$$

The guiding principle in applying this formula is the same as with (1): The integrand in (4) can be viewed as the length of the horizontal cross section at an arbitrary point $y$ on the $y$-axis, in which case Formula (4) states that the area can be obtained by integrating the length of the horizontal cross section over the interval $[c, d]$ on the $y$-axis (Figure 6.1.12).

In Example 4, we split the region into two parts to facilitate integrating with respect to $x$. In the next example we will see that splitting this region can be avoided if we integrate with respect to $y$.

- Example 5 Find the area of the region enclosed by $x=y^{2}$ and $y=x-2$, integrating with respect to $y$.

Solution. As indicated in Figure 6.1.10 the left boundary is $x=y^{2}$, the right boundary is $y=x-2$, and the region extends over the interval $-1 \leq y \leq 2$. However, to apply (4) the equations for the boundaries must be written so that $x$ is expressed explicitly as a function of $y$. Thus, we rewrite $y=x-2$ as $x=y+2$. It now follows from (4) that

$$
A=\int_{-1}^{2}\left[(y+2)-y^{2}\right] d y=\left[\frac{y^{2}}{2}+2 y-\frac{y^{3}}{3}\right]_{-1}^{2}=\frac{9}{2}
$$

which agrees with the result obtained in Example 4.

## QUICK CHECK EXERCISES 6.1 (See page 421 for answers.)

1. An integral expression for the area of the region between the curves $y=20-3 x^{2}$ and $y=e^{x}$ and bounded on the sides by $x=0$ and $x=2$ is $\qquad$
2. An integral expression for the area of the parallelogram bounded by $y=2 x+8, y=2 x-3, x=-1$, and $x=5$ is $\qquad$ The value of this integral is $\qquad$ .
3. (a) The points of intersection for the circle $x^{2}+y^{2}=4$ and the line $y=x+2$ are $\qquad$ and $\qquad$
(b) Expressed as a definite integral with respect to $x$, gives the area of the region insid
$x^{2}+y^{2}=4$ and above the line $y=x+2$.
(c) Expressed as a definite integral with respect to $y$,
$\qquad$ gives the area of the region described in part (b).
4. The area of the region enclosed by the curves $y=x^{2}$ and $y=\sqrt[3]{x}$ is $\qquad$

## EXERCISE SET 6.1 $\sim$ Graphing Utility c CAS

1-4 Find the area of the shaded region.
1.

2.

3.

4.


5-6 Find the area of the shaded region by (a) integrating with respect to $x$ and (b) integrating with respect to $y$.

5.
6.


7-18 Sketch the region enclosed by the curves and find its area.
7. $y=x^{2}, y=\sqrt{x}, x=\frac{1}{4}, x=1$
8. $y=x^{3}-4 x, y=0, x=0, x=2$
9. $y=\cos 2 x, y=0, x=\pi / 4, x=\pi / 2$
10. $y=\sec ^{2} x, y=2, x=-\pi / 4, x=\pi / 4$
11. $x=\sin y, x=0, y=\pi / 4, \quad y=3 \pi / 4$
12. $x^{2}=y, x=y-2$
13. $y=e^{x}, y=e^{2 x}, x=0, x=\ln 2$
14. $x=1 / y, x=0, y=1, y=e$
15. $y=\frac{2}{1+x^{2}}, y=|x|$
16. $y=\frac{1}{\sqrt{1-x^{2}}}, y=2$
17. $y=2+|x-1|, \quad y=-\frac{1}{5} x+7$
18. $y=x, y=4 x, y=-x+2$

19-26 Use a graphing utility, where helpful, to find the area of the region enclosed by the curves.
19. $y=x^{3}-4 x^{2}+3 x, y=0$
20. $y=x^{3}-2 x^{2}, y=2 x^{2}-3 x$
21. $y=\sin x, y=\cos x, x=0, x=2 \pi$
22. $y=x^{3}-4 x, y=0$
23. $x=y^{3}-y, x=0$
24. $x=y^{3}-4 y^{2}+3 y, x=y^{2}-y$
25. $y=x e^{x^{2}}, y=2|x|$
26. $y=\frac{1}{x \sqrt{1-(\ln x)^{2}}}, y=\frac{3}{x}$

27-30 True-False Determine whether the statement is true or false. Explain your answer. [In each exercise, assume that $f$ and $g$ are distinct continuous functions on $[a, b]$ and that $A$ denotes the area of the region bounded by the graphs of $y=f(x)$, $y=g(x), x=a$, and $x=b$.]
27. If $f$ and $g$ differ by a positive constant $c$, then $A=c(b-a)$.
28. If

$$
\int_{a}^{b}[f(x)-g(x)] d x=-3
$$

then $A=3$.
29. If

$$
\int_{a}^{b}[f(x)-g(x)] d x=0
$$

then the graphs of $y=f(x)$ and $y=g(x)$ cross at least once on $[a, b]$.
30. If

$$
A=\left|\int_{a}^{b}[f(x)-g(x)] d x\right|
$$

then the graphs of $y=f(x)$ and $y=g(x)$ don't cross on $[a, b]$.31. Estimate the value of $k(0<k<1)$ so that the region enclosed by $y=1 / \sqrt{1-x^{2}}, y=x, x=0$, and $x=k$ has an area of 1 square unit.32. Estimate the area of the region in the first quadrant enclosed by $y=\sin 2 x$ and $y=\sin ^{-1} x$.
C 33. Use a CAS to find the area enclosed by $y=3-2 x$ and $y=x^{6}+2 x^{5}-3 x^{4}+x^{2}$.
C 34. Use a CAS to find the exact area enclosed by the curves $y=x^{5}-2 x^{3}-3 x$ and $y=x^{3}$.
35. Find a horizontal line $y=k$ that divides the area between $y=x^{2}$ and $y=9$ into two equal parts.
36. Find a vertical line $x=k$ that divides the area enclosed by $x=\sqrt{y}, x=2$, and $y=0$ into two equal parts.
37. (a) Find the area of the region enclosed by the parabola $y=2 x-x^{2}$ and the $x$-axis.
(b) Find the value of $m$ so that the line $y=m x$ divides the region in part (a) into two regions of equal area.
38. Find the area between the curve $y=\sin x$ and the line segment joining the points $(0,0)$ and $(5 \pi / 6,1 / 2)$ on the curve.

39-43 Use Newton's Method (Section 4.7), where needed, to approximate the $x$-coordinates of the intersections of the curves to at least four decimal places, and then use those approximations to approximate the area of the region.
39. The region that lies below the curve $y=\sin x$ and above the line $y=0.2 x$, where $x \geq 0$.
40. The region enclosed by the graphs of $y=x^{2}$ and $y=\cos x$.
41. The region enclosed by the graphs of $y=(\ln x) / x$ and $y=x-2$.
42. The region enclosed by the graphs of $y=3-2 \cos x$ and $y=2 /\left(1+x^{2}\right)$.
43. The region enclosed by the graphs of $y=x^{2}-1$ and $y=2 \sin x$.
44. Referring to the accompanying figure, use a CAS to estimate the value of $k$ so that the areas of the shaded regions are equal.
Source: This exercise is based on Problem A1 that was posed in the Fifty-Fourth Annual William Lowell Putnam Mathematical Competition.


4 Figure Ex-44

## FOCUS ON CONCEPTS

45. Two racers in adjacent lanes move with velocity functions $v_{1}(t) \mathrm{m} / \mathrm{s}$ and $v_{2}(t) \mathrm{m} / \mathrm{s}$, respectively. Suppose that the racers are even at time $t=60 \mathrm{~s}$. Interpret the
value of the integral

$$
\int_{0}^{60}\left[v_{2}(t)-v_{1}(t)\right] d t
$$

in this context.
46. The accompanying figure shows acceleration versus time curves for two cars that move along a straight track, accelerating from rest at the starting line. What does the area $A$ between the curves over the interval $0 \leq t \leq T$ represent? Justify your answer.

< Figure Ex-46
47. Suppose that $f$ and $g$ are integrable on $[a, b]$, but neither $f(x) \geq g(x)$ nor $g(x) \geq f(x)$ holds for all $x$ in $[a, b]$ [i.e., the curves $y=f(x)$ and $y=g(x)$ are intertwined].
(a) What is the geometric significance of the integral

$$
\int_{a}^{b}[f(x)-g(x)] d x ?
$$

(b) What is the geometric significance of the integral

$$
\int_{a}^{b}|f(x)-g(x)| d x ?
$$

48. Let $A(n)$ be the area in the first quadrant enclosed by the curves $y=\sqrt[n]{x}$ and $y=x$.
(a) By considering how the graph of $y=\sqrt[n]{x}$ changes as $n$ increases, make a conjecture about the limit of $A(n)$ as $n \rightarrow+\infty$.
(b) Confirm your conjecture by calculating the limit.
49. Find the area of the region enclosed between the curve $x^{1 / 2}+y^{1 / 2}=a^{1 / 2}$ and the coordinate axes.
50. Show that the area of the ellipse in the accompanying figure is $\pi a b$. [Hint: Use a formula from geometry.]


Figure Ex-50
51. Writing Suppose that $f$ and $g$ are continuous on $[a, b]$ but that the graphs of $y=f(x)$ and $y=g(x)$ cross several times. Describe a step-by-step procedure for determining the area bounded by the graphs of $y=f(x), y=g(x)$, $x=a$, and $x=b$.
52. Writing Suppose that $R$ and $S$ are two regions in the $x y$ plane that lie between a pair of lines $L_{1}$ and $L_{2}$ that are parallel to the $y$-axis. Assume that each line between $L_{1}$ and $L_{2}$ that is parallel to the $y$-axis intersects $R$ and $S$ in
line segments of equal length. Give an informal argument that the area of $R$ is equal to the area of $S$. (Make reasonable assumptions about the boundaries of $R$ and $S$.)

## QUICK CHECK ANSWERS 6.1

1. $\int_{0}^{2}\left[\left(20-3 x^{2}\right)-e^{x}\right] d x$
2. $\int_{-1}^{5}[(2 x+8)-(2 x-3)] d x ; 66$
3. (a) $(-2,0)$;
$(0,2)$
(b) $\int_{-2}^{0}\left[\sqrt{4-x^{2}}-(x+2)\right] d x$
(c) $\int_{0}^{2}\left[(y-2)+\sqrt{4-y^{2}}\right] d y$
4. $\frac{5}{12}$

### 6.2 VOLUMES BY SLICING; DISKS AND WASHERS

In the last section we showed that the area of a plane region bounded by two curves can be obtained by integrating the length of a general cross section over an appropriate interval. In this section we will see that the same basic principle can be used to find volumes of certain three-dimensional solids.

## VOLUMES BY SLICING

Recall that the underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the area. Under appropriate conditions, the same strategy can be used to find the volume of a solid. The idea is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the volume (Figure 6.2.1).

$\Delta$ Figure 6.2.1


In a thin slab, the cross sections do not vary much in size and shape.
$\triangle$ Figure 6.2.2

What makes this method work is the fact that a thin slab has a cross section that does not vary much in size or shape, which, as we will see, makes its volume easy to approximate (Figure 6.2.2). Moreover, the thinner the slab, the less variation in its cross sections and the better the approximation. Thus, once we approximate the volumes of the slabs, we can set up a Riemann sum whose limit is the volume of the entire solid. We will give the details shortly, but first we need to discuss how to find the volume of a solid whose cross sections do not vary in size and shape (i.e., are congruent).

One of the simplest examples of a solid with congruent cross sections is a right circular cylinder of radius $r$, since all cross sections taken perpendicular to the central axis are circular regions of radius $r$. The volume $V$ of a right circular cylinder of radius $r$ and height $h$ can be expressed in terms of the height and the area of a cross section as

$$
\begin{equation*}
V=\pi r^{2} h=[\text { area of a cross section }] \times[\text { height }] \tag{1}
\end{equation*}
$$

This is a special case of a more general volume formula that applies to solids called right cylinders. A right cylinder is a solid that is generated when a plane region is translated along a line or axis that is perpendicular to the region (Figure 6.2.3).

$\Delta$ Figure 6.2.3

$\Delta$ Figure 6.2.4

$\triangle$ Figure 6.2.5

If a right cylinder is generated by translating a region of area $A$ through a distance $h$, then $h$ is called the height (or sometimes the width) of the cylinder, and the volume $V$ of the cylinder is defined to be

$$
\begin{equation*}
V=A \cdot h=[\text { area of a cross section }] \times[\text { height }] \tag{2}
\end{equation*}
$$

(Figure 6.2.4). Note that this is consistent with Formula (1) for the volume of a right circular cylinder.

We now have all of the tools required to solve the following problem.
6.2.1 PRObLEM Let $S$ be a solid that extends along the $x$-axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the $x$-axis at $x=a$ and $x=b$ (Figure 6.2.5). Find the volume $V$ of the solid, assuming that its cross-sectional area $A(x)$ is known at each $x$ in the interval $[a, b]$.

To solve this problem we begin by dividing the interval $[a, b]$ into $n$ subintervals, thereby dividing the solid into $n$ slabs as shown in the left part of Figure 6.2.6. If we assume that the width of the $k$ th subinterval is $\Delta x_{k}$, then the volume of the $k$ th slab can be approximated by the volume $A\left(x_{k}^{*}\right) \Delta x_{k}$ of a right cylinder of width (height) $\Delta x_{k}$ and cross-sectional area $A\left(x_{k}^{*}\right)$, where $x_{k}^{*}$ is a point in the $k$ th subinterval (see the right part of Figure 6.2.6).
$>$ Figure 6.2.6


Adding these approximations yields the following Riemann sum that approximates the volume $V$ :

$$
V \approx \sum_{k=1}^{n} A\left(x_{k}^{*}\right) \Delta x_{k}
$$

Taking the limit as $n$ increases and the widths of all the subintervals approach zero yields the definite integral

$$
V=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} A\left(x_{k}^{*}\right) \Delta x_{k}=\int_{a}^{b} A(x) d x
$$

In summary, we have the following result.
6.2.2 vOLUME FORMULA Let $S$ be a solid bounded by two parallel planes perpendicular to the $x$-axis at $x=a$ and $x=b$. If, for each $x$ in $[a, b]$, the cross-sectional area of $S$ perpendicular to the $x$-axis is $A(x)$, then the volume of the solid is

$$
\begin{equation*}
V=\int_{a}^{b} A(x) d x \tag{3}
\end{equation*}
$$

provided $A(x)$ is integrable.

There is a similar result for cross sections perpendicular to the $y$-axis.
6.2.3 VOLUME FORMULA Let $S$ be a solid bounded by two parallel planes perpendicular to the $y$-axis at $y=c$ and $y=d$. If, for each $y$ in $[c, d]$, the cross-sectional area of $S$ perpendicular to the $y$-axis is $A(y)$, then the volume of the solid is

$$
\begin{equation*}
V=\int_{c}^{d} A(y) d y \tag{4}
\end{equation*}
$$

provided $A(y)$ is integrable.

In words, these formulas state:

The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.

- Example 1 Derive the formula for the volume of a right pyramid whose altitude is $h$ and whose base is a square with sides of length $a$.

Solution. As illustrated in Figure 6.2.7a, we introduce a rectangular coordinate system in which the $y$-axis passes through the apex and is perpendicular to the base, and the $x$-axis passes through the base and is parallel to a side of the base.

At any $y$ in the interval $[0, h]$ on the $y$-axis, the cross section perpendicular to the $y$ axis is a square. If $s$ denotes the length of a side of this square, then by similar triangles (Figure 6.2.7b)

$$
\frac{\frac{1}{2} s}{\frac{1}{2} a}=\frac{h-y}{h} \quad \text { or } \quad s=\frac{a}{h}(h-y)
$$

Thus, the area $A(y)$ of the cross section at $y$ is

$$
A(y)=s^{2}=\frac{a^{2}}{h^{2}}(h-y)^{2}
$$

and by (4) the volume is

$$
\begin{aligned}
V=\int_{0}^{h} A(y) d y & =\int_{0}^{h} \frac{a^{2}}{h^{2}}(h-y)^{2} d y=\frac{a^{2}}{h^{2}} \int_{0}^{h}(h-y)^{2} d y \\
& =\frac{a^{2}}{h^{2}}\left[-\frac{1}{3}(h-y)^{3}\right]_{y=0}^{h}=\frac{a^{2}}{h^{2}}\left[0+\frac{1}{3} h^{3}\right]=\frac{1}{3} a^{2} h
\end{aligned}
$$

That is, the volume is $\frac{1}{3}$ of the area of the base times the altitude.

## SOLIDS OF REVOLUTION

A solid of revolution is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the axis of revolution. Many familiar solids are of this type (Figure 6.2.8).


VOLUMES BY DISKS PERPENDICULAR TO THE x-AXIS
We will be interested in the following general problem.
6.2.4 PROblem Let $f$ be continuous and nonnegative on $[a, b]$, and let $R$ be the region that is bounded above by $y=f(x)$, below by the $x$-axis, and on the sides by the lines $x=a$ and $x=b$ (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region $R$ about the $x$-axis.


Figure 6.2.9

(b)

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the $x$-axis at the point $x$ is a circular disk of radius $f(x)$ (Figure 6.2.9b). The area of this region is

$$
A(x)=\pi[f(x)]^{2}
$$

Thus, from (3) the volume of the solid is

$$
\begin{equation*}
V=\int_{a}^{b} \pi[f(x)]^{2} d x \tag{5}
\end{equation*}
$$

Because the cross sections are disk shaped, the application of this formula is called the method of disks.

Example 2 Find the volume of the solid that is obtained when the region under the curve $y=\sqrt{x}$ over the interval [1,4] is revolved about the $x$-axis (Figure 6.2.10).

Solution. From (5), the volume is

$$
\left.V=\int_{a}^{b} \pi[f(x)]^{2} d x=\int_{1}^{4} \pi x d x=\frac{\pi x^{2}}{2}\right]_{1}^{4}=8 \pi-\frac{\pi}{2}=\frac{15 \pi}{2}
$$

Example 3 Derive the formula for the volume of a sphere of radius $r$.
Solution. As indicated in Figure 6.2.11, a sphere of radius $r$ can be generated by revolving the upper semicircular disk enclosed between the $x$-axis and

$$
x^{2}+y^{2}=r^{2}
$$

about the $x$-axis. Since the upper half of this circle is the graph of $y=f(x)=\sqrt{r^{2}-x^{2}}$, it follows from (5) that the volume of the sphere is

$$
V=\int_{a}^{b} \pi[f(x)]^{2} d x=\int_{-r}^{r} \pi\left(r^{2}-x^{2}\right) d x=\pi\left[r^{2} x-\frac{x^{3}}{3}\right]_{-r}^{r}=\frac{4}{3} \pi r^{3}
$$

## VOLUMES BY WASHERS PERPENDICULAR TO THE x-AXIS

Not all solids of revolution have solid interiors; some have holes or channels that create interior surfaces, as in Figure 6.2.8d. So we will also be interested in problems of the following type.
6.2.5 PROBLEM Let $f$ and $g$ be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all $x$ in the interval $[a, b]$. Let $R$ be the region that is bounded above by $y=f(x)$, below by $y=g(x)$, and on the sides by the lines $x=a$ and $x=b$ (Figure 6.2.12a). Find the volume of the solid of revolution that is generated by revolving the region $R$ about the $x$-axis (Figure 6.2.12b).

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the $x$-axis at the point $x$ is the annular or "washer-shaped"
region with inner radius $g(x)$ and outer radius $f(x)$ (Figure 6.2.12b); its area is

$$
A(x)=\pi[f(x)]^{2}-\pi[g(x)]^{2}=\pi\left([f(x)]^{2}-[g(x)]^{2}\right)
$$

Thus, from (3) the volume of the solid is

$$
\begin{equation*}
V=\int_{a}^{b} \pi\left([f(x)]^{2}-[g(x)]^{2}\right) d x \tag{6}
\end{equation*}
$$

Because the cross sections are washer shaped, the application of this formula is called the method of washers.

- Example 4 Find the volume of the solid generated when the region between the graphs of the equations $f(x)=\frac{1}{2}+x^{2}$ and $g(x)=x$ over the interval [ 0,2 ] is revolved about the $x$-axis.

Solution. First sketch the region (Figure 6.2.13a); then imagine revolving it about the $x$-axis (Figure 6.2.13b). From (6) the volume is

$$
\begin{aligned}
V & =\int_{a}^{b} \pi\left([f(x)]^{2}-[g(x)]^{2}\right) d x=\int_{0}^{2} \pi\left(\left[\frac{1}{2}+x^{2}\right]^{2}-x^{2}\right) d x \\
& =\int_{0}^{2} \pi\left(\frac{1}{4}+x^{4}\right) d x=\pi\left[\frac{x}{4}+\frac{x^{5}}{5}\right]_{0}^{2}=\frac{69 \pi}{10}
\end{aligned}
$$




Unequal scales on axes

$$
\begin{aligned}
& \text { Region defined } \\
& \text { by } f \text { and } g
\end{aligned}
$$

$$
\begin{aligned}
& \text { The resulting } \\
& \text { solid of revolution }
\end{aligned}
$$

(a)
(b)

## VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE y-AXIS

The methods of disks and washers have analogs for regions that are revolved about the $y$ axis (Figures 6.2.14 and 6.2.15). Using the method of slicing and Formula (4), you should be able to deduce the following formulas for the volumes of the solids in the figures.

$$
\begin{equation*}
V=\int_{c}^{d} \pi[u(y)]^{2} d y \quad V=\int_{c}^{d} \pi\left([w(y)]^{2}-[v(y)]^{2}\right) d y \tag{7-8}
\end{equation*}
$$


$\Delta$ Figure 6.2.15

- Example 5 Find the volume of the solid generated when the region enclosed by $y=\sqrt{x}, y=2$, and $x=0$ is revolved about the $y$-axis.

Solution. First sketch the region and the solid (Figure 6.2.16). The cross sections taken perpendicular to the $y$-axis are disks, so we will apply (7). But first we must rewrite $y=\sqrt{x}$ as $x=y^{2}$. Thus, from (7) with $u(y)=y^{2}$, the volume is

$$
\left.V=\int_{c}^{d} \pi[u(y)]^{2} d y=\int_{0}^{2} \pi y^{4} d y=\frac{\pi y^{5}}{5}\right]_{0}^{2}=\frac{32 \pi}{5}
$$




## OTHER AXES OF REVOLUTION

It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. Instead of developing a new formula for each situation, we will appeal to Formulas (3) and (4) and integrate an appropriate cross-sectional area to find the volume.

- Example 6 Find the volume of the solid generated when the region under the curve $y=x^{2}$ over the interval $[0,2]$ is rotated about the line $y=-1$.

Solution. First sketch the region and the axis of revolution; then imagine revolving the region about the axis (Figure 6.2.17). At each $x$ in the interval $0 \leq x \leq 2$, the cross section of the solid perpendicular to the axis $y=-1$ is a washer with outer radius $x^{2}+1$ and inner radius 1 . Since the area of this washer is

$$
A(x)=\pi\left(\left[x^{2}+1\right]^{2}-1^{2}\right)=\pi\left(x^{4}+2 x^{2}\right)
$$

it follows by (3) that the volume of the solid is

$$
V=\int_{0}^{2} A(x) d x=\int_{0}^{2} \pi\left(x^{4}+2 x^{2}\right) d x=\pi\left[\frac{1}{5} x^{5}+\frac{2}{3} x^{3}\right]_{0}^{2}=\frac{176 \pi}{15}
$$




## QUICK CHECK EXERCISES 6.2 (See page 431 for answers.)

1. A solid $S$ extends along the $x$-axis from $x=1$ to $x=3$. For $x$ between 1 and 3, the cross-sectional area of $S$ perpendicular to the $x$-axis is $3 x^{2}$. An integral expression for the volume of $S$ is $\qquad$ The value of this integral is
2. A solid $S$ is generated by revolving the region between the $x$-axis and the curve $y=\sqrt{\sin x}(0 \leq x \leq \pi)$ about the $x$ axis.
(a) For $x$ between 0 and $\pi$, the cross-sectional area of $S$ perpendicular to the $x$-axis at $x$ is $A(x)=$ $\qquad$ .
(b) An integral expression for the volume of $S$ is $\qquad$
(c) The value of the integral in part (b) is $\qquad$ —.
3. A solid $S$ is generated by revolving the region enclosed by the line $y=2 x+1$ and the curve $y=x^{2}+1$ about the $x$-axis.
(a) For $x$ between $\qquad$ and $\qquad$ the crosssectional area of $S$ perpendicular to the $x$-axis at $x$ is $A(x)=$ $\qquad$
(b) An integral expression for the volume of $S$ is $\qquad$
4. A solid $S$ is generated by revolving the region enclosed by the line $y=x+1$ and the curve $y=x^{2}+1$ about the $y$ axis.
(a) For $y$ between $\qquad$ and $\qquad$ the crosssectional area of $S$ perpendicular to the $y$-axis at $y$ is $A(y)=$ $\qquad$ —.
(b) An integral expression for the volume of $S$ is $\qquad$

## EXERCISE SET 6.2 <br> CAS

1-8 Find the volume of the solid that results when the shaded region is revolved about the indicated axis.
1.

2.

3.

4.


9. Find the volume of the solid whose base is the region bounded between the curve $y=x^{2}$ and the $x$-axis from $x=0$ to $x=2$ and whose cross sections taken perpendicular to the $x$-axis are squares.
10. Find the volume of the solid whose base is the region bounded between the curve $y=\sec x$ and the $x$-axis from $x=\pi / 4$ to $x=\pi / 3$ and whose cross sections taken perpendicular to the $x$-axis are squares.

11-18 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the $x$-axis.
11. $y=\sqrt{25-x^{2}}, y=3$
12. $y=9-x^{2}, y=0$
13. $x=\sqrt{y}, x=y / 4$
14. $y=\sin x, y=\cos x, x=0, x=\pi / 4$
[Hint: Use the identity $\cos 2 x=\cos ^{2} x-\sin ^{2} x$.]
15. $y=e^{x}, y=0, x=0, x=\ln 3$
16. $y=e^{-2 x}, y=0, x=0, x=1$
17. $y=\frac{1}{\sqrt{4+x^{2}}}, x=-2, x=2, y=0$
18. $y=\frac{e^{3 x}}{\sqrt{1+e^{6 x}}}, x=0, x=1, y=0$
19. Find the volume of the solid whose base is the region bounded between the curve $y=x^{3}$ and the $y$-axis from $y=0$ to $y=1$ and whose cross sections taken perpendicular to the $y$-axis are squares.
20. Find the volume of the solid whose base is the region enclosed between the curve $x=1-y^{2}$ and the $y$-axis and whose cross sections taken perpendicular to the $y$-axis are squares.

21-26 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the $y$-axis.
21. $x=\csc y, \quad y=\pi / 4, \quad y=3 \pi / 4, \quad x=0$
22. $y=x^{2}, x=y^{2}$
23. $x=y^{2}, x=y+2$
24. $x=1-y^{2}, x=2+y^{2}, y=-1, \quad y=1$
25. $y=\ln x, x=0, y=0, y=1$
26. $y=\sqrt{\frac{1-x^{2}}{x^{2}}} \quad(x>0), x=0, y=0, y=2$

27-30 True-False Determine whether the statement is true or false. Explain your answer. [In these exercises, assume that a solid $S$ of volume $V$ is bounded by two parallel planes perpendicular to the $x$-axis at $x=a$ and $x=b$ and that for each $x$ in $[a, b], A(x)$ denotes the cross-sectional area of $S$ perpendicular to the $x$-axis.]
27. If each cross section of $S$ perpendicular to the $x$-axis is a square, then $S$ is a rectangular parallelepiped (i.e., is box shaped).
28. If each cross section of $S$ is a disk or a washer, then $S$ is a solid of revolution.
29. If $x$ is in centimeters (cm), then $A(x)$ must be a quadratic function of $x$, since units of $A(x)$ will be square centimeters ( $\mathrm{cm}^{2}$ ).
30. The average value of $A(x)$ on the interval $[a, b]$ is given by $V /(b-a)$.
31. Find the volume of the solid that results when the region above the $x$-axis and below the ellipse

$$
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1 \quad(a>0, b>0)
$$

is revolved about the $x$-axis.
32. Let $V$ be the volume of the solid that results when the region enclosed by $y=1 / x, y=0, x=2$, and $x=b(0<b<2)$ is revolved about the $x$-axis. Find the value of $b$ for which $V=3$.
33. Find the volume of the solid generated when the region enclosed by $y=\sqrt{x+1}, y=\sqrt{2 x}$, and $y=0$ is revolved about the $x$-axis. [Hint: Split the solid into two parts.]
34. Find the volume of the solid generated when the region enclosed by $y=\sqrt{x}, y=6-x$, and $y=0$ is revolved about the $x$-axis. [Hint: Split the solid into two parts.]

## FOCUS ON CONCEPTS

35. Suppose that $f$ is a continuous function on $[a, b]$, and let $R$ be the region between the curve $y=f(x)$ and the line $y=k$ from $x=a$ to $x=b$. Using the method of disks, derive with explanation a formula for the volume of a solid generated by revolving $R$ about the line $y=k$. State and explain additional assumptions, if any, that you need about $f$ for your formula.
36. Suppose that $v$ and $w$ are continuous functions on $[c, d]$, and let $R$ be the region between the curves $x=v(y)$ and $x=w(y)$ from $y=c$ to $y=d$. Using the method of washers, derive with explanation a formula for the volume of a solid generated by revolving $R$ about the line
$x=k$. State and explain additional assumptions, if any, that you need about $v$ and $w$ for your formula.
37. Consider the solid generated by revolving the shaded region in Exercise 1 about the line $y=2$.
(a) Make a conjecture as to which is larger: the volume of this solid or the volume of the solid in Exercise 1. Explain the basis of your conjecture.
(b) Check your conjecture by calculating this volume and comparing it to the volume obtained in Exercise 1.
38. Consider the solid generated by revolving the shaded region in Exercise 4 about the line $x=2.5$.
(a) Make a conjecture as to which is larger: the volume of this solid or the volume of the solid in Exercise 4. Explain the basis of your conjecture.
(b) Check your conjecture by calculating this volume and comparing it to the volume obtained in Exercise 4.
39. Find the volume of the solid that results when the region enclosed by $y=\sqrt{x}, y=0$, and $x=9$ is revolved about the line $x=9$.
40. Find the volume of the solid that results when the region in Exercise 39 is revolved about the line $y=3$.
41. Find the volume of the solid that results when the region enclosed by $x=y^{2}$ and $x=y$ is revolved about the line $y=-1$.
42. Find the volume of the solid that results when the region in Exercise 41 is revolved about the line $x=-1$.
43. Find the volume of the solid that results when the region enclosed by $y=x^{2}$ and $y=x^{3}$ is revolved about the line $x=1$.
44. Find the volume of the solid that results when the region in Exercise 43 is revolved about the line $y=-1$.
45. A nose cone for a space reentry vehicle is designed so that a cross section, taken $x \mathrm{ft}$ from the tip and perpendicular to the axis of symmetry, is a circle of radius $\frac{1}{4} x^{2} \mathrm{ft}$. Find the volume of the nose cone given that its length is 20 ft .
46. A certain solid is 1 ft high, and a horizontal cross section taken $x \mathrm{ft}$ above the bottom of the solid is an annulus of inner radius $x^{2} \mathrm{ft}$ and outer radius $\sqrt{x} \mathrm{ft}$. Find the volume of the solid.
47. Find the volume of the solid whose base is the region bounded between the curves $y=x$ and $y=x^{2}$, and whose cross sections perpendicular to the $x$-axis are squares.
48. The base of a certain solid is the region enclosed by $y=\sqrt{x}$, $y=0$, and $x=4$. Every cross section perpendicular to the $x$-axis is a semicircle with its diameter across the base. Find the volume of the solid.
49. In parts (a)-(c) find the volume of the solid whose base is enclosed by the circle $x^{2}+y^{2}=1$ and whose cross sections taken perpendicular to the $x$-axis are
(a) semicircles
(b) squares
(c) equilateral triangles.

50. As shown in the accompanying figure, a cathedral dome is designed with three semicircular supports of radius $r$ so that each horizontal cross section is a regular hexagon. Show that the volume of the dome is $r^{3} \sqrt{3}$.


## 4Figure Ex-50

C 51-54 Use a CAS to estimate the volume of the solid that results when the region enclosed by the curves is revolved about the stated axis.
51. $y=\sin ^{8} x, y=2 x / \pi, x=0, x=\pi / 2 ; x$-axis
52. $y=\pi^{2} \sin x \cos ^{3} x, y=4 x^{2}, x=0, x=\pi / 4 ; x$-axis
53. $y=e^{x}, x=1, y=1 ; y$-axis
54. $y=x \sqrt{\tan ^{-1} x}, y=x ; x$-axis
55. The accompanying figure shows a spherical cap of radius $\rho$ and height $h$ cut from a sphere of radius $r$. Show that the volume $V$ of the spherical cap can be expressed as
(a) $V=\frac{1}{3} \pi h^{2}(3 r-h)$
(b) $V=\frac{1}{6} \pi h\left(3 \rho^{2}+h^{2}\right)$.


## < Figure Ex-55

56. If fluid enters a hemispherical bowl with a radius of 10 ft at a rate of $\frac{1}{2} \mathrm{ft}^{3} / \mathrm{min}$, how fast will the fluid be rising when the depth is 5 ft ? [Hint: See Exercise 55.]
57. The accompanying figure (on the next page) shows the dimensions of a small lightbulb at 10 equally spaced points.
(a) Use formulas from geometry to make a rough estimate of the volume enclosed by the glass portion of the bulb.
(b) Use the average of left and right endpoint approximations to approximate the volume.


- Figure Ex- 57

58. Use the result in Exercise 55 to find the volume of the solid that remains when a hole of radius $r / 2$ is drilled through the center of a sphere of radius $r$, and then check your answer by integrating.
59. As shown in the accompanying figure, a cocktail glass with a bowl shaped like a hemisphere of diameter 8 cm contains a cherry with a diameter of 2 cm . If the glass is filled to a depth of $h \mathrm{~cm}$, what is the volume of liquid it contains? [Hint: First consider the case where the cherry is partially submerged, then the case where it is totally submerged.]

< Figure Ex-59
60. Find the volume of the torus that results when the region enclosed by the circle of radius $r$ with center at $(h, 0), h>r$, is revolved about the $y$-axis. [Hint: Use an appropriate formula from plane geometry to help evaluate the definite integral.]
61. A wedge is cut from a right circular cylinder of radius $r$ by two planes, one perpendicular to the axis of the cylinder and the other making an angle $\theta$ with the first. Find the volume of the wedge by slicing perpendicular to the $y$-axis as shown in the accompanying figure.

< Figure Ex-61
62. Find the volume of the wedge described in Exercise 61 by slicing perpendicular to the $x$-axis.
63. Two right circular cylinders of radius $r$ have axes that intersect at right angles. Find the volume of the solid common to the two cylinders. [Hint: One-eighth of the solid is sketched in the accompanying figure.]
64. In 1635 Bonaventura Cavalieri, a student of Galileo, stated the following result, called Cavalieri's principle: If two solids have the same height, and if the areas of their cross sections taken parallel to and at equal distances from their bases are always equal, then the solids have the same volume. Use this result to find the volume of the oblique cylinder in the accompanying figure. (See Exercise 52 of Section 6.1 for a planar version of Cavalieri's principle.)

$\triangle$ Figure Ex-63

$\triangle$ Figure Ex-64
65. Writing Use the results of this section to derive Cavalieri's principle (Exercise 64).
66. Writing Write a short paragraph that explains how Formulas (4)-(8) may all be viewed as consequences of Formula (3).

## QUICK CHECK ANSWERS 6.2

1. $\int_{1}^{3} 3 x^{2} d x ; 26$ 2. (a) $\pi \sin x$
(b) $\int_{0}^{\pi} \pi \sin x d x$
(c) $2 \pi$
2. (a) $0 ; 2 ; \pi\left[(2 x+1)^{2}-\left(x^{2}+1\right)^{2}\right]=\pi\left[-x^{4}+2 x^{2}+4 x\right]$
(b) $\int_{0}^{2} \pi\left[-x^{4}+2 x^{2}+4 x\right] d x$
3. (a) 1 ; $2 ; \pi\left[(y-1)-(y-1)^{2}\right]=\pi\left[-y^{2}+3 y-2\right]$
(b) $\int_{1}^{2} \pi\left[-y^{2}+3 y-2\right] d y$

### 6.3 VOLUMES BY CYLINDRICAL SHELLS

The methods for computing volumes that have been discussed so far depend on our ability to compute the cross-sectional area of the solid and to integrate that area across the solid. In this section we will develop another method for finding volumes that may be applicable when the cross-sectional area cannot be found or the integration is too difficult.

## CYLINDRICAL SHELLS

In this section we will be interested in the following problem.
6.3.1 PROBLEM Let $f$ be continuous and nonnegative on $[a, b](0 \leq a<b)$, and let $R$ be the region that is bounded above by $y=f(x)$, below by the $x$-axis, and on the sides by the lines $x=a$ and $x=b$. Find the volume $V$ of the solid of revolution $S$ that is generated by revolving the region $R$ about the $y$-axis (Figure 6.3.1).



Sometimes problems of the above type can be solved by the method of disks or washers perpendicular to the $y$-axis, but when that method is not applicable or the resulting integral is difficult, the method of cylindrical shells, which we will discuss here, will often work.

A cylindrical shell is a solid enclosed by two concentric right circular cylinders (Figure 6.3.2). The volume $V$ of a cylindrical shell with inner radius $r_{1}$, outer radius $r_{2}$, and height $h$ can be written as

$$
\begin{aligned}
V & =\text { [area of cross section }] \cdot[\text { height }] \\
& =\left(\pi r_{2}^{2}-\pi r_{1}^{2}\right) h \\
& =\pi\left(r_{2}+r_{1}\right)\left(r_{2}-r_{1}\right) h \\
& =2 \pi \cdot\left[\frac{1}{2}\left(r_{1}+r_{2}\right)\right] \cdot h \cdot\left(r_{2}-r_{1}\right)
\end{aligned}
$$

But $\frac{1}{2}\left(r_{1}+r_{2}\right)$ is the average radius of the shell and $r_{2}-r_{1}$ is its thickness, so

$$
\begin{equation*}
V=2 \pi \cdot[\text { average radius }] \cdot[\text { height }] \cdot[\text { thickness }] \tag{1}
\end{equation*}
$$

We will now show how this formula can be used to solve Problem 6.3.1. The underlying idea is to divide the interval $[a, b]$ into $n$ subintervals, thereby subdividing the region $R$ into $n$ strips, $R_{1}, R_{2}, \ldots, R_{n}$ (Figure 6.3.3a). When the region $R$ is revolved about the $y$-axis, these strips generate "tube-like" solids $S_{1}, S_{2}, \ldots, S_{n}$ that are nested one inside the other and together comprise the entire solid $S$ (Figure 6.3.3b). Thus, the volume $V$ of the solid can be obtained by adding together the volumes of the tubes; that is,

$$
V=V\left(S_{1}\right)+V\left(S_{2}\right)+\cdots+V\left(S_{n}\right)
$$

$>$ Figure 6.3.3

(a)

(b)

As a rule, the tubes will have curved upper surfaces, so there will be no simple formulas for their volumes. However, if the strips are thin, then we can approximate each strip by a rectangle (Figure 6.3.4a). These rectangles, when revolved about the $y$-axis, will produce cylindrical shells whose volumes closely approximate the volumes of the tubes generated by the original strips (Figure 6.3.4b). We will show that by adding the volumes of the cylindrical shells we can obtain a Riemann sum that approximates the volume $V$, and by taking the limit of the Riemann sums we can obtain an integral for the exact volume $V$.

(a)

(b)

$\Delta$ Figure 6.3.5

To implement this idea, suppose that the $k$ th strip extends from $x_{k-1}$ to $x_{k}$ and that the width of this strip is

$$
\Delta x_{k}=x_{k}-x_{k-1}
$$

If we let $x_{k}^{*}$ be the midpoint of the interval $\left[x_{k-1}, x_{k}\right]$, and if we construct a rectangle of height $f\left(x_{k}^{*}\right)$ over the interval, then revolving this rectangle about the $y$-axis produces a cylindrical shell of average radius $x_{k}^{*}$, height $f\left(x_{k}^{*}\right)$, and thickness $\Delta x_{k}$ (Figure 6.3.5). From (1), the volume $V_{k}$ of this cylindrical shell is

$$
V_{k}=2 \pi x_{k}^{*} f\left(x_{k}^{*}\right) \Delta x_{k}
$$

Adding the volumes of the $n$ cylindrical shells yields the following Riemann sum that approximates the volume $V$ :

$$
V \approx \sum_{k=1}^{n} 2 \pi x_{k}^{*} f\left(x_{k}^{*}\right) \Delta x_{k}
$$

Taking the limit as $n$ increases and the widths of all the subintervals approach zero yields the definite integral

$$
V=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} 2 \pi x_{k}^{*} f\left(x_{k}^{*}\right) \Delta x_{k}=\int_{a}^{b} 2 \pi x f(x) d x
$$

In summary, we have the following result.


Cutaway view of the solid
(b)
$\Delta$ Figure 6.3.6
6.3.2 VOLUME BY CYLINDRICAL SHELLS AbOUT THE $\boldsymbol{y}$-AXIS Let $f$ be continuous and nonnegative on $[a, b](0 \leq a<b)$, and let $R$ be the region that is bounded above by $y=f(x)$, below by the $x$-axis, and on the sides by the lines $x=a$ and $x=b$. Then the volume $V$ of the solid of revolution that is generated by revolving the region $R$ about the $y$-axis is given by

$$
\begin{equation*}
V=\int_{a}^{b} 2 \pi x f(x) d x \tag{2}
\end{equation*}
$$

- Example 1 Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y=\sqrt{x}, x=1, x=4$, and the $x$-axis is revolved about the $y$-axis.

Solution. First sketch the region (Figure 6.3.6a); then imagine revolving it about the $y$-axis (Figure 6.3.6b). Since $f(x)=\sqrt{x}, a=1$, and $b=4$, Formula (2) yields

$$
V=\int_{1}^{4} 2 \pi x \sqrt{x} d x=2 \pi \int_{1}^{4} x^{3 / 2} d x=\left[2 \pi \cdot \frac{2}{5} x^{5 / 2}\right]_{1}^{4}=\frac{4 \pi}{5}[32-1]=\frac{124 \pi}{5}
$$

## VARIATIONS OF THE METHOD OF CYLINDRICAL SHELLS

The method of cylindrical shells is applicable in a variety of situations that do not fit the conditions required by Formula (2). For example, the region may be enclosed between two curves, or the axis of revolution may be some line other than the $y$-axis. However, rather than develop a separate formula for every possible situation, we will give a general way of thinking about the method of cylindrical shells that can be adapted to each new situation as it arises.

For this purpose, we will need to reexamine the integrand in Formula (2): At each $x$ in the interval $[a, b]$, the vertical line segment from the $x$-axis to the curve $y=f(x)$ can be viewed as the cross section of the region $R$ at $x$ (Figure 6.3.7a). When the region $R$ is revolved about the $y$-axis, the cross section at $x$ sweeps out the surface of a right circular cylinder of height $f(x)$ and radius $x$ (Figure 6.3.7b). The area of this surface is

$$
2 \pi x f(x)
$$

(Figure $6.3 .7 c$ ), which is the integrand in (2). Thus, Formula (2) can be viewed informally in the following way.
6.3.3 AN INFORMAL VIEWPOINT ABOUT CYLINDRICAL SHELLS The volume $V$ of a solid of revolution that is generated by revolving a region $R$ about an axis can be obtained by integrating the area of the surface generated by an arbitrary cross section of $R$ taken parallel to the axis of revolution.

(a)

(b)

(c)

The following examples illustrate how to apply this result in situations where Formula (2) is not applicable.

- Example 2 Use cylindrical shells to find the volume of the solid generated when the region $R$ in the first quadrant enclosed between $y=x$ and $y=x^{2}$ is revolved about the $y$-axis (Figure 6.3.8a).

Solution. As illustrated in part (b) of Figure 6.3.8, at each $x$ in $[0,1]$ the cross section of $R$ parallel to the $y$-axis generates a cylindrical surface of height $x-x^{2}$ and radius $x$. Since the area of this surface is

$$
2 \pi x\left(x-x^{2}\right)
$$

the volume of the solid is

$$
V=\int_{0}^{1} 2 \pi x\left(x-x^{2}\right) d x=2 \pi \int_{0}^{1}\left(x^{2}-x^{3}\right) d x
$$

$$
=2 \pi\left[\frac{x^{3}}{3}-\frac{x^{4}}{4}\right]_{0}^{1}=2 \pi\left[\frac{1}{3}-\frac{1}{4}\right]=\frac{\pi}{6}
$$


$\Delta$ Figure 6.3.8

Note that the volume found in Example 3 agrees with the volume of the same solid found by the method of washers in Example 6 of Section 6.2. Confirm that the volume in Example 2 found by the method of cylindrical shells can also be obtained by the method of washers.

Example 3 Use cylindrical shells to find the volume of the solid generated when the region $R$ under $y=x^{2}$ over the interval [0,2] is revolved about the line $y=-1$.

Solution. First draw the axis of revolution; then imagine revolving the region about the axis (Figure 6.3.9a). As illustrated in Figure 6.3.9b, at each $y$ in the interval $0 \leq y \leq 4$, the cross section of $R$ parallel to the $x$-axis generates a cylindrical surface of height $2-\sqrt{y}$ and radius $y+1$. Since the area of this surface is

$$
2 \pi(y+1)(2-\sqrt{y})
$$

it follows that the volume of the solid is

$$
\begin{aligned}
\int_{0}^{4} 2 \pi(y+1)(2-\sqrt{y}) d y & =2 \pi \int_{0}^{4}\left(2 y-y^{3 / 2}+2-y^{1 / 2}\right) d y \\
& =2 \pi\left[y^{2}-\frac{2}{5} y^{5 / 2}+2 y-\frac{2}{3} y^{3 / 2}\right]_{0}^{4}=\frac{176 \pi}{15}
\end{aligned}
$$


(a)

(b)

Figure 6.3.9

## QUICK CHECK EXERCISES 6.3 (See page 438 for answers.)

1. Let $R$ be the region between the $x$-axis and the curve $y=1+\sqrt{x}$ for $1 \leq x \leq 4$.
(a) For $x$ between 1 and 4, the area of the cylindrical surface generated by revolving the vertical cross section of $R$ at $x$ about the $y$-axis is $\qquad$
(b) Using cylindrical shells, an integral expression for the volume of the solid generated by revolving $R$ about the $y$-axis is $\qquad$ —.
2. Let $R$ be the region described in Quick Check Exercise 1. (a) For $x$ between 1 and 4, the area of the cylindrical sur-
face generated by revolving the vertical cross section of $R$ at $x$ about the line $x=5$ is $\qquad$ _.
(b) Using cylindrical shells, an integral expression for the volume of the solid generated by revolving $R$ about the line $x=5$ is $\qquad$
3. A solid $S$ is generated by revolving the region enclosed by the curves $x=(y-2)^{2}$ and $x=4$ about the $x$-axis. Using cylindrical shells, an integral expression for the volume of $S$ is $\qquad$ —.

## EXERCISE SET 6.3 C CAS

1-4 Use cylindrical shells to find the volume of the solid generated when the shaded region is revolved about the indicated axis.
1.

3.

2.

4.


5-12 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the $y$-axis.
5. $y=x^{3}, x=1, y=0$
6. $y=\sqrt{x}, x=4, x=9, y=0$
7. $y=1 / x, y=0, x=1, x=3$
8. $y=\cos \left(x^{2}\right), x=0, x=\frac{1}{2} \sqrt{\pi}, y=0$
9. $y=2 x-1, y=-2 x+3, x=2$
10. $y=2 x-x^{2}, y=0$
11. $y=\frac{1}{x^{2}+1}, x=0, x=1, y=0$
12. $y=e^{x^{2}}, x=1, x=\sqrt{3}, y=0$

13-16 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the $x$-axis.
13. $y^{2}=x, y=1, x=0$
14. $x=2 y, y=2, y=3, x=0$
15. $y=x^{2}, x=1, y=0$
16. $x y=4, x+y=5$

17-20 True-False Determine whether the statement is true or false. Explain your answer.
17. The volume of a cylindrical shell is equal to the product of the thickness of the shell with the surface area of a cylinder whose height is that of the shell and whose radius is equal to the average of the inner and outer radii of the shell.
18. The method of cylindrical shells is a special case of the method of integration of cross-sectional area that was discussed in Section 6.2.
19. In the method of cylindrical shells, integration is over an interval on a coordinate axis that is perpendicular to the axis of revolution of the solid.
20. The Riemann sum approximation

$$
V \approx \sum_{k=1}^{n} 2 \pi x_{k}^{*} f\left(x_{k}^{*}\right) \Delta x_{k} \quad\left(\text { where } x_{k}^{*}=\frac{x_{k}+x_{k-1}}{2}\right)
$$

for the volume of a solid of revolution is exact when $f$ is a constant function.
C 21. Use a CAS to find the volume of the solid generated when the region enclosed by $y=e^{x}$ and $y=0$ for $1 \leq x \leq 2$ is revolved about the $y$-axis.
(C) 22. Use a CAS to find the volume of the solid generated when the region enclosed by $y=\cos x, y=0$, and $x=0$ for $0 \leq x \leq \pi / 2$ is revolved about the $y$-axis.
[C 23. Consider the region to the right of the $y$-axis, to the left of the vertical line $x=k(0<k<\pi)$, and between the curve $y=\sin x$ and the $x$-axis. Use a CAS to estimate the value of $k$ so that the solid generated by revolving the region about the $y$-axis has a volume of 8 cubic units.

## FOCUS ON CONCEPTS

24. Let $R_{1}$ and $R_{2}$ be regions of the form shown in the accompanying figure. Use cylindrical shells to find a formula for the volume of the solid that results when
(a) region $R_{1}$ is revolved about the $y$-axis
(b) region $R_{2}$ is revolved about the $x$-axis.


$\Delta$ Figure Ex-24
25. (a) Use cylindrical shells to find the volume of the solid that is generated when the region under the curve

$$
y=x^{3}-3 x^{2}+2 x
$$

over $[0,1]$ is revolved about the $y$-axis.
(b) For this problem, is the method of cylindrical shells easier or harder than the method of slicing discussed in the last section? Explain.
26. Let $f$ be continuous and nonnegative on $[a, b]$, and let $R$ be the region that is enclosed by $y=f(x)$ and $y=0$ for $a \leq x \leq b$. Using the method of cylindrical shells, derive with explanation a formula for the volume of the solid generated by revolving $R$ about the line $x=k$, where $k \leq a$.

27-28 Using the method of cylindrical shells, set up but do not evaluate an integral for the volume of the solid generated when the region $R$ is revolved about (a) the line $x=1$ and (b) the line $y=-1$.
27. $R$ is the region bounded by the graphs of $y=x, y=0$, and $x=1$.
28. $R$ is the region in the first quadrant bounded by the graphs of $y=\sqrt{1-x^{2}}, y=0$, and $x=0$.
29. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y=1 / x^{3}$, $x=1, x=2, y=0$ is revolved about the line $x=-1$.
30. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y=x^{3}$, $y=1, x=0$ is revolved about the line $y=1$.
31. Use cylindrical shells to find the volume of the cone generated when the triangle with vertices $(0,0),(0, r),(h, 0)$, where $r>0$ and $h>0$, is revolved about the $x$-axis.
32. The region enclosed between the curve $y^{2}=k x$ and the line $x=\frac{1}{4} k$ is revolved about the line $x=\frac{1}{2} k$. Use cylindrical shells to find the volume of the resulting solid. (Assume $k>0$.)
33. As shown in the accompanying figure, a cylindrical hole is drilled all the way through the center of a sphere. Show that the volume of the remaining solid depends only on the length $L$ of the hole, not on the size of the sphere.


4Figure Ex-33
34. Use cylindrical shells to find the volume of the torus obtained by revolving the circle $x^{2}+y^{2}=a^{2}$ about the line
$x=b$, where $b>a>0$. [Hint: It may help in the integration to think of an integral as an area.]
35. Let $V_{x}$ and $V_{y}$ be the volumes of the solids that result when the region enclosed by $y=1 / x, y=0, x=\frac{1}{2}$, and $x=b$ $\left(b>\frac{1}{2}\right)$ is revolved about the $x$-axis and $y$-axis, respectively. Is there a value of $b$ for which $V_{x}=V_{y}$ ?
36. (a) Find the volume $V$ of the solid generated when the region bounded by $y=1 /\left(1+x^{4}\right), y=0, x=1$, and $x=b(b>1)$ is revolved about the $y$-axis.
(b) Find $\lim _{b \rightarrow+\infty} V$.
37. Writing Faced with the problem of computing the volume of a solid of revolution, how would you go about deciding whether to use the method of disks/washers or the method of cylindrical shells?
38. Writing With both the method of disks/washers and with the method of cylindrical shells, we integrate an "area" to get the volume of a solid of revolution. However, these two approaches differ in very significant ways. Write a brief paragraph that discusses these differences.

## QUICK CHECK ANSWERS 6.3

1. (a) $2 \pi x(1+\sqrt{x})$
(b) $\int_{1}^{4} 2 \pi x(1+\sqrt{x}) d x$
2. (a) $2 \pi(5-x)(1+\sqrt{x})$
(b) $\int_{1}^{4} 2 \pi(5-x)(1+\sqrt{x}) d x$
3. $\int_{0}^{4} 2 \pi y\left[4-(y-2)^{2}\right] d y$

### 6.4 LENGTH OF A PLANE CURVE


$\Delta$ Figure 6.4.1

Intuitively, you might think of the arc length of a curve as the number obtained by aligning a piece of string with the curve and then measuring the length of the string after it is straightened out.

In this section we will use the tools of calculus to study the problem of finding the length of a plane curve.

## ARC LENGTH

Our first objective is to define what we mean by the length (also called the arc length) of a plane curve $y=f(x)$ over an interval $[a, b]$ (Figure 6.4.1). Once that is done we will be able to focus on the problem of computing arc lengths. To avoid some complications that would otherwise occur, we will impose the requirement that $f^{\prime}$ be continuous on $[a, b]$, in which case we will say that $y=f(x)$ is a smooth curve on $[a, b]$ or that $f$ is a smooth function on $[a, b]$. Thus, we will be concerned with the following problem.
6.4.1 ARC LENGTH PROblem Suppose that $y=f(x)$ is a smooth curve on the interval $[a, b]$. Define and find a formula for the arc length $L$ of the curve $y=f(x)$ over the interval $[a, b]$.

To define the arc length of a curve we start by breaking the curve into small segments. Then we approximate the curve segments by line segments and add the lengths of the line segments to form a Riemann sum. Figure 6.4.2 illustrates how such line segments tend to become better and better approximations to a curve as the number of segments increases. As the number of segments increases, the corresponding Riemann sums approach a definite integral whose value we will take to be the arc length $L$ of the curve.

To implement our idea for solving Problem 6.4.1, divide the interval $[a, b]$ into $n$ subintervals by inserting points $x_{1}, x_{2}, \ldots, x_{n-1}$ between $a=x_{0}$ and $b=x_{n}$. As shown in Figure 6.4.3a, let $P_{0}, P_{1}, \ldots, P_{n}$ be the points on the curve with $x$-coordinates $a=x_{0}$,



Figure 6.4.3
(a)
(b)
$x_{1}, x_{2}, \ldots, x_{n-1}, b=x_{n}$ and join these points with straight line segments. These line segments form a polygonal path that we can regard as an approximation to the curve $y=f(x)$. As indicated in Figure 6.4.3b, the length $L_{k}$ of the $k$ th line segment in the polygonal path is

$$
\begin{equation*}
L_{k}=\sqrt{\left(\Delta x_{k}\right)^{2}+\left(\Delta y_{k}\right)^{2}}=\sqrt{\left(\Delta x_{k}\right)^{2}+\left[f\left(x_{k}\right)-f\left(x_{k-1}\right)\right]^{2}} \tag{1}
\end{equation*}
$$

If we now add the lengths of these line segments, we obtain the following approximation to the length $L$ of the curve

$$
\begin{equation*}
L \approx \sum_{k=1}^{n} L_{k}=\sum_{k=1}^{n} \sqrt{\left(\Delta x_{k}\right)^{2}+\left[f\left(x_{k}\right)-f\left(x_{k-1}\right)\right]^{2}} \tag{2}
\end{equation*}
$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a point $x_{k}^{*}$ between $x_{k-1}$ and $x_{k}$ such that

$$
\frac{f\left(x_{k}\right)-f\left(x_{k-1}\right)}{x_{k}-x_{k-1}}=f^{\prime}\left(x_{k}^{*}\right) \quad \text { or } \quad f\left(x_{k}\right)-f\left(x_{k-1}\right)=f^{\prime}\left(x_{k}^{*}\right) \Delta x_{k}
$$

and hence we can rewrite (2) as

$$
L \approx \sum_{k=1}^{n} \sqrt{\left(\Delta x_{k}\right)^{2}+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}\left(\Delta x_{k}\right)^{2}}=\sum_{k=1}^{n} \sqrt{1+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}} \Delta x_{k}
$$

Thus, taking the limit as $n$ increases and the widths of all the subintervals approach zero yields the following integral that defines the arc length $L$ :

$$
L=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} \sqrt{1+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}} \Delta x_{k}=\int_{a}^{b} \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x
$$

In summary, we have the following definition.
6.4.2 DEFINITION If $y=f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length $L$ of this curve over $[a, b]$ is defined as

$$
\begin{equation*}
L=\int_{a}^{b} \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x \tag{3}
\end{equation*}
$$

This result provides both a definition and a formula for computing arc lengths. Where convenient, (3) can also be expressed as

$$
\begin{equation*}
L=\int_{a}^{b} \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x=\int_{a}^{b} \sqrt{1+\left(\frac{d y}{d x}\right)^{2}} d x \tag{4}
\end{equation*}
$$

Moreover, for a curve expressed in the form $x=g(y)$, where $g^{\prime}$ is continuous on $[c, d]$, the arc length $L$ from $y=c$ to $y=d$ can be expressed as

$$
\begin{equation*}
L=\int_{c}^{d} \sqrt{1+\left[g^{\prime}(y)\right]^{2}} d y=\int_{c}^{d} \sqrt{1+\left(\frac{d x}{d y}\right)^{2}} d y \tag{5}
\end{equation*}
$$

- Example 1 Find the arc length of the curve $y=x^{3 / 2}$ from $(1,1)$ to $(2,2 \sqrt{2})$ (Figure 6.4.4) in two ways: (a) using Formula (4) and (b) using Formula (5).

Solution (a).

$$
\frac{d y}{d x}=\frac{3}{2} x^{1 / 2}
$$

and since the curve extends from $x=1$ to $x=2$, it follows from (4) that

$$
L=\int_{1}^{2} \sqrt{1+\left(\frac{3}{2} x^{1 / 2}\right)^{2}} d x=\int_{1}^{2} \sqrt{1+\frac{9}{4} x} d x
$$

To evaluate this integral we make the $u$-substitution

$$
u=1+\frac{9}{4} x, \quad d u=\frac{9}{4} d x
$$

and then change the $x$-limits of integration $(x=1, x=2)$ to the corresponding $u$-limits $\left(u=\frac{13}{4}, u=\frac{22}{4}\right)$ :

$$
\begin{aligned}
\left.L=\frac{4}{9} \int_{13 / 4}^{22 / 4} u^{1 / 2} d u=\frac{8}{27} u^{3 / 2}\right]_{13 / 4}^{22 / 4} & =\frac{8}{27}\left[\left(\frac{22}{4}\right)^{3 / 2}-\left(\frac{13}{4}\right)^{3 / 2}\right] \\
& =\frac{22 \sqrt{22}-13 \sqrt{13}}{27} \approx 2.09
\end{aligned}
$$

Solution (b). To apply Formula (5) we must first rewrite the equation $y=x^{3 / 2}$ so that $x$ is expressed as a function of $y$. This yields $x=y^{2 / 3}$ and

$$
\frac{d x}{d y}=\frac{2}{3} y^{-1 / 3}
$$

Since the curve extends from $y=1$ to $y=2 \sqrt{2}$, it follows from (5) that

$$
L=\int_{1}^{2 \sqrt{2}} \sqrt{1+\frac{4}{9} y^{-2 / 3}} d y=\frac{1}{3} \int_{1}^{2 \sqrt{2}} y^{-1 / 3} \sqrt{9 y^{2 / 3}+4} d y
$$

The arc from the point $(1,1)$ to the point $(2,2 \sqrt{2})$ in Figure 6.4.4 is nearly a straight line, so the arc length should be only slightly larger than the straightline distance between these points. Show that this is so.

## TECHNOLOGY MASTERY

If your calculating utility has a numerical integration capability, use it to confirm that the arc length $L$ in Example 2 is approximately $L \approx 3.8202$.

To evaluate this integral we make the $u$-substitution

$$
u=9 y^{2 / 3}+4, \quad d u=6 y^{-1 / 3} d y
$$

and change the $y$-limits of integration $(y=1, y=2 \sqrt{2})$ to the corresponding $u$-limits ( $u=13, u=22$ ). This gives

$$
\left.L=\frac{1}{18} \int_{13}^{22} u^{1 / 2} d u=\frac{1}{27} u^{3 / 2}\right]_{13}^{22}=\frac{1}{27}\left[(22)^{3 / 2}-(13)^{3 / 2}\right]=\frac{22 \sqrt{22}-13 \sqrt{13}}{27}
$$

The answer in part (b) agrees with that in part (a); however, the integration in part (b) is more tedious. In problems where there is a choice between using (4) or (5), it is often the case that one of the formulas leads to a simpler integral than the other.

## FINDING ARC LENGTH BY NUMERICAL METHODS

In the next chapter we will develop some techniques of integration that will enable us to find exact values of more integrals encountered in arc length calculations; however, generally speaking, most such integrals are impossible to evaluate in terms of elementary functions. In these cases one usually approximates the integral using a numerical method such as the midpoint rule discussed in Section 5.4.

- Example 2 From (4), the arc length of $y=\sin x$ from $x=0$ to $x=\pi$ is given by the integral

$$
L=\int_{0}^{\pi} \sqrt{1+(\cos x)^{2}} d x
$$

This integral cannot be evaluated in terms of elementary functions; however, using a calculating utility with a numerical integration capability yields the approximation $L \approx 3.8202$.

## QUICK CHECK EXERCISES 6.4 (See page 443 for answers.)

1. A function $f$ is smooth on $[a, b]$ if $f^{\prime}$ is $\qquad$ on $[a, b]$.
2. If a function $f$ is smooth on $[a, b]$, then the length of the curve $y=f(x)$ over $[a, b]$ is $\qquad$ —.
3. The distance between points $(1,0)$ and $(e, 1)$ is $\qquad$
4. Let $L$ be the length of the curve $y=\ln x$ from $(1,0)$ to $(e, 1)$.
(a) Integrating with respect to $x$, an integral expression for $L$ is $\qquad$
(b) Integrating with respect to $y$, an integral expression for $L$ is $\qquad$ —.

## EXERCISE SET 6.4 C CAS

1. Use the Theorem of Pythagoras to find the length of the line segment $y=2 x$ from $(1,2)$ to $(2,4)$, and confirm that the value is consistent with the length computed using
(a) Formula (4)
(b) Formula (5).
2. Use the Theorem of Pythagoras to find the length of the line segment $y=5 x$ from $(0,0)$ and $(1,5)$, and confirm that the value is consistent with the length computed using
(a) Formula (4)
(b) Formula (5).

3-8 Find the exact arc length of the curve over the interval.
3. $y=3 x^{3 / 2}-1$ from $x=0$ to $x=1$
4. $x=\frac{1}{3}\left(y^{2}+2\right)^{3 / 2}$ from $y=0$ to $y=1$
5. $y=x^{2 / 3}$ from $x=1$ to $x=8$
6. $y=\left(x^{6}+8\right) /\left(16 x^{2}\right)$ from $x=2$ to $x=3$
7. $24 x y=y^{4}+48$ from $y=2$ to $y=4$
8. $x=\frac{1}{8} y^{4}+\frac{1}{4} y^{-2}$ from $y=1$ to $y=4$

9-12 True-False Determine whether the statement is true or false. Explain your answer.
9. The graph of $y=\sqrt{1-x^{2}}$ is a smooth curve on $[-1,1]$.

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10. The approximation

$$
L \approx \sum_{k=1}^{n} \sqrt{\left(\Delta x_{k}\right)^{2}+\left[f\left(x_{k}\right)-f\left(x_{k-1}\right)\right]^{2}}
$$

for arc length is not expressed in the form of a Riemann sum.
11. The approximation

$$
L \approx \sum_{k=1}^{n} \sqrt{1+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}} \Delta x_{k}
$$

for arc length is exact when $f$ is a linear function of $x$.
12. In our definition of the arc length for the graph of $y=f(x)$, we need $f^{\prime}(x)$ to be a continuous function in order for $f$ to satisfy the hypotheses of the Mean-Value Theorem (4.8.2).

C 13-14 Express the exact arc length of the curve over the given interval as an integral that has been simplified to eliminate the radical, and then evaluate the integral using a CAS.
13. $y=\ln (\sec x)$ from $x=0$ to $x=\pi / 4$
14. $y=\ln (\sin x)$ from $x=\pi / 4$ to $x=\pi / 2$

## FOCUS ON CONCEPTS

15. Consider the curve $y=x^{2 / 3}$.
(a) Sketch the portion of the curve between $x=-1$ and $x=8$.
(b) Explain why Formula (4) cannot be used to find the arc length of the curve sketched in part (a).
(c) Find the arc length of the curve sketched in part (a).
16. The curve segment $y=x^{2}$ from $x=1$ to $x=2$ may also be expressed as the graph of $x=\sqrt{y}$ from $y=1$ to $y=4$. Set up two integrals that give the arc length of this curve segment, one by integrating with respect to $x$, and the other by integrating with respect to $y$. Demonstrate a substitution that verifies that these two integrals are equal.
17. Consider the curve segments $y=x^{2}$ from $x=\frac{1}{2}$ to $x=2$ and $y=\sqrt{x}$ from $x=\frac{1}{4}$ to $x=4$.
(a) Graph the two curve segments and use your graphs to explain why the lengths of these two curve segments should be equal.
(b) Set up integrals that give the arc lengths of the curve segments by integrating with respect to $x$. Demonstrate a substitution that verifies that these two integrals are equal.
(c) Set up integrals that give the arc lengths of the curve segments by integrating with respect to $y$.
(d) Approximate the arc length of each curve segment using Formula (2) with $n=10$ equal subintervals.
(e) Which of the two approximations in part (d) is more accurate? Explain.
(f) Use the midpoint approximation with $n=10$ subintervals to approximate each arc length integral in part (b).
(g) Use a calculating utility with numerical integration capabilities to approximate the arc length integrals in part (b) to four decimal places.
18. Follow the directions of Exercise 17 for the curve segments $y=x^{8 / 3}$ from $x=10^{-3}$ to $x=1$ and $y=x^{3 / 8}$ from $x=10^{-8}$ to $x=1$.
19. Follow the directions of Exercise 17 for the curve segment $y=\tan x$ from $x=0$ to $x=\pi / 3$ and for the curve segment $y=\tan ^{-1} x$ from $x=0$ to $x=\sqrt{3}$.
20. Let $y=f(x)$ be a smooth curve on the closed interval $[a, b]$. Prove that if $m$ and $M$ are nonnegative numbers such that $m \leq\left|f^{\prime}(x)\right| \leq M$ for all $x$ in $[a, b]$, then the arc length $L$ of $y=f(x)$ over the interval $[a, b]$ satisfies the inequalities

$$
(b-a) \sqrt{1+m^{2}} \leq L \leq(b-a) \sqrt{1+M^{2}}
$$

21. Use the result of Exercise 20 to show that the arc length $L$ of $y=\sec x$ over the interval $0 \leq x \leq \pi / 3$ satisfies

$$
\frac{\pi}{3} \leq L \leq \frac{\pi}{3} \sqrt{13}
$$

C 22. A basketball player makes a successful shot from the free throw line. Suppose that the path of the ball from the moment of release to the moment it enters the hoop is described by

$$
y=2.15+2.09 x-0.41 x^{2}, \quad 0 \leq x \leq 4.6
$$

where $x$ is the horizontal distance (in meters) from the point of release, and $y$ is the vertical distance (in meters) above the floor. Use a CAS or a scientific calculator with a numerical integration capability to approximate the distance the ball travels from the moment it is released to the moment it enters the hoop. Round your answer to two decimal places.
C 23. Find a positive value of $k$ (to two decimal places) such that the curve $y=k \sin x$ has an arc length of $L=5$ units over the interval from $x=0$ to $x=\pi$. [Hint: Find an integral for the arc length $L$ in terms of $k$, and then use a CAS or a scientific calculator with a numerical integration capability to find integer values of $k$ at which the values of $L-5$ have opposite signs. Complete the solution by using the Intermediate-Value Theorem (1.5.7) to approximate the value of $k$ to two decimal places.]
C 24. As shown in the accompanying figure on the next page, a horizontal beam with dimensions 2 in $\times 6$ in $\times 16 \mathrm{ft}$ is fixed at both ends and is subjected to a uniformly distributed load of $120 \mathrm{lb} / \mathrm{ft}$. As a result of the load, the centerline of the beam undergoes a deflection that is described by

$$
y=-1.67 \times 10^{-8}\left(x^{4}-2 L x^{3}+L^{2} x^{2}\right)
$$

( $0 \leq x \leq 192$ ), where $L=192$ in is the length of the unloaded beam, $x$ is the horizontal distance along the beam measured in inches from the left end, and $y$ is the deflection of the centerline in inches.
(a) Graph $y$ versus $x$ for $0 \leq x \leq 192$.
(b) Find the maximum deflection of the centerline.
(c) Use a CAS or a calculator with a numerical integration capability to find the length of the centerline of the loaded beam. Round your answer to two decimal places.

25. A golfer makes a successful chip shot to the green. Suppose that the path of the ball from the moment it is struck to the moment it hits the green is described by

$$
y=12.54 x-0.41 x^{2}
$$

where $x$ is the horizontal distance (in yards) from the point where the ball is struck, and $y$ is the vertical distance (in yards) above the fairway. Use a CAS or a calculating utility with a numerical integration capability to find the distance the ball travels from the moment it is struck to the moment it hits the green. Assume that the fairway and green are at the same level and round your answer to two decimal places.

26-34 These exercises assume familiarity with the basic concepts of parametric curves. If needed, an introduction to this material is provided in Web Appendix I.
C 26. Assume that no segment of the curve

$$
x=x(t), \quad y=y(t), \quad(a \leq t \leq b)
$$

is traced more than once as $t$ increases from $a$ to $b$. Divide the interval $[a, b]$ into $n$ subintervals by inserting points $t_{1}, t_{2}, \ldots, t_{n-1}$ between $a=t_{0}$ and $b=t_{n}$. Let $L$ denote the arc length of the curve. Give an informal argument for the approximation

$$
L \approx \sum_{k=1}^{n} \sqrt{\left[x\left(t_{k}\right)-x\left(t_{k-1}\right)\right]^{2}+\left[y\left(t_{k}\right)-y\left(t_{k-1}\right)\right]^{2}}
$$

If $d x / d t$ and $d y / d t$ are continuous functions for $a \leq t \leq b$, then it can be shown that as $\max \Delta t_{k} \rightarrow 0$, this sum converges to

$$
L=\int_{a}^{b} \sqrt{\left(\frac{d x}{d t}\right)^{2}+\left(\frac{d y}{d t}\right)^{2}} d t
$$

27-32 Use the arc length formula from Exercise 26 to find the arc length of the curve.
27. $x=\frac{1}{3} t^{3}, \quad y=\frac{1}{2} t^{2} \quad(0 \leq t \leq 1)$
28. $x=(1+t)^{2}, \quad y=(1+t)^{3} \quad(0 \leq t \leq 1)$
29. $x=\cos 2 t, \quad y=\sin 2 t \quad(0 \leq t \leq \pi / 2)$
30. $x=\cos t+t \sin t, \quad y=\sin t-t \cos t \quad(0 \leq t \leq \pi)$
31. $x=e^{t} \cos t, \quad y=e^{t} \sin t \quad(0 \leq t \leq \pi / 2)$
32. $x=e^{t}(\sin t+\cos t), \quad y=e^{t}(\cos t-\sin t) \quad(1 \leq t \leq 4)$

C
33. (a) Show that the total arc length of the ellipse

$$
x=2 \cos t, \quad y=\sin t \quad(0 \leq t \leq 2 \pi)
$$

is given by

$$
4 \int_{0}^{\pi / 2} \sqrt{1+3 \sin ^{2} t} d t
$$

(b) Use a CAS or a scientific calculator with a numerical integration capability to approximate the arc length in part (a). Round your answer to two decimal places.
(c) Suppose that the parametric equations in part (a) describe the path of a particle moving in the $x y$-plane, where $t$ is time in seconds and $x$ and $y$ are in centimeters. Use a CAS or a scientific calculator with a numerical integration capability to approximate the distance traveled by the particle from $t=1.5 \mathrm{~s}$ to $t=4.8 \mathrm{~s}$. Round your answer to two decimal places.
34. Show that the total arc length of the ellipse $x=a \cos t$, $y=b \sin t, 0 \leq t \leq 2 \pi$ for $a>b>0$ is given by

$$
4 a \int_{0}^{\pi / 2} \sqrt{1-k^{2} \cos ^{2} t} d t
$$

where $k=\sqrt{a^{2}-b^{2}} / a$.
35. Writing In our discussion of Arc Length Problem 6.4.1, we derived the approximation

$$
L \approx \sum_{k=1}^{n} \sqrt{1+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}} \Delta x_{k}
$$

Discuss the geometric meaning of this approximation. (Be sure to address the appearance of the derivative $f^{\prime}$.)
36. Writing Give examples in which Formula (4) for arc length cannot be applied directly, and describe how you would go about finding the arc length of the curve in each case. (Discuss both the use of alternative formulas and the use of numerical methods.)

## QUICK CHECK ANSWERS 6.4

1. continuous
2. $\int_{a}^{b} \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x \quad$ 3. $\sqrt{(e-1)^{2}+1}$
3. (a) $\int_{1}^{e} \sqrt{1+(1 / x)^{2}} d x$ (b) $\int_{0}^{1} \sqrt{1+e^{2 y}} d y$

### 6.5 AREA OF A SURFACE OF REVOLUTION

In this section we will consider the problem of finding the area of a surface that is generated by revolving a plane curve about a line.

## SURFACE AREA

A surface of revolution is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter, and the lateral surface of a right circular cylinder can be generated by revolving a line segment about an axis that is parallel to it (Figure 6.5.1).

Some Surfaces of Revolution



$\triangle$ Figure 6.5.2
6.5.1 SURFACE AREA PROBLEM Suppose that $f$ is a smooth, nonnegative function on $[a, b]$ and that a surface of revolution is generated by revolving the portion of the curve $y=f(x)$ between $x=a$ and $x=b$ about the $x$-axis (Figure 6.5.2). Define what is meant by the area $S$ of the surface, and find a formula for computing it.

To motivate an appropriate definition for the area $S$ of a surface of revolution, we will decompose the surface into small sections whose areas can be approximated by elementary formulas, add the approximations of the areas of the sections to form a Riemann sum that approximates $S$, and then take the limit of the Riemann sums to obtain an integral for the exact value of $S$.

To implement this idea, divide the interval $[a, b]$ into $n$ subintervals by inserting points $x_{1}$, $x_{2}, \ldots, x_{n-1}$ between $a=x_{0}$ and $b=x_{n}$. As illustrated in Figure 6.5.3 $a$, the corresponding points on the graph of $f$ define a polygonal path that approximates the curve $y=f(x)$ over the interval $[a, b]$. As illustrated in Figure $6.5 .3 b$, when this polygonal path is revolved about the $x$-axis, it generates a surface consisting of $n$ parts, each of which is a portion of a right circular cone called a frustum (from the Latin meaning "bit" or "piece"). Thus, the area of each part of the approximating surface can be obtained from the formula

$$
\begin{equation*}
S=\pi\left(r_{1}+r_{2}\right) l \tag{1}
\end{equation*}
$$

for the lateral area $S$ of a frustum of slant height $l$ and base radii $r_{1}$ and $r_{2}$ (Figure 6.5.4). As suggested by Figure 6.5.5, the $k$ th frustum has radii $f\left(x_{k-1}\right)$ and $f\left(x_{k}\right)$ and height $\Delta x_{k}$. Its slant height is the length $L_{k}$ of the $k$ th line segment in the polygonal path, which from Formula (1) of Section 6.4 is

$$
L_{k}=\sqrt{\left(\Delta x_{k}\right)^{2}+\left[f\left(x_{k}\right)-f\left(x_{k-1}\right)\right]^{2}}
$$


(a)
(b)

$\Delta$ Figure 6.5.3


Frustum
$\Delta$ Figure 6.5.4

$\Delta$ Figure 6.5.5

This makes the lateral area $S_{k}$ of the $k$ th frustum

$$
S_{k}=\pi\left[f\left(x_{k-1}\right)+f\left(x_{k}\right)\right] \sqrt{\left(\Delta x_{k}\right)^{2}+\left[f\left(x_{k}\right)-f\left(x_{k-1}\right)\right]^{2}}
$$

If we add these areas, we obtain the following approximation to the area $S$ of the entire surface:

$$
\begin{equation*}
S \approx \sum_{k=1}^{n} \pi\left[f\left(x_{k-1}\right)+f\left(x_{k}\right)\right] \sqrt{\left(\Delta x_{k}\right)^{2}+\left[f\left(x_{k}\right)-f\left(x_{k-1}\right)\right]^{2}} \tag{2}
\end{equation*}
$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a point $x_{k}^{*}$ between $x_{k-1}$ and $x_{k}$ such that

$$
\frac{f\left(x_{k}\right)-f\left(x_{k-1}\right)}{x_{k}-x_{k-1}}=f^{\prime}\left(x_{k}^{*}\right) \quad \text { or } \quad f\left(x_{k}\right)-f\left(x_{k-1}\right)=f^{\prime}\left(x_{k}^{*}\right) \Delta x_{k}
$$

and hence we can rewrite (2) as

$$
\begin{align*}
S & \approx \sum_{k=1}^{n} \pi\left[f\left(x_{k-1}\right)+f\left(x_{k}\right)\right] \sqrt{\left(\Delta x_{k}\right)^{2}+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}\left(\Delta x_{k}\right)^{2}} \\
& =\sum_{k=1}^{n} \pi\left[f\left(x_{k-1}\right)+f\left(x_{k}\right)\right] \sqrt{1+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}} \Delta x_{k} \tag{3}
\end{align*}
$$

However, this is not yet a Riemann sum because it involves the variables $x_{k-1}$ and $x_{k}$. To eliminate these variables from the expression, observe that the average value of the numbers $f\left(x_{k-1}\right)$ and $f\left(x_{k}\right)$ lies between these numbers, so the continuity of $f$ and the Intermediate-Value Theorem (1.5.7) imply that there is a point $x_{k}^{* *}$ between $x_{k-1}$ and $x_{k}$ such that

$$
\frac{1}{2}\left[f\left(x_{k-1}\right)+f\left(x_{k}\right)\right]=f\left(x_{k}^{* *}\right)
$$

Thus, (2) can be expressed as

$$
S \approx \sum_{k=1}^{n} 2 \pi f\left(x_{k}^{* *}\right) \sqrt{1+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}} \Delta x_{k}
$$

Although this expression is close to a Riemann sum in form, it is not a true Riemann sum because it involves two variables $x_{k}^{*}$ and $x_{k}^{* *}$, rather than $x_{k}^{*}$ alone. However, it is proved in advanced calculus courses that this has no effect on the limit because of the continuity of $f$. Thus, we can assume that $x_{k}^{* *}=x_{k}^{*}$ when taking the limit, and this suggests that $S$ can be defined as

$$
S=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} 2 \pi f\left(x_{k}^{*}\right) \sqrt{1+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}} \Delta x_{k}=\int_{a}^{b} 2 \pi f(x) \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x
$$

In summary, we have the following definition.
6.5.2 DEFINITION If $f$ is a smooth, nonnegative function on $[a, b]$, then the surface area $S$ of the surface of revolution that is generated by revolving the portion of the curve $y=f(x)$ between $x=a$ and $x=b$ about the $x$-axis is defined as

$$
S=\int_{a}^{b} 2 \pi f(x) \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x
$$

This result provides both a definition and a formula for computing surface areas. Where convenient, this formula can also be expressed as

$$
\begin{equation*}
S=\int_{a}^{b} 2 \pi f(x) \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x=\int_{a}^{b} 2 \pi y \sqrt{1+\left(\frac{d y}{d x}\right)^{2}} d x \tag{4}
\end{equation*}
$$

Moreover, if $g$ is nonnegative and $x=g(y)$ is a smooth curve on the interval $[c, d]$, then the area of the surface that is generated by revolving the portion of a curve $x=g(y)$ between $y=c$ and $y=d$ about the $y$-axis can be expressed as

$$
\begin{equation*}
S=\int_{c}^{d} 2 \pi g(y) \sqrt{1+\left[g^{\prime}(y)\right]^{2}} d y=\int_{c}^{d} 2 \pi x \sqrt{1+\left(\frac{d x}{d y}\right)^{2}} d y \tag{5}
\end{equation*}
$$


$\Delta$ Figure 6.5.6

© Figure 6.5.7

- Example 1 Find the area of the surface that is generated by revolving the portion of the curve $y=x^{3}$ between $x=0$ and $x=1$ about the $x$-axis.

Solution. First sketch the curve; then imagine revolving it about the $x$-axis (Figure 6.5.6). Since $y=x^{3}$, we have $d y / d x=3 x^{2}$, and hence from (4) the surface area $S$ is

$$
\begin{aligned}
S & =\int_{0}^{1} 2 \pi y \sqrt{1+\left(\frac{d y}{d x}\right)^{2}} d x \\
& =\int_{0}^{1} 2 \pi x^{3} \sqrt{1+\left(3 x^{2}\right)^{2}} d x \\
& =2 \pi \int_{0}^{1} x^{3}\left(1+9 x^{4}\right)^{1 / 2} d x \\
& =\frac{2 \pi}{36} \int_{1}^{10} u^{1 / 2} d u \quad \begin{array}{r}
u=1+9 x^{4} \\
d u=36 x^{3} d x
\end{array} \\
& \left.=\frac{2 \pi}{36} \cdot \frac{2}{3} u^{3 / 2}\right]_{u=1}^{10}=\frac{\pi}{27}\left(10^{3 / 2}-1\right) \approx 3.56
\end{aligned}
$$

- Example 2 Find the area of the surface that is generated by revolving the portion of the curve $y=x^{2}$ between $x=1$ and $x=2$ about the $y$-axis.

Solution. First sketch the curve; then imagine revolving it about the $y$-axis (Figure 6.5.7). Because the curve is revolved about the $y$-axis we will apply Formula (5). Toward this end, we rewrite $y=x^{2}$ as $x=\sqrt{y}$ and observe that the $y$-values corresponding to $x=1$ and
$x=2$ are $y=1$ and $y=4$. Since $x=\sqrt{y}$, we have $d x / d y=1 /(2 \sqrt{y})$, and hence from (5) the surface area $S$ is

$$
\begin{aligned}
S & =\int_{1}^{4} 2 \pi x \sqrt{1+\left(\frac{d x}{d y}\right)^{2}} d y \\
& =\int_{1}^{4} 2 \pi \sqrt{y} \sqrt{1+\left(\frac{1}{2 \sqrt{y}}\right)^{2}} d y \\
& =\pi \int_{1}^{4} \sqrt{4 y+1} d y \\
& =\frac{\pi}{4} \int_{5}^{17} u^{1 / 2} d u \quad \begin{array}{c}
u=4 y+1 \\
d u=4 d y
\end{array} \\
& \left.=\frac{\pi}{4} \cdot \frac{2}{3} u^{3 / 2}\right]_{u=5}^{17}=\frac{\pi}{6}\left(17^{3 / 2}-5^{3 / 2}\right) \approx 30.85
\end{aligned}
$$

1. If $f$ is a smooth, nonnegative function on $[a, b]$, then the surface area $S$ of the surface of revolution generated by revolving the portion of the curve $y=f(x)$ between $x=a$ and $x=b$ about the $x$-axis is $\qquad$
2. The lateral area of the frustum with slant height $\sqrt{10}$ and base radii $r_{1}=1$ and $r_{2}=2$ is $\qquad$
3. An integral expression for the area of the surface generated by rotating the line segment joining $(3,1)$ and $(6,2)$ about the $x$-axis is $\qquad$
4. An integral expression for the area of the surface generated by rotating the line segment joining $(3,1)$ and $(6,2)$ about the $y$-axis is $\qquad$

## EXERCISE SET 6.5 c CAS

1-4 Find the area of the surface generated by revolving the given curve about the $x$-axis.

1. $y=7 x, 0 \leq x \leq 1$
2. $y=\sqrt{x}, 1 \leq x \leq 4$
3. $y=\sqrt{4-x^{2}},-1 \leq x \leq 1$
4. $x=\sqrt[3]{y}, 1 \leq y \leq 8$

5-8 Find the area of the surface generated by revolving the given curve about the $y$-axis.
5. $x=9 y+1,0 \leq y \leq 2$
6. $x=y^{3}, 0 \leq y \leq 1$
7. $x=\sqrt{9-y^{2}},-2 \leq y \leq 2$
8. $x=2 \sqrt{1-y},-1 \leq y \leq 0$
c 9-12 Use a CAS to find the exact area of the surface generated by revolving the curve about the stated axis.
9. $y=\sqrt{x}-\frac{1}{3} x^{3 / 2}, 1 \leq x \leq 3$; $x$-axis
10. $y=\frac{1}{3} x^{3}+\frac{1}{4} x^{-1}, 1 \leq x \leq 2$; $x$-axis
11. $8 x y^{2}=2 y^{6}+1,1 \leq y \leq 2 ; y$-axis
12. $x=\sqrt{16-y}, 0 \leq y \leq 15 ; y$-axis
[C 13-16 Use a CAS or a calculating utility with a numerical integration capability to approximate the area of the surface generated by revolving the curve about the stated axis. Round your answer to two decimal places.
13. $y=\sin x, 0 \leq x \leq \pi ; x$-axis
14. $x=\tan y, 0 \leq y \leq \pi / 4 ; y$-axis
15. $y=e^{x}, 0 \leq x \leq 1$; $x$-axis
16. $y=e^{x}, 1 \leq y \leq e ; y$-axis

17-20 True-False Determine whether the statement is true or false. Explain your answer.
17. The lateral surface area $S$ of a right circular cone with height $h$ and base radius $r$ is $S=\pi r \sqrt{r^{2}+h^{2}}$.
18. The lateral surface area of a frustum of slant height $l$ and base radii $r_{1}$ and $r_{2}$ is equal to the lateral surface area of a right circular cylinder of height $l$ and radius equal to the average of $r_{1}$ and $r_{2}$.
19. The approximation

$$
S \approx \sum_{k=1}^{n} 2 \pi f\left(x_{k}^{* *}\right) \sqrt{1+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}} \Delta x_{k}
$$

for surface area is exact if $f$ is a positive-valued constant function.
20. The expression

$$
\sum_{k=1}^{n} 2 \pi f\left(x_{k}^{* *}\right) \sqrt{1+\left[f^{\prime}\left(x_{k}^{*}\right)\right]^{2}} \Delta x_{k}
$$

is not a true Riemann sum for

$$
\int_{a}^{b} 2 \pi f(x) \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x
$$

21-22 Approximate the area of the surface using Formula (2) with $n=20$ subintervals of equal width. Round your answer to two decimal places.
21. The surface of Exercise 13.
22. The surface of Exercise 16 .

## FOCUS ON CONCEPTS

23. Assume that $y=f(x)$ is a smooth curve on the interval $[a, b]$ and assume that $f(x) \geq 0$ for $a \leq x \leq b$. Derive a formula for the surface area generated when the curve $y=f(x), a \leq x \leq b$, is revolved about the line $y=-k(k>0)$.
24. Would it be circular reasoning to use Definition 6.5.2 to find the surface area of a frustum of a right circular cone? Explain your answer.
25. Show that the area of the surface of a sphere of radius $r$ is $4 \pi r^{2}$. [Hint: Revolve the semicircle $y=\sqrt{r^{2}-x^{2}}$ about the $x$-axis.]
26. The accompanying figure shows a spherical cap of height $h$ cut from a sphere of radius $r$. Show that the surface area $S$ of the cap is $S=2 \pi r h$. [Hint: Revolve an appropriate portion of the circle $x^{2}+y^{2}=r^{2}$ about the $y$-axis.]


## Figure Ex-26

27. The portion of a sphere that is cut by two parallel planes is called a zone. Use the result of Exercise 26 to show that the surface area of a zone depends on the radius of the sphere and the distance between the planes, but not on the location of the zone.
28. Let $y=f(x)$ be a smooth curve on the interval $[a, b]$ and assume that $f(x) \geq 0$ for $a \leq x \leq b$. By the Extreme-Value

Theorem (4.4.2), the function $f$ has a maximum value $K$ and a minimum value $k$ on $[a, b]$. Prove: If $L$ is the arc length of the curve $y=f(x)$ between $x=a$ and $x=b$, and if $S$ is the area of the surface that is generated by revolving this curve about the $x$-axis, then

$$
2 \pi k L \leq S \leq 2 \pi K L
$$

29. Use the results of Exercise 28 above and Exercise 21 in Section 6.4 to show that the area $S$ of the surface generated by revolving the curve $y=\sec x, 0 \leq x \leq \pi / 3$, about the $x$-axis satisfies

$$
\frac{2 \pi^{2}}{3} \leq S \leq \frac{4 \pi^{2}}{3} \sqrt{13}
$$

30. Let $y=f(x)$ be a smooth curve on $[a, b]$ and assume that $f(x) \geq 0$ for $a \leq x \leq b$. Let $A$ be the area under the curve $y=f(x)$ between $x=a$ and $x=b$, and let $S$ be the area of the surface obtained when this section of curve is revolved about the $x$-axis.
(a) Prove that $2 \pi A \leq S$.
(b) For what functions $f$ is $2 \pi A=S$ ?

31-37 These exercises assume familiarity with the basic concepts of parametric curves. If needed, an introduction to this material is provided in Web Appendix I.
31-32 For these exercises, divide the interval $[a, b]$ into $n$ subintervals by inserting points $t_{1}, t_{2}, \ldots, t_{n-1}$ between $a=t_{0}$ and $b=t_{n}$, and assume that $x^{\prime}(t)$ and $y^{\prime}(t)$ are continuous functions and that no segment of the curve

$$
x=x(t), \quad y=y(t) \quad(a \leq t \leq b)
$$

is traced more than once.
31. Let $S$ be the area of the surface generated by revolving the curve $x=x(t), y=y(t)(a \leq t \leq b)$ about the $x$-axis. Explain how $S$ can be approximated by

$$
\begin{aligned}
S \approx & \sum_{k=1}^{n}\left(\pi\left[y\left(t_{k-1}\right)+y\left(t_{k}\right)\right]\right. \\
& \left.\times \sqrt{\left[x\left(t_{k}\right)-x\left(t_{k-1}\right)\right]^{2}+\left[y\left(t_{k}\right)-y\left(t_{k-1}\right)\right]^{2}}\right)
\end{aligned}
$$

Using results from advanced calculus, it can be shown that as max $\Delta t_{k} \rightarrow 0$, this sum converges to

$$
\begin{equation*}
S=\int_{a}^{b} 2 \pi y(t) \sqrt{\left[x^{\prime}(t)\right]^{2}+\left[y^{\prime}(t)\right]^{2}} d t \tag{A}
\end{equation*}
$$

32. Let $S$ be the area of the surface generated by revolving the curve $x=x(t), y=y(t)(a \leq t \leq b)$ about the $y$-axis. Explain how $S$ can be approximated by

$$
\begin{aligned}
S \approx & \sum_{k=1}^{n}\left(\pi\left[x\left(t_{k-1}\right)+x\left(t_{k}\right)\right]\right. \\
& \left.\times \sqrt{\left[x\left(t_{k}\right)-x\left(t_{k-1}\right)\right]^{2}+\left[y\left(t_{k}\right)-y\left(t_{k-1}\right)\right]^{2}}\right)
\end{aligned}
$$

Using results from advanced calculus, it can be shown that as max $\Delta t_{k} \rightarrow 0$, this sum converges to

$$
\begin{equation*}
S=\int_{a}^{b} 2 \pi x(t) \sqrt{\left[x^{\prime}(t)\right]^{2}+\left[y^{\prime}(t)\right]^{2}} d t \tag{B}
\end{equation*}
$$

33-37 Use Formulas (A) and (B) from Exercises 31 and 32.
33. Find the area of the surface generated by revolving the parametric curve $x=t^{2}, y=2 t(0 \leq t \leq 4)$ about the $x$-axis.
34. Use a CAS to find the area of the surface generated by revolving the parametric curve

$$
x=\cos ^{2} t, \quad y=5 \sin t \quad(0 \leq t \leq \pi / 2)
$$

about the $x$-axis.
35. Find the area of the surface generated by revolving the parametric curve $x=t, y=2 t^{2}(0 \leq t \leq 1)$ about the $y$-axis.
36. Find the area of the surface generated by revolving the parametric curve $x=\cos ^{2} t, y=\sin ^{2} t(0 \leq t \leq \pi / 2)$ about the $y$-axis.
37. By revolving the semicircle

$$
x=r \cos t, \quad y=r \sin t \quad(0 \leq t \leq \pi)
$$

about the $x$-axis, show that the surface area of a sphere of radius $r$ is $4 \pi r^{2}$.
38. Writing Compare the derivation of Definition 6.5 .2 with that of Definition 6.4.2. Discuss the geometric features that result in similarities in the two definitions.
39. Writing Discuss what goes wrong if we replace the frustums of right circular cones by right circular cylinders in the derivation of Definition 6.5.2.

QUICK CHECK ANSWERS 6.5

1. $\int_{a}^{b} 2 \pi f(x) \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x$
2. $3 \sqrt{10} \pi$
3. $\int_{3}^{6}(2 \pi)\left(\frac{x}{3}\right) \sqrt{\frac{10}{9}} d x=\int_{3}^{6} \frac{2 \sqrt{10} \pi}{9} x d x$
4. $\int_{1}^{2}(2 \pi)(3 y) \sqrt{10} d y$

### 6.6 WORK

In this section we will use the integration tools developed in the preceding chapter to study some of the basic principles of "work," which is one of the fundamental concepts in physics and engineering.

## THE ROLE OF WORK IN PHYSICS AND ENGINEERING

In this section we will be concerned with two related concepts, work and energy. To put these ideas in a familiar setting, when you push a stalled car for a certain distance you are performing work, and the effect of your work is to make the car move. The energy of motion caused by the work is called the kinetic energy of the car. The exact connection between work and kinetic energy is governed by a principle of physics called the workenergy relationship. Although we will touch on this idea in this section, a detailed study of the relationship between work and energy will be left for courses in physics and engineering. Our primary goal here will be to explain the role of integration in the study of work.

## WORK DONE BY A CONSTANT FORCE APPLIED IN THE DIRECTION OF MOTION

When a stalled car is pushed, the speed that the car attains depends on the force $F$ with which it is pushed and the distance $d$ over which that force is applied (Figure 6.6.1). Force and distance appear in the following definition of work.

Figure 6.6.1


If you push against an immovable object, such as a brick wall, you may tire yourself out, but you will not perform any work. Why?


Vasili Alexeev shown lifting a recordbreaking 562 lb in the 1976 Olympics. In eight successive years he won Olympic gold medals, captured six world championships, and broke 80 world records. In 1999 he was honored in Greece as the best sportsman of the 20th Century.
6.6.1 DEFINITION If a constant force of magnitude $F$ is applied in the direction of motion of an object, and if that object moves a distance $d$, then we define the work $W$ performed by the force on the object to be

$$
\begin{equation*}
W=F \cdot d \tag{1}
\end{equation*}
$$

Common units for measuring force are newtons (N) in the International System of Units (SI), dynes (dyn) in the centimeter-gram-second (CGS) system, and pounds (lb) in the British Engineering (BE) system. One newton is the force required to give a mass of 1 kg an acceleration of $1 \mathrm{~m} / \mathrm{s}^{2}$, one dyne is the force required to give a mass of 1 g an acceleration of $1 \mathrm{~cm} / \mathrm{s}^{2}$, and one pound of force is the force required to give a mass of 1 slug an acceleration of $1 \mathrm{ft} / \mathrm{s}^{2}$.

It follows from Definition 6.6 .1 that work has units of force times distance. The most common units of work are newton-meters ( $\mathrm{N} \cdot \mathrm{m}$ ), dyne-centimeters (dyn $\cdot \mathrm{cm}$ ), and footpounds ( $\mathrm{ft} \cdot \mathrm{lb}$ ). As indicated in Table 6.6.1, one newton-meter is also called a joule ( J ), and one dyne-centimeter is also called an erg. One foot-pound is approximately 1.36 J .

Table 6.6.1


- Example 1 An object moves 5 ft along a line while subjected to a constant force of 100 lb in its direction of motion. The work done is

$$
W=F \cdot d=100 \cdot 5=500 \mathrm{ft} \cdot \mathrm{lb}
$$

An object moves 25 m along a line while subjected to a constant force of 4 N in its direction of motion. The work done is

$$
W=F \cdot d=4 \cdot 25=100 \mathrm{~N} \cdot \mathrm{~m}=100 \mathrm{~J}
$$

- Example 2 In the 1976 Olympics, Vasili Alexeev astounded the world by lifting a record-breaking 562 lb from the floor to above his head (about 2 m ). Equally astounding was the feat of strongman Paul Anderson, who in 1957 braced himself on the floor and used his back to lift 6270 lb of lead and automobile parts a distance of 1 cm . Who did more work?

Solution. To lift an object one must apply sufficient force to overcome the gravitational force that the Earth exerts on that object. The force that the Earth exerts on an object is that object's weight; thus, in performing their feats, Alexeev applied a force of 562 lb over a distance of 2 m and Anderson applied a force of 6270 lb over a distance of 1 cm . Pounds are units in the BE system, meters are units in SI, and centimeters are units in the CGS system. We will need to decide on the measurement system we want to use and be consistent. Let us agree to use SI and express the work of the two men in joules. Using the conversion factor in Table 6.6.1 we obtain

$$
\begin{aligned}
& 562 \mathrm{lb} \approx 562 \mathrm{lb} \times 4.45 \mathrm{~N} / \mathrm{lb} \approx 2500 \mathrm{~N} \\
& 6270 \mathrm{lb} \approx 6270 \mathrm{lb} \times 4.45 \mathrm{~N} / \mathrm{lb} \approx 27,900 \mathrm{~N}
\end{aligned}
$$

Using these values and the fact that $1 \mathrm{~cm}=0.01 \mathrm{~m}$ we obtain

$$
\begin{aligned}
& \text { Alexeev's work }=(2500 \mathrm{~N}) \times(2 \mathrm{~m})=5000 \mathrm{~J} \\
& \text { Anderson's work }=(27,900 \mathrm{~N}) \times(0.01 \mathrm{~m})=279 \mathrm{~J}
\end{aligned}
$$

Therefore, even though Anderson's lift required a tremendous upward force, it was applied over such a short distance that Alexeev did more work.

## WORK DONE BY A VARIABLE FORCE APPLIED IN THE DIRECTION OF MOTION

Many important problems are concerned with finding the work done by a variable force that is applied in the direction of motion. For example, Figure $6.6 .2 a$ shows a spring in its natural state (neither compressed nor stretched). If we want to pull the block horizontally (Figure 6.6.2b), then we would have to apply more and more force to the block to overcome the increasing force of the stretching spring. Thus, our next objective is to define what is meant by the work performed by a variable force and to find a formula for computing it. This will require calculus.
6.6.2 PROBLEM Suppose that an object moves in the positive direction along a coordinate line while subjected to a variable force $F(x)$ that is applied in the direction of motion. Define what is meant by the work $W$ performed by the force on the object as the object moves from $x=a$ to $x=b$, and find a formula for computing the work.

The basic idea for solving this problem is to break up the interval $[a, b]$ into subintervals that are sufficiently small that the force does not vary much on each subinterval. This will allow us to treat the force as constant on each subinterval and to approximate the work on each subinterval using Formula (1). By adding the approximations to the work on the subintervals, we will obtain a Riemann sum that approximates the work $W$ over the entire interval, and by taking the limit of the Riemann sums we will obtain an integral for $W$.

To implement this idea, divide the interval $[a, b]$ into $n$ subintervals by inserting points $x_{1}, x_{2}, \ldots, x_{n-1}$ between $a=x_{0}$ and $b=x_{n}$. We can use Formula (1) to approximate the work $W_{k}$ done in the $k$ th subinterval by choosing any point $x_{k}^{*}$ in this interval and regarding the force to have a constant value $F\left(x_{k}^{*}\right)$ throughout the interval. Since the width of the $k$ th subinterval is $x_{k}-x_{k-1}=\Delta x_{k}$, this yields the approximation

$$
W_{k} \approx F\left(x_{k}^{*}\right) \Delta x_{k}
$$

Adding these approximations yields the following Riemann sum that approximates the work $W$ done over the entire interval:

$$
W \approx \sum_{k=1}^{n} F\left(x_{k}^{*}\right) \Delta x_{k}
$$

Taking the limit as $n$ increases and the widths of all the subintervals approach zero yields the definite integral

$$
W=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} F\left(x_{k}^{*}\right) \Delta x_{k}=\int_{a}^{b} F(x) d x
$$

In summary, we have the following result.
6.6.3 DEFINITION Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a variable force $F(x)$ that is applied in the direction of motion. Then we define the work $W$ performed by the force on the object to be

$$
\begin{equation*}
W=\int_{a}^{b} F(x) d x \tag{2}
\end{equation*}
$$

Hooke's law [Robert Hooke (1635-1703), English physicist] states that under appropriate conditions a spring that is stretched $x$ units beyond its natural length pulls back with a force

$$
F(x)=k x
$$

where $k$ is a constant (called the spring constant or spring stiffness). The value of $k$ depends on such factors as the thickness of the spring and the material used in its composition. Since $k=F(x) / x$, the constant $k$ has units of force per unit length.

## - Example 3 A spring exerts a force of 5 N when stretched 1 m beyond its natural length.

(a) Find the spring constant $k$.
(b) How much work is required to stretch the spring 1.8 m beyond its natural length?

Solution (a). From Hooke's law,

$$
F(x)=k x
$$

From the data, $F(x)=5 \mathrm{~N}$ when $x=1 \mathrm{~m}$, so $5=k \cdot 1$. Thus, the spring constant is $k=5$ newtons per meter $(\mathrm{N} / \mathrm{m})$. This means that the force $F(x)$ required to stretch the spring $x$ meters is

$$
\begin{equation*}
F(x)=5 x \tag{3}
\end{equation*}
$$


$\triangle$ Figure 6.6.3

$\Delta$ Figure 6.6.4

Solution (b). Place the spring along a coordinate line as shown in Figure 6.6.3. We want to find the work $W$ required to stretch the spring over the interval from $x=0$ to $x=1.8$. From (2) and (3) the work $W$ required is

$$
\left.W=\int_{a}^{b} F(x) d x=\int_{0}^{1.8} 5 x d x=\frac{5 x^{2}}{2}\right]_{0}^{1.8}=8.1 \mathrm{~J}
$$

- Example 4 An astronaut's weight (or more precisely, Earth weight) is the force exerted on the astronaut by the Earth's gravity. As the astronaut moves upward into space, the gravitational pull of the Earth decreases, and hence so does his or her weight. If the Earth is assumed to be a sphere of radius 4000 mi , then it can be shown using physics that an astronaut who weighs 150 lb on Earth will have a weight of

$$
w(x)=\frac{2,400,000,000}{x^{2}} \mathrm{lb}, \quad x \geq 4000
$$

at a distance of $x$ mi from the Earth's center (Exercise 25). Use this formula to determine the work in foot-pounds required to lift the astronaut to a point that is 800 mi above the surface of the Earth (Figure 6.6.4).

Solution. Since the Earth has a radius of 4000 mi , the astronaut is lifted from a point that is 4000 mi from the Earth's center to a point that is 4800 mi from the Earth's center. Thus,
from (2), the work $W$ required to lift the astronaut is

$$
\begin{aligned}
W & =\int_{4000}^{4800} \frac{2,400,000,000}{x^{2}} d x \\
& \left.=-\frac{2,400,000,000}{x}\right]_{4000}^{4800} \\
& =-500,000+600,000 \\
& =100,000 \mathrm{mile}-\mathrm{pounds} \\
& =(100,000 \mathrm{mi} \cdot \mathrm{lb}) \times(5280 \mathrm{ft} / \mathrm{mi}) \\
& =5.28 \times 10^{8} \mathrm{ft} \cdot \mathrm{lb}
\end{aligned}
$$

## CALCULATING WORK FROM BASIC PRINCIPLES

Some problems cannot be solved by mechanically substituting into formulas, and one must return to basic principles to obtain solutions. This is illustrated in the next example.

- Example 5 Figure $6.6 .5 a$ shows a conical container of radius 10 ft and height 30 ft . Suppose that this container is filled with water to a depth of 15 ft . How much work is required to pump all of the water out through a hole in the top of the container?

Solution. Our strategy will be to divide the water into thin layers, approximate the work required to move each layer to the top of the container, add the approximations for the layers to obtain a Riemann sum that approximates the total work, and then take the limit of the Riemann sums to produce an integral for the total work.

To implement this idea, introduce an $x$-axis as shown in Figure 6.6.5a, and divide the water into $n$ layers with $\Delta x_{k}$ denoting the thickness of the $k$ th layer. This division induces a partition of the interval $[15,30]$ into $n$ subintervals. Although the upper and lower surfaces of the $k$ th layer are at different distances from the top, the difference will be small if the layer is thin, and we can reasonably assume that the entire layer is concentrated at a single point $x_{k}^{*}$ (Figure 6.6.5a). Thus, the work $W_{k}$ required to move the $k$ th layer to the top of the container is approximately

$$
\begin{equation*}
W_{k} \approx F_{k} x_{k}^{*} \tag{4}
\end{equation*}
$$

where $F_{k}$ is the force required to lift the $k$ th layer. But the force required to lift the $k$ th layer is the force needed to overcome gravity, and this is the same as the weight of the layer. If the layer is very thin, we can approximate the volume of the $k$ th layer with the volume of a cylinder of height $\Delta x_{k}$ and radius $r_{k}$, where (by similar triangles)

$$
\frac{r_{k}}{x_{k}^{*}}=\frac{10}{30}=\frac{1}{3}
$$

or, equivalently, $r_{k}=x_{k}^{*} / 3$ (Figure 6.6.5b). Therefore, the volume of the $k$ th layer of water is approximately

$$
\pi r_{k}^{2} \Delta x_{k}=\pi\left(x_{k}^{*} / 3\right)^{2} \Delta x_{k}=\frac{\pi}{9}\left(x_{k}^{*}\right)^{2} \Delta x_{k}
$$

Since the weight density of water is $62.4 \mathrm{lb} / \mathrm{ft}^{3}$, it follows that

$$
F_{k} \approx \frac{62.4 \pi}{9}\left(x_{k}^{*}\right)^{2} \Delta x_{k}
$$

Thus, from (4)

$$
W_{k} \approx\left(\frac{62.4 \pi}{9}\left(x_{k}^{*}\right)^{2} \Delta x_{k}\right) x_{k}^{*}=\frac{62.4 \pi}{9}\left(x_{k}^{*}\right)^{3} \Delta x_{k}
$$

and hence the work $W$ required to move all $n$ layers has the approximation

$$
W=\sum_{k=1}^{n} W_{k} \approx \sum_{k=1}^{n} \frac{62.4 \pi}{9}\left(x_{k}^{*}\right)^{3} \Delta x_{k}
$$

To find the exact value of the work we take the limit as max $\Delta x_{k} \rightarrow 0$. This yields

$$
\begin{aligned}
W & =\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} \frac{62.4 \pi}{9}\left(x_{k}^{*}\right)^{3} \Delta x_{k}=\int_{15}^{30} \frac{62.4 \pi}{9} x^{3} d x \\
& \left.=\frac{62.4 \pi}{9}\left(\frac{x^{4}}{4}\right)\right]_{15}^{30}=1,316,250 \pi \approx 4,135,000 \mathrm{ft} \cdot \mathrm{lb}
\end{aligned}
$$

- Figure 6.6.5

(a)

(b)


Mike Brinson/Getty Images
The work performed by the skater's stick in a brief interval of time produces the blinding speed of the hockey puck.

## THE WORK-ENERGY RELATIONSHIP

When you see an object in motion, you can be certain that somehow work has been expended to create that motion. For example, when you drop a stone from a building, the stone gathers speed because the force of the Earth's gravity is performing work on it, and when a hockey player strikes a puck with a hockey stick, the work performed on the puck during the brief period of contact with the stick creates the enormous speed of the puck across the ice. However, experience shows that the speed obtained by an object depends not only on the amount of work done, but also on the mass of the object. For example, the work required to throw a 5 oz baseball $50 \mathrm{mi} / \mathrm{h}$ would accelerate a 10 lb bowling ball to less than $9 \mathrm{mi} / \mathrm{h}$.

Using the method of substitution for definite integrals, we will derive a simple equation that relates the work done on an object to the object's mass and velocity. Furthermore, this equation will allow us to motivate an appropriate definition for the "energy of motion" of an object. As in Definition 6.6.3, we will assume that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a force $F(x)$ that is applied in the direction of motion. We let $m$ denote the mass of the object, and we let $x=x(t), v=v(t)=x^{\prime}(t)$, and $a=a(t)=v^{\prime}(t)$ denote the respective position, velocity, and acceleration of the object at time $t$. We will need the following important result from physics that relates the force acting on an object with the mass and acceleration of the object.
6.6.4 NEWTON'S SECOND LAW OF MOTION If an object with mass $m$ is subjected to a force $F$, then the object undergoes an acceleration $a$ that satisfies the equation

$$
\begin{equation*}
F=m a \tag{5}
\end{equation*}
$$

It follows from Newton's Second Law of Motion that

$$
F(x(t))=m a(t)=m v^{\prime}(t)
$$

Assume that

$$
x\left(t_{0}\right)=a \quad \text { and } \quad x\left(t_{1}\right)=b
$$

with

$$
v\left(t_{0}\right)=v_{i} \quad \text { and } \quad v\left(t_{1}\right)=v_{f}
$$

the initial and final velocities of the object, respectively. Then

$$
\begin{aligned}
W & =\int_{a}^{b} F(x) d x=\int_{x\left(t_{0}\right)}^{x\left(t_{1}\right)} F(x) d x \\
& =\int_{t_{0}}^{t_{1}} F(x(t)) x^{\prime}(t) d t \quad \text { By Theorem 5.9.1 with } x=x(t), d x=x^{\prime}(t) d t \\
& =\int_{t_{0}}^{t_{1}} m v^{\prime}(t) v(t) d t=\int_{t_{0}}^{t_{1}} m v(t) v^{\prime}(t) d t \\
& =\int_{v\left(t_{0}\right)}^{v\left(t_{1}\right)} m v d v \quad \text { By Theorem 5.9.1 with } v=v(t), d v=v^{\prime}(t) d t \\
& =\int_{v_{i}}^{v_{f}} m v d v=\left.\frac{1}{2} m v^{2}\right|_{v_{i}} ^{v_{f}}=\frac{1}{2} m v_{f}^{2}-\frac{1}{2} m v_{i}^{2}
\end{aligned}
$$

We see from the equation

$$
\begin{equation*}
W=\frac{1}{2} m v_{f}^{2}-\frac{1}{2} m v_{i}^{2} \tag{6}
\end{equation*}
$$

that the work done on the object is equal to the change in the quantity $\frac{1}{2} m v^{2}$ from its initial value to its final value. We will refer to Equation (6) as the work-energy relationship. If we define the "energy of motion" or kinetic energy of our object to be given by

$$
\begin{equation*}
K=\frac{1}{2} m v^{2} \tag{7}
\end{equation*}
$$

then Equation (6) tells us that the work done on an object is equal to the change in the object's kinetic energy. Loosely speaking, we may think of work done on an object as being "transformed" into kinetic energy of the object. The units of kinetic energy are the same as the units of work. For example, in SI kinetic energy is measured in joules (J).

- Example 6 A space probe of mass $m=5.00 \times 10^{4} \mathrm{~kg}$ travels in deep space subjected only to the force of its own engine. Starting at a time when the speed of the probe is $v=1.10 \times 10^{4} \mathrm{~m} / \mathrm{s}$, the engine is fired continuously over a distance of $2.50 \times 10^{6} \mathrm{~m}$ with a constant force of $4.00 \times 10^{5} \mathrm{~N}$ in the direction of motion. What is the final speed of the probe?

Solution. Since the force applied by the engine is constant and in the direction of motion, the work $W$ expended by the engine on the probe is

$$
W=\text { force } \times \text { distance }=\left(4.00 \times 10^{5} \mathrm{~N}\right) \times\left(2.50 \times 10^{6} \mathrm{~m}\right)=1.00 \times 10^{12} \mathrm{~J}
$$

From (6), the final kinetic energy $K_{f}=\frac{1}{2} m v_{f}^{2}$ of the probe can be expressed in terms of the work $W$ and the initial kinetic energy $K_{i}=\frac{1}{2} m v_{i}^{2}$ as

$$
K_{f}=W+K_{i}
$$

Thus, from the known mass and initial speed we have

$$
K_{f}=\left(1.00 \times 10^{12} \mathrm{~J}\right)+\frac{1}{2}\left(5.00 \times 10^{4} \mathrm{~kg}\right)\left(1.10 \times 10^{4} \mathrm{~m} / \mathrm{s}\right)^{2}=4.025 \times 10^{12} \mathrm{~J}
$$

The final kinetic energy is $K_{f}=\frac{1}{2} m v_{f}^{2}$, so the final speed of the probe is

$$
v_{f}=\sqrt{\frac{2 K_{f}}{m}}=\sqrt{\frac{2\left(4.025 \times 10^{12}\right)}{5.00 \times 10^{4}}} \approx 1.27 \times 10^{4} \mathrm{~m} / \mathrm{s}
$$

## QUICK CHECK EXERCISES 6.6 (See page 458 for answers.)

1. If a constant force of 5 lb moves an object 10 ft , then the work done by the force on the object is $\qquad$ -
2. A newton-meter is also called a $\qquad$ A dynecentimeter is also called an $\qquad$
3. Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$. The work per-
formed on the object by a variable force $F(x)$ applied in the direction of motion is $W=$ $\qquad$
4. A force $F(x)=10-2 x \mathrm{~N}$ applied in the positive $x$-direction moves an object 3 m from $x=2$ to $x=5$. The work done by the force on the object is $\qquad$ —.

## EXERCISE SET 6.6

## FOCUS ON CONCEPTS

1. A variable force $F(x)$ in the positive $x$-direction is graphed in the accompanying figure. Find the work done by the force on a particle that moves from $x=0$ to $x=3$.

2. A variable force $F(x)$ in the positive $x$-direction is graphed in the accompanying figure. Find the work done by the force on a particle that moves from $x=0$ to $x=5$.

3. For the variable force $F(x)$ in Exercise 2, consider the distance $d$ for which the work done by the force on the particle when the particle moves from $x=0$ to $x=d$ is half of the work done when the particle moves from $x=0$ to $x=5$. By inspecting the graph of $F$, is $d$ more or less than 2.5 ? Explain, and then find the exact value of $d$.
4. Suppose that a variable force $F(x)$ is applied in the positive $x$-direction so that an object moves from $x=a$ to $x=b$. Relate the work done by the force on the object and the average value of $F$ over $[a, b]$, and illustrate this relationship graphically.
5. A constant force of 10 lb in the positive $x$-direction is applied to a particle whose velocity versus time curve is shown in the accompanying figure. Find the work done by the force on the particle from time $t=0$ to $t=5$.

$$
\begin{aligned}
& \text { F Figure Ex-5 }
\end{aligned}
$$

6. A spring exerts a force of 6 N when it is stretched from its natural length of 4 m to a length of $4 \frac{1}{2} \mathrm{~m}$. Find the work required to stretch the spring from its natural length to a length of 6 m .
7. A spring exerts a force of 100 N when it is stretched 0.2 m beyond its natural length. How much work is required to stretch the spring 0.8 m beyond its natural length?
8. A spring whose natural length is 15 cm exerts a force of 45 N when stretched to a length of 20 cm .
(a) Find the spring constant (in newtons/meter).
(b) Find the work that is done in stretching the spring 3 cm beyond its natural length.
(c) Find the work done in stretching the spring from a length of 20 cm to a length of 25 cm .
9. Assume that $10 \mathrm{ft} \cdot \mathrm{lb}$ of work is required to stretch a spring 1 ft beyond its natural length. What is the spring constant?

10-13 True-False Determine whether the statement is true or false. Explain your answer.
10. In order to support the weight of a parked automobile, the surface of a driveway must do work against the force of gravity on the vehicle.
11. A force of 10 lb in the direction of motion of an object that moves 5 ft in 2 s does six times the work of a force of 10 lb in the direction of motion of an object that moves 5 ft in 12 s .
12. It follows from Hooke's law that in order to double the distance a spring is stretched beyond its natural length, four times as much work is required.
13. In the International System of Units, work and kinetic energy have the same units.
14. A cylindrical tank of radius 5 ft and height 9 ft is two-thirds filled with water. Find the work required to pump all the water over the upper rim.
15. Solve Exercise 14 assuming that the tank is half-filled with water.
16. A cone-shaped water reservoir is 20 ft in diameter across the top and 15 ft deep. If the reservoir is filled to a depth of 10 ft , how much work is required to pump all the water to the top of the reservoir?
17. The vat shown in the accompanying figure contains water to a depth of 2 m . Find the work required to pump all the water to the top of the vat. [Use $9810 \mathrm{~N} / \mathrm{m}^{3}$ as the weight density of water.]
18. The cylindrical tank shown in the accompanying figure is filled with a liquid weighing $50 \mathrm{lb} / \mathrm{ft}^{3}$. Find the work required to pump all the liquid to a level 1 ft above the top of the tank.

$\triangle$ Figure Ex-17

$\triangle$ Figure Ex-18
19. A swimming pool is built in the shape of a rectangular parallelepiped 10 ft deep, 15 ft wide, and 20 ft long.
(a) If the pool is filled to 1 ft below the top, how much work is required to pump all the water into a drain at the top edge of the pool?
(b) A one-horsepower motor can do $550 \mathrm{ft} \cdot \mathrm{lb}$ of work per second. What size motor is required to empty the pool in 1 hour?
20. How much work is required to fill the swimming pool in Exercise 19 to 1 ft below the top if the water is pumped in through an opening located at the bottom of the pool?
21. A 100 ft length of steel chain weighing $15 \mathrm{lb} / \mathrm{ft}$ is dangling from a pulley. How much work is required to wind the chain onto the pulley?
22. A 3 lb bucket containing 20 lb of water is hanging at the end of a 20 ft rope that weighs $4 \mathrm{oz} / \mathrm{ft}$. The other end of the rope is attached to a pulley. How much work is required to wind the length of rope onto the pulley, assuming that the rope is wound onto the pulley at a rate of $2 \mathrm{ft} / \mathrm{s}$ and that as the bucket is being lifted, water leaks from the bucket at a rate of $0.5 \mathrm{lb} / \mathrm{s}$ ?
23. A rocket weighing 3 tons is filled with 40 tons of liquid fuel. In the initial part of the flight, fuel is burned off at a constant rate of 2 tons per 1000 ft of vertical height. How much work in foot-tons ( ft -ton) is done lifting the rocket 3000 ft ?
24. It follows from Coulomb's law in physics that two like electrostatic charges repel each other with a force inversely proportional to the square of the distance between them. Suppose that two charges $A$ and $B$ repel with a force of $k$ newtons when they are positioned at points $A(-a, 0)$ and $B(a, 0)$, where $a$ is measured in meters. Find the work $W$ required to move charge $A$ along the $x$-axis to the origin if charge $B$ remains stationary.
25. It is a law of physics that the gravitational force exerted by the Earth on an object above the Earth's surface varies inversely as the square of its distance from the Earth's center. Thus, an object's weight $w(x)$ is related to its distance $x$ from the Earth's center by a formula of the form

$$
w(x)=\frac{k}{x^{2}}
$$

where $k$ is a constant of proportionality that depends on the mass of the object.
(a) Use this fact and the assumption that the Earth is a sphere of radius 4000 mi to obtain the formula for $w(x)$ in Example 4.
(b) Find a formula for the weight $w(x)$ of a satellite that is $x$ mi from the Earth's surface if its weight on Earth is 6000 lb .
(c) How much work is required to lift the satellite from the surface of the Earth to an orbital position that is 1000 mi high?
26. (a) The formula $w(x)=k / x^{2}$ in Exercise 25 is applicable to all celestial bodies. Assuming that the Moon is a sphere of radius 1080 mi , find the force that the Moon exerts on an astronaut who is $x \mathrm{mi}$ from the surface of the Moon if her weight on the Moon's surface is 20 lb .
(b) How much work is required to lift the astronaut to a point that is 10.8 mi above the Moon's surface?
27. The world's first commercial high-speed magnetic levitation (MAGLEV) train, a 30 km double-track project connecting Shanghai, China, to Pudong International Airport, began full revenue service in 2003. Suppose that a MAGLEV train has a mass $m=4.00 \times 10^{5} \mathrm{~kg}$ and that starting at a time when the train has a speed of $20 \mathrm{~m} / \mathrm{s}$ the engine applies a force of $6.40 \times 10^{5} \mathrm{~N}$ in the direction of motion over a distance of $3.00 \times 10^{3} \mathrm{~m}$. Use the work-energy relationship (6) to find the final speed of the train.
28. Assume that a Mars probe of mass $m=2.00 \times 10^{3} \mathrm{~kg}$ is subjected only to the force of its own engine. Starting at a time when the speed of the probe is $v=1.00 \times 10^{4} \mathrm{~m} / \mathrm{s}$, the engine is fired continuously over a distance of $1.50 \times 10^{5} \mathrm{~m}$ with a constant force of $2.00 \times 10^{5} \mathrm{~N}$ in the direction of motion. Use the work-energy relationship (6) to find the final speed of the probe.
29. On August 10, 1972 a meteorite with an estimated mass of $4 \times 10^{6} \mathrm{~kg}$ and an estimated speed of $15 \mathrm{~km} / \mathrm{s}$ skipped across the atmosphere above the western United States and Canada but fortunately did not hit the Earth.
(a) Assuming that the meteorite had hit the Earth with a speed of $15 \mathrm{~km} / \mathrm{s}$, what would have been its change in kinetic energy in joules ( J )?
(b) Express the energy as a multiple of the explosive energy of 1 megaton of TNT, which is $4.2 \times 10^{15} \mathrm{~J}$.
(c) The energy associated with the Hiroshima atomic bomb was 13 kilotons of TNT. To how many such bombs would the meteorite impact have been equivalent?
30. Writing After reading Examples 3-5, a student classifies work problems as either "pushing/pulling" or "pumping."

Describe these categories in your own words and discuss the methods used to solve each type. Give examples to illustrate that these categories are not mutually exclusive.
31. Writing How might you recognize that a problem can be solved by means of the work-energy relationship? That is, what sort of "givens" and "unknowns" would suggest such a solution? Discuss two or three examples.

## QUICK CHECK ANSWERS 6.6

1. $50 \mathrm{ft} \cdot \mathrm{lb} \quad$ 2. joule; erg $\quad$ 3. $\int_{a}^{b} F(x) d x \quad$ 4. 9 J

### 6.7 MOMENTS, CENTERS OF GRAVITY, AND CENTROIDS


$\triangle$ Figure 6.7.1

The units in Equation (1) are consistent since mass $=($ mass $/$ area $) \times$ area.

$\Delta$ Figure 6.7.2

Suppose that a rigid physical body is acted on by a constant gravitational field. Because the body is composed of many particles, each of which is affected by gravity, the action of the gravitational field on the body consists of a large number of forces distributed over the entire body. However, it is a fact of physics that these individual forces can be replaced by a single force acting at a point called the center of gravity of the body. In this section we will show how integrals can be used to locate centers of gravity.

## DENSITY AND MASS OF A LAMINA

Let us consider an idealized flat object that is thin enough to be viewed as a two-dimensional plane region (Figure 6.7.1). Such an object is called a lamina. A lamina is called homogeneous if its composition is uniform throughout and inhomogeneous otherwise. We will consider homogeneous laminas in this section. Inhomogeneous laminas will be discussed in Chapter 14. The density of a homogeneous lamina is defined to be its mass per unit area. Thus, the density $\delta$ of a homogeneous lamina of mass $M$ and area $A$ is given by $\delta=M / A$. Notice that the mass $M$ of a homogeneous lamina can be expressed as

$$
\begin{equation*}
M=\delta A \tag{1}
\end{equation*}
$$

Example 1 A triangular lamina with vertices $(0,0),(0,1)$, and $(1,0)$ has density $\delta=3$. Find its total mass.

Solution. Referring to (1) and Figure 6.7.2, the mass $M$ of the lamina is

$$
M=\delta A=3 \cdot \frac{1}{2}=\frac{3}{2}(\text { unit of mass })
$$

## CENTER OF GRAVITY OF A LAMINA

Assume that the acceleration due to the force of gravity is constant and acts downward, and suppose that a lamina occupies a region $R$ in a horizontal $x y$-plane. It can be shown that there exists a unique point $(\bar{x}, \bar{y})$ (which may or may not belong to $R$ ) such that the effect
of gravity on the lamina is "equivalent" to that of a single force acting at the point $(\bar{x}, \bar{y})$. This point is called the center of gravity of the lamina, and if it is in $R$, then the lamina will balance horizontally on the point of a support placed at $(\bar{x}, \bar{y})$. For example, the center of gravity of a homogeneous disk is at the center of the disk, and the center of gravity of a homogeneous rectangular region is at the center of the rectangle. For an irregularly shaped homogeneous lamina, locating the center of gravity requires calculus.
6.7.1 PROBLEM Let $f$ be a positive continuous function on the interval $[a, b]$. Suppose that a homogeneous lamina with constant density $\delta$ occupies a region $R$ in a horizontal $x y$-plane bounded by the graphs of $y=f(x), y=0, x=a$, and $x=b$. Find the coordinates $(\bar{x}, \bar{y})$ of the center of gravity of the lamina.

To motivate the solution, consider what happens if we try to balance the lamina on a knife-edge parallel to the $x$-axis. Suppose the lamina in Figure 6.7.3 is placed on a knifeedge along a line $y=c$ that does not pass through the center of gravity. Because the lamina behaves as if its entire mass is concentrated at the center of gravity $(\bar{x}, \bar{y})$, the lamina will be rotationally unstable and the force of gravity will cause a rotation about $y=c$. Similarly, the lamina will undergo a rotation if placed on a knife-edge along $y=d$. However, if the knife-edge runs along the line $y=\bar{y}$ through the center of gravity, the lamina will be in perfect balance. Similarly, the lamina will be in perfect balance on a knife-edge along the line $x=\bar{x}$ through the center of gravity. This suggests that the center of gravity of a lamina can be determined as the intersection of two lines of balance, one parallel to the $x$-axis and the other parallel to the $y$-axis. In order to find these lines of balance, we will need some preliminary results about rotations.


- Figure 6.7.3

Children on a seesaw learn by experience that a lighter child can balance a heavier one by sitting farther from the fulcrum or pivot point. This is because the tendency for an object to produce rotation is proportional not only to its mass but also to the distance between the object and the fulcrum. To make this more precise, consider an $x$-axis, which we view as a weightless beam. If a mass $m$ is located on the axis at $x$, then the tendency for that mass to produce a rotation of the beam about a point $a$ on the axis is measured by the following quantity, called the moment of $\boldsymbol{m}$ about $\boldsymbol{x}=\boldsymbol{a}$ :

$$
\left[\begin{array}{c}
\text { moment of } m \\
\text { about } a
\end{array}\right]=m(x-a)
$$



Positive moment about $a$ (clockwise rotation)


Negative moment about $a$ (counterclockwise rotation)
$\triangle$ Figure 6.7.4

$\Delta$ Figure 6.7.6

(a)

(b)

The number $x-a$ is called the lever arm. Depending on whether the mass is to the right or left of $a$, the lever arm is either the distance between $x$ and $a$ or the negative of this distance (Figure 6.7.4). Positive lever arms result in positive moments and clockwise rotations, and negative lever arms result in negative moments and counterclockwise rotations.

Suppose that masses $m_{1}, m_{2}, \ldots, m_{n}$ are located at $x_{1}, x_{2}, \ldots, x_{n}$ on a coordinate axis and a fulcrum is positioned at the point $a$ (Figure 6.7.5). Depending on whether the sum of the moments about $a$,

$$
\sum_{k=1}^{n} m_{k}\left(x_{k}-a\right)=m_{1}\left(x_{1}-a\right)+m_{2}\left(x_{2}-a\right)+\cdots+m_{n}\left(x_{n}-a\right)
$$

is positive, negative, or zero, a weightless beam along the axis will rotate clockwise about $a$, rotate counterclockwise about $a$, or balance perfectly. In the last case, the system of masses is said to be in equilibrium.


The preceding ideas can be extended to masses distributed in two-dimensional space. If we imagine the $x y$-plane to be a weightless sheet supporting a mass $m$ located at a point $(x, y)$, then the tendency for the mass to produce a rotation of the sheet about the line $x=a$ is $m(x-a)$, called the moment of $\boldsymbol{m}$ about $\boldsymbol{x}=\boldsymbol{a}$, and the tendency for the mass to produce a rotation about the line $y=c$ is $m(y-c)$, called the moment of $\boldsymbol{m}$ about $\boldsymbol{y}=\boldsymbol{c}$ (Figure 6.7.6). In summary,

$$
\left[\begin{array}{c}
\text { moment of } m  \tag{2-3}\\
\text { about the } \\
\text { line } x=a
\end{array}\right]=m(x-a) \quad \text { and } \quad\left[\begin{array}{c}
\text { moment of } m \\
\text { about the } \\
\text { line } y=c
\end{array}\right]=m(y-c)
$$

If a number of masses are distributed throughout the $x y$-plane, then the plane (viewed as a weightless sheet) will balance on a knife-edge along the line $x=a$ if the sum of the moments about the line is zero. Similarly, the plane will balance on a knife-edge along the line $y=c$ if the sum of the moments about that line is zero.

We are now ready to solve Problem 6.7.1. The basic idea for solving this problem is to divide the lamina into strips whose areas may be approximated by the areas of rectangles. These area approximations, along with Formulas (2) and (3), will allow us to create a Riemann sum that approximates the moment of the lamina about a horizontal or vertical line. By taking the limit of Riemann sums we will then obtain an integral for the moment of a lamina about a horizontal or vertical line. We observe that since the lamina balances on the lines $x=\bar{x}$ and $y=\bar{y}$, the moment of the lamina about those lines should be zero. This observation will enable us to calculate $\bar{x}$ and $\bar{y}$.

To implement this idea, we divide the interval $[a, b]$ into $n$ subintervals by inserting the points $x_{1}, x_{2}, \ldots, x_{n-1}$ between $a=x_{0}$ and $b=x_{n}$. This has the effect of dividing the lamina $R$ into $n$ strips $R_{1}, R_{2}, \ldots, R_{n}$ (Figure 6.7.7a). Suppose that the $k$ th strip extends from $x_{k-1}$ to $x_{k}$ and that the width of this strip is

$$
\Delta x_{k}=x_{k}-x_{k-1}
$$

We will let $x_{k}^{*}$ be the midpoint of the $k$ th subinterval and we will approximate $R_{k}$ by a rectangle of width $\Delta x_{k}$ and height $f\left(x_{k}^{*}\right)$. From (1), the mass $\Delta M_{k}$ of this rectangle is $\Delta M_{k}=\delta f\left(x_{k}^{*}\right) \Delta x_{k}$, and we will assume that the rectangle behaves as if its entire mass is concentrated at its center $\left(x_{k}^{*}, y_{k}^{*}\right)=\left(x_{k}^{*}, \frac{1}{2} f\left(x_{k}^{*}\right)\right)$ (Figure 6.7.7b). It then follows from (2) and (3) that the moments of $R_{k}$ about the lines $x=\bar{x}$ and $y=\bar{y}$ may be approximated
by $\left(x_{k}^{*}-\bar{x}\right) \Delta M_{k}$ and $\left(y_{k}^{*}-\bar{y}\right) \Delta M_{k}$, respectively. Adding these approximations yields the following Riemann sums that approximate the moment of the entire lamina about the lines $x=\bar{x}$ and $y=\bar{y}$ :

$$
\begin{aligned}
\sum_{k=1}^{n}\left(x_{k}^{*}-\bar{x}\right) \Delta M_{k} & =\sum_{k=1}^{n}\left(x_{k}^{*}-\bar{x}\right) \delta f\left(x_{k}^{*}\right) \Delta x_{k} \\
\sum_{k=1}^{n}\left(y_{k}^{*}-\bar{y}\right) \Delta M_{k} & =\sum_{k=1}^{n}\left(\frac{f\left(x_{k}^{*}\right)}{2}-\bar{y}\right) \delta f\left(x_{k}^{*}\right) \Delta x_{k}
\end{aligned}
$$

Taking the limits as $n$ increases and the widths of all the rectangles approach zero yields the definite integrals

$$
\int_{a}^{b}(x-\bar{x}) \delta f(x) d x \quad \text { and } \quad \int_{a}^{b}\left(\frac{f(x)}{2}-\bar{y}\right) \delta f(x) d x
$$

that represent the moments of the lamina about the lines $x=\bar{x}$ and $y=\bar{y}$. Since the lamina balances on those lines, the moments of the lamina about those lines should be zero:

$$
\int_{a}^{b}(x-\bar{x}) \delta f(x) d x=\int_{a}^{b}\left(\frac{f(x)}{2}-\bar{y}\right) \delta f(x) d x=0
$$

Since $\bar{x}$ and $\bar{y}$ are constant, these equations can be rewritten as

$$
\begin{aligned}
\int_{a}^{b} \delta x f(x) d x & =\bar{x} \int_{a}^{b} \delta f(x) d x \\
\int_{a}^{b} \frac{1}{2} \delta(f(x))^{2} d x & =\bar{y} \int_{a}^{b} \delta f(x) d x
\end{aligned}
$$

from which we obtain the following formulas for the center of gravity of the lamina:

$$
\bar{x}=\frac{\int_{a}^{b} \delta x f(x) d x}{\int_{a}^{b} \delta f(x) d x}, \quad \bar{y}=\frac{\int_{a}^{b} \frac{1}{2} \delta(f(x))^{2} d x}{\int_{a}^{b} \delta f(x) d x}
$$

Observe that in both formulas the denominator is the mass $M$ of the lamina. The numerator in the formula for $\bar{x}$ is denoted by $M_{y}$ and is called the first moment of the lamina about the $y$-axis; the numerator of the formula for $\bar{y}$ is denoted by $M_{x}$ and is called the first moment of the lamina about the $\boldsymbol{x}$-axis. Thus, we can write (4) and (5) as

Alternative Formulas for Center of Gravity $(\bar{x}, \bar{y})$ of a Lamina

$$
\begin{align*}
& \bar{x}=\frac{M_{y}}{M}=\frac{1}{\text { mass of } R} \int_{a}^{b} \delta x f(x) d x  \tag{6}\\
& \bar{y}=\frac{M_{x}}{M}=\frac{1}{\operatorname{mass} \text { of } R} \int_{a}^{b} \frac{1}{2} \delta(f(x))^{2} d x \tag{7}
\end{align*}
$$

Example 2 Find the center of gravity of the triangular lamina with vertices $(0,0)$, $(0,1)$, and $(1,0)$ and density $\delta=3$.

Solution. The lamina is shown in Figure 6.7.2. In Example 1 we found the mass of the lamina to be

$$
M=\frac{3}{2}
$$

Since the density factor has canceled, we may interpret the centroid as a geometric property of the region, and distinguish it from the center of gravity, which is a physical property of an idealized object that occupies the region.


Figure 6.7.8

The moment of the lamina about the $y$-axis is

$$
\begin{aligned}
M_{y} & =\int_{0}^{1} \delta x f(x) d x=\int_{0}^{1} 3 x(-x+1) d x \\
& \left.=\int_{0}^{1}\left(-3 x^{2}+3 x\right) d x=\left(-x^{3}+\frac{3}{2} x^{2}\right)\right]_{0}^{1}=-1+\frac{3}{2}=\frac{1}{2}
\end{aligned}
$$

and the moment about the $x$-axis is

$$
\begin{aligned}
M_{x} & =\int_{0}^{1} \frac{1}{2} \delta(f(x))^{2} d x=\int_{0}^{1} \frac{3}{2}(-x+1)^{2} d x \\
& \left.=\int_{0}^{1} \frac{3}{2}\left(x^{2}-2 x+1\right) d x=\frac{3}{2}\left(\frac{1}{3} x^{3}-x^{2}+x\right)\right]_{0}^{1}=\frac{3}{2}\left(\frac{1}{3}\right)=\frac{1}{2}
\end{aligned}
$$

From (6) and (7),

$$
\bar{x}=\frac{M_{y}}{M}=\frac{1 / 2}{3 / 2}=\frac{1}{3}, \quad \bar{y}=\frac{M_{x}}{M}=\frac{1 / 2}{3 / 2}=\frac{1}{3}
$$

so the center of gravity is $\left(\frac{1}{3}, \frac{1}{3}\right)$.

In the case of a homogeneous lamina, the center of gravity of a lamina occupying the region $R$ is called the centroid of the region $\boldsymbol{R}$. Since the lamina is homogeneous, $\delta$ is constant. The factor $\delta$ in (4) and (5) may thus be moved through the integral signs and canceled, and (4) and (5) can be expressed as

$$
\begin{align*}
& \text { Centroid of a Region } \boldsymbol{R} \\
& \bar{x}=\frac{\int_{a}^{b} x f(x) d x}{\int_{a}^{b} f(x) d x}=\frac{1}{\operatorname{area} \text { of } R} \int_{a}^{b} x f(x) d x  \tag{8}\\
& \bar{y}=\frac{\int_{a}^{b} \frac{1}{2}(f(x))^{2} d x}{\int_{a}^{b} f(x) d x}=\frac{1}{\operatorname{area} \text { of } R} \int_{a}^{b} \frac{1}{2}(f(x))^{2} d x \tag{9}
\end{align*}
$$

- Example 3 Find the centroid of the semicircular region in Figure 6.7.8.

Solution. By symmetry, $\bar{x}=0$ since the $y$-axis is obviously a line of balance. To find $\bar{y}$, first note that the equation of the semicircle is $y=f(x)=\sqrt{a^{2}-x^{2}}$. From (9),

$$
\begin{aligned}
\bar{y} & =\frac{1}{\text { area of } R} \int_{-a}^{a} \frac{1}{2}(f(x))^{2} d x=\frac{1}{\frac{1}{2} \pi a^{2}} \int_{-a}^{a} \frac{1}{2}\left(a^{2}-x^{2}\right) d x \\
& \left.=\frac{1}{\pi a^{2}}\left(a^{2} x-\frac{1}{3} x^{3}\right)\right]_{-a}^{a} \\
& =\frac{1}{\pi a^{2}}\left[\left(a^{3}-\frac{1}{3} a^{3}\right)-\left(-a^{3}+\frac{1}{3} a^{3}\right)\right] \\
& =\frac{1}{\pi a^{2}}\left(\frac{4 a^{3}}{3}\right)=\frac{4 a}{3 \pi}
\end{aligned}
$$

so the centroid is $(0,4 a / 3 \pi)$.

$\Delta$ Figure 6.7.9


Aigure 6.7.10

## OTHER TYPES OF REGIONS

The strategy used to find the center of gravity of the region in Problem 6.7.1 can be used to find the center of gravity of regions that are not of that form.

Consider a homogeneous lamina that occupies the region $R$ between two continuous functions $f(x)$ and $g(x)$ over the interval $[a, b]$, where $f(x) \geq g(x)$ for $a \leq x \leq b$. To find the center of gravity of this lamina we can subdivide it into $n$ strips using lines parallel to the $y$-axis. If $x_{k}^{*}$ is the midpoint of the $k$ th strip, the strip can be approximated by a rectangle of width $\Delta x_{k}$ and height $f\left(x_{k}^{*}\right)-g\left(x_{k}^{*}\right)$. We assume that the entire mass of the $k$ th rectangle is concentrated at its center $\left(x_{k}^{*}, y_{k}^{*}\right)=\left(x_{k}^{*}, \frac{1}{2}\left(f\left(x_{k}^{*}\right)+g\left(x_{k}^{*}\right)\right)\right)$ (Figure 6.7.9). Continuing the argument as in the solution of Problem 6.7.1, we find that the center of gravity of the lamina is

$$
\begin{align*}
& \bar{x}=\frac{\int_{a}^{b} x(f(x)-g(x)) d x}{\int_{a}^{b}(f(x)-g(x)) d x}=\frac{1}{\operatorname{area} \text { of } R} \int_{a}^{b} x(f(x)-g(x)) d x  \tag{10}\\
& \bar{y}=\frac{\int_{a}^{b} \frac{1}{2}\left([f(x)]^{2}-[g(x)]^{2}\right) d x}{\int_{a}^{b}(f(x)-g(x)) d x}=\frac{1}{\operatorname{area} \text { of } R} \int_{a}^{b} \frac{1}{2}\left([f(x)]^{2}-[g(x)]^{2}\right) d x \tag{11}
\end{align*}
$$

Note that the density of the lamina does not appear in Equations (10) and (11). This reflects the fact that the centroid is a geometric property of $R$.

- Example 4 Find the centroid of the region $R$ enclosed between the curves $y=x^{2}$ and $y=x+6$.

Solution. To begin, we note that the two curves intersect when $x=-2$ and $x=3$ and that $x+6 \geq x^{2}$ over that interval (Figure 6.7.10). The area of $R$ is

$$
\int_{-2}^{3}\left[(x+6)-x^{2}\right] d x=\frac{125}{6}
$$

From (10) and (11),

$$
\begin{aligned}
\bar{x} & =\frac{1}{\text { area of } R} \int_{-2}^{3} x\left[(x+6)-x^{2}\right] d x \\
& \left.=\frac{6}{125}\left(\frac{1}{3} x^{3}+3 x^{2}-\frac{1}{4} x^{4}\right)\right]_{-2}^{3} \\
& =\frac{6}{125} \cdot \frac{125}{12}=\frac{1}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{y} & =\frac{1}{\text { area of } R} \int_{-2}^{3} \frac{1}{2}\left((x+6)^{2}-\left(x^{2}\right)^{2}\right) d x \\
& =\frac{6}{125} \int_{-2}^{3} \frac{1}{2}\left(x^{2}+12 x+36-x^{4}\right) d x \\
& \left.=\frac{6}{125} \cdot \frac{1}{2}\left(\frac{1}{3} x^{3}+6 x^{2}+36 x-\frac{1}{5} x^{5}\right)\right]_{-2}^{3} \\
& =\frac{6}{125} \cdot \frac{250}{3}=4
\end{aligned}
$$

so the centroid of $R$ is $\left(\frac{1}{2}, 4\right)$.

$\Delta$ Figure 6.7.11

$\Delta$ Figure 6.7.12

Suppose that $w$ is a continuous function of $y$ on an interval $[c, d]$ with $w(y) \geq 0$ for $c \leq y \leq d$. Consider a lamina that occupies a region $R$ bounded above by $y=d$, below by $y=c$, on the left by the $y$-axis, and on the right by $x=w(y)$ (Figure 6.7.11). To find the center of gravity of this lamina, we note that the roles of $x$ and $y$ in Problem 6.7.1 have been reversed. We now imagine the lamina to be subdivided into $n$ strips using lines parallel to the $x$-axis. We let $y_{k}^{*}$ be the midpoint of the $k$ th subinterval and approximate the strip by a rectangle of width $\Delta y_{k}$ and height $w\left(y_{k}^{*}\right)$. We assume that the entire mass of the $k$ th rectangle is concentrated at its center $\left(x_{k}^{*}, y_{k}^{*}\right)=\left(\frac{1}{2} w\left(y_{k}^{*}\right), y_{k}^{*}\right)$ (Figure 6.7.11). Continuing the argument as in the solution of Problem 6.7.1, we find that the center of gravity of the lamina is

$$
\begin{align*}
& \bar{x}=\frac{\int_{c}^{d} \frac{1}{2}(w(y))^{2} d y}{\int_{c}^{d} w(y) d y}=\frac{1}{\operatorname{area} \text { of } R} \int_{c}^{d} \frac{1}{2}(w(y))^{2} d y  \tag{12}\\
& \bar{y}=\frac{\int_{c}^{d} y w(y) d y}{\int_{c}^{d} w(y) d y}=\frac{1}{\operatorname{area} \text { of } R} \int_{c}^{d} y w(y) d y \tag{13}
\end{align*}
$$

Once again, the absence of the density in Equations (12) and (13) reflects the geometric nature of the centroid.

- Example 5 Find the centroid of the region $R$ enclosed between the curves $y=\sqrt{x}$, $y=1, y=2$, and the $y$-axis (Figure 6.7.12).

Solution. Note that $x=w(y)=y^{2}$ and that the area of $R$ is

$$
\int_{1}^{2} y^{2} d y=\frac{7}{3}
$$

From (12) and (13),

$$
\begin{aligned}
& \left.\bar{x}=\frac{1}{\operatorname{area} \text { of } R} \int_{1}^{2} \frac{1}{2}\left(y^{2}\right)^{2} d y=\frac{3}{7} \cdot \frac{1}{10} y^{5}\right]_{1}^{2}=\frac{3}{7} \cdot \frac{31}{10}=\frac{93}{70} \\
& \left.\bar{y}=\frac{1}{\operatorname{area~of~} R} \int_{1}^{2} y\left(y^{2}\right) d y=\frac{3}{7} \cdot \frac{1}{4} y^{4}\right]_{1}^{2}=\frac{3}{7} \cdot \frac{15}{4}=\frac{45}{28}
\end{aligned}
$$

so the centroid of $R$ is $(93 / 70,45 / 28) \approx(1.329,1.607)$.

## THEOREM OF PAPPUS

The following theorem, due to the Greek mathematician Pappus, gives an important relationship between the centroid of a plane region $R$ and the volume of the solid generated when the region is revolved about a line.
6.7.2 THEOREM (Theorem of Pappus) If $R$ is a bounded plane region and $L$ is a line that lies in the plane of $R$ such that $R$ is entirely on one side of $L$, then the volume of the solid formed by revolving $R$ about $L$ is given by

$$
\text { volume }=(\text { area of } R) \cdot\binom{\text { distance traveled }}{\text { by the centroid }}
$$



Figure 6.7.13

PROOF We prove this theorem in the special case where $L$ is the $y$-axis, the region $R$ is in the first quadrant, and the region $R$ is of the form given in Problem 6.7.1. (A more general proof will be outlined in the Exercises of Section 14.8.) In this case, the volume $V$ of the solid formed by revolving $R$ about $L$ can be found by the method of cylindrical shells (Section 6.3) to be

$$
V=2 \pi \int_{a}^{b} x f(x) d x
$$

Thus, it follows from (8) that

$$
V=2 \pi \bar{x}[\text { area of } R]
$$

This completes the proof since $2 \pi \bar{x}$ is the distance traveled by the centroid when $R$ is revolved about the $y$-axis.

- Example 6 Use Pappus' Theorem to find the volume $V$ of the torus generated by revolving a circular region of radius $b$ about a line at a distance $a$ (greater than $b$ ) from the center of the circle (Figure 6.7.13).

Solution. By symmetry, the centroid of a circular region is its center. Thus, the distance traveled by the centroid is $2 \pi a$. Since the area of a circle of radius $b$ is $\pi b^{2}$, it follows from Pappus' Theorem that the volume of the torus is

$$
V=(2 \pi a)\left(\pi b^{2}\right)=2 \pi^{2} a b^{2}
$$

## QUICK CHECK EXERCISES 6.7 (See page 467 for answers.)

1. The total mass of a homogeneous lamina of area $A$ and density $\delta$ is $\qquad$
2. A homogeneous lamina of mass $M$ and density $\delta$ occupies a region in the $x y$-plane bounded by the graphs of $y=f(x)$, $y=0, x=a$, and $x=b$, where $f$ is a nonnegative continuous function defined on an interval $[a, b]$. The $x$-coordinate of the center of gravity of the lamina is $M_{y} / M$, where $M_{y}$ is called the $\qquad$ and is given by the integral $\qquad$
3. Let $R$ be the region between the graphs of $y=x^{2}$ and $y=2-x$ for $0 \leq x \leq 1$. The area of $R$ is $\frac{7}{6}$ and the centroid of $R$ is
4. If the region $R$ in Quick Check Exercise 3 is used to generate a solid $G$ by rotating $R$ about a horizontal line 6 units above its centroid, then the volume of $G$ is $\qquad$

## EXERCISE SET 6.7 C CAS

## FOCUS ON CONCEPTS

1. Masses $m_{1}=5, m_{2}=10$, and $m_{3}=20$ are positioned on a weightless beam as shown in the accompanying figure.
(a) Suppose that the fulcrum is positioned at $x=5$. Without computing the sum of moments about 5, determine whether the sum is positive, zero, or negative. Explain.
ments, refinements, and proofs of results by earlier mathematicians. Pappus' Theorem, stated without proof in Book VII of The Collection, was probably known and proved in earlier times. This result is sometimes called Guldin's Theorem in recognition of the Swiss mathematician, Paul Guldin (1577-1643), who rediscovered it independently.
(b) Where should the fulcrum be placed so that the beam is in equilibrium?


Pappus of Alexandria (4th century A.D.) Greek mathematician. Pappus lived during the early Christian era when mathematical activity was in a period of decline. His main contributions to mathematics appeared in a series of eight books called The Collection (written about 340 A.D.). This work, which survives only partially, contained some original results but was devoted mostly to state-
2. Masses $m_{1}=10, m_{2}=3, m_{3}=4$, and $m$ are positioned on a weightless beam, with the fulcrum positioned at point 4 , as shown in the accompanying figure.
(a) Suppose that $m=14$. Without computing the sum of the moments about 4, determine whether the sum is positive, zero, or negative. Explain.
(b) For what value of $m$ is the beam in equilibrium?

$\triangle$ Figure Ex-2
3-6 Find the centroid of the region by inspection and confirm your answer by integrating.
3.

4.

5.

6.


7-20 Find the centroid of the region.
7.

8.

9.

10.

11. The triangle with vertices $(0,0),(2,0)$, and $(0,1)$.
12. The triangle with vertices $(0,0),(1,1)$, and $(2,0)$.
13. The region bounded by the graphs of $y=x^{2}$ and $x+y=6$.
14. The region bounded on the left by the $y$-axis, on the right by the line $x=2$, below by the parabola $y=x^{2}$, and above by the line $y=x+6$.
15. The region bounded by the graphs of $y=x^{2}$ and $y=x+2$.
16. The region bounded by the graphs of $y=x^{2}$ and $y=1$.
17. The region bounded by the graphs of $y=\sqrt{x}$ and $y=x^{2}$.
18. The region bounded by the graphs of $x=1 / y, x=0$, $y=1$, and $y=2$.
19. The region bounded by the graphs of $y=x, x=1 / y^{2}$, and $y=2$.
20. The region bounded by the graphs of $x y=4$ and $x+y=5$.

## FOCUS ON CONCEPTS

21. Use symmetry considerations to argue that the centroid of an isosceles triangle lies on the median to the base of the triangle.
22. Use symmetry considerations to argue that the centroid of an ellipse lies at the intersection of the major and minor axes of the ellipse.

23-26 Find the mass and center of gravity of the lamina with density $\delta$.
23. A lamina bounded by the $x$-axis, the line $x=1$, and the curve $y=\sqrt{x} ; \delta=2$.
24. A lamina bounded by the graph of $x=y^{4}$ and the line $x=1$; $\delta=15$.
25. A lamina bounded by the graph of $y=|x|$ and the line $y=1 ; \delta=3$.
26. A lamina bounded by the $x$-axis and the graph of the equation $y=1-x^{2} ; \delta=3$.
(C 27-30 Use a CAS to find the mass and center of gravity of the lamina with density $\delta$.
27. A lamina bounded by $y=\sin x, y=0, x=0$, and $x=\pi$; $\delta=4$.
28. A lamina bounded by $y=e^{x}, y=0, x=0$, and $x=1$; $\delta=1 /(e-1)$.
29. A lamina bounded by the graph of $y=\ln x$, the $x$-axis, and the line $x=2 ; \delta=1$.
30. A lamina bounded by the graphs of $y=\cos x, y=\sin x$, $x=0$, and $x=\pi / 4 ; \delta=1+\sqrt{2}$.

31-34 True-False Determine whether the statement is true or false. Explain your answer. [In Exercise 34, assume that the (rotated) square lies in the $x y$-plane to the right of the $y$-axis.]
31. The centroid of a rectangle is the intersection of the diagonals of the rectangle.
32. The centroid of a rhombus is the intersection of the diagonals of the rhombus.
33. The centroid of an equilateral triangle is the intersection of the medians of the triangle.
34. By rotating a square about its center, it is possible to change the volume of the solid of revolution generated by revolving the square about the $y$-axis.
35. Find the centroid of the triangle with vertices $(0,0),(a, b)$, and $(a,-b)$.
36. Prove that the centroid of a triangle is the point of intersection of the three medians of the triangle. [Hint: Choose coordinates so that the vertices of the triangle are located at $(0,-a),(0, a)$, and $(b, c)$.]
37. Find the centroid of the isosceles trapezoid with vertices $(-a, 0),(a, 0),(-b, c)$, and $(b, c)$.
38. Prove that the centroid of a parallelogram is the point of intersection of the diagonals of the parallelogram. [Hint: Choose coordinates so that the vertices of the parallelogram are located at $(0,0),(0, a),(b, c)$, and $(b, a+c)$.]
39. Use the Theorem of Pappus and the fact that the volume of a sphere of radius $a$ is $V=\frac{4}{3} \pi a^{3}$ to show that the centroid of the lamina that is bounded by the $x$-axis and the semicircle $y=\sqrt{a^{2}-x^{2}}$ is $(0,4 a /(3 \pi))$. (This problem was solved directly in Example 3.)
40. Use the Theorem of Pappus and the result of Exercise 39 to find the volume of the solid generated when the region
bounded by the $x$-axis and the semicircle $y=\sqrt{a^{2}-x^{2}}$ is revolved about
(a) the line $y=-a$
(b) the line $y=x-a$.
41. Use the Theorem of Pappus and the fact that the area of an ellipse with semiaxes $a$ and $b$ is $\pi a b$ to find the volume of the elliptical torus generated by revolving the ellipse

$$
\frac{(x-k)^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1
$$

about the $y$-axis. Assume that $k>a$.
42. Use the Theorem of Pappus to find the volume of the solid that is generated when the region enclosed by $y=x^{2}$ and $y=8-x^{2}$ is revolved about the $x$-axis.
43. Use the Theorem of Pappus to find the centroid of the triangular region with vertices $(0,0),(a, 0)$, and $(0, b)$, where $a>0$ and $b>0$. [Hint: Revolve the region about the $x$ axis to obtain $\bar{y}$ and about the $y$-axis to obtain $\bar{x}$.]
44. Writing Suppose that a region $R$ in the plane is decomposed into two regions $R_{1}$ and $R_{2}$ whose areas are $A_{1}$ and $A_{2}$, respectively, and whose centroids are ( $\bar{x}_{1}, \bar{y}_{1}$ ) and ( $\bar{x}_{2}, \bar{y}_{2}$ ), respectively. Investigate the problem of expressing the centroid of $R$ in terms of $A_{1}, A_{2},\left(\bar{x}_{1}, \bar{y}_{1}\right)$, and $\left(\bar{x}_{2}, \bar{y}_{2}\right)$. Write a short report on your investigations, supporting your reasoning with plausible arguments. Can you extend your results to decompositions of $R$ into more than two regions?
45. Writing How might you recognize that a problem can be solved by means of the Theorem of Pappus? That is, what sort of "givens" and "unknowns" would suggest such a solution? Discuss two or three examples.

1. $\delta A \quad$ 2. first moment about the $y$-axis; $\int_{a}^{b} \delta x f(x) d x$
2. $\left(\frac{5}{14}, \frac{32}{35}\right)$
3. $14 \pi$

### 6.8 FLUID PRESSURE AND FORCE

In this section we will use the integration tools developed in the preceding chapter to study the pressures and forces exerted by fluids on submerged objects.

## WHAT IS A FLUID?

Afluid is a substance that flows to conform to the boundaries of any container in which it is placed. Fluids include liquids, such as water, oil, and mercury, as well as gases, such as helium, oxygen, and air. The study of fluids falls into two categories: fluid statics (the study of fluids at rest) and fluid dynamics (the study of fluids in motion). In this section we will be concerned only with fluid statics; toward the end of this text we will investigate problems in fluid dynamics.


Jupiter Images Corp.
Snowshoes prevent the woman from sinking by spreading her weight over a large area to reduce her pressure on the snow.

## THE CONCEPT OF PRESSURE

The effect that a force has on an object depends on how that force is spread over the surface of the object. For example, when you walk on soft snow with boots, the weight of your body crushes the snow and you sink into it. However, if you put on a pair of snowshoes to spread the weight of your body over a greater surface area, then the weight of your body has less of a crushing effect on the snow. The concept that accounts for both the magnitude of a force and the area over which it is applied is called pressure.
6.8.1 DEFINITION If a force of magnitude $F$ is applied to a surface of area $A$, then we define the pressure $P$ exerted by the force on the surface to be

$$
\begin{equation*}
P=\frac{F}{A} \tag{1}
\end{equation*}
$$

It follows from this definition that pressure has units of force per unit area. The most common units of pressure are newtons per square meter $\left(\mathrm{N} / \mathrm{m}^{2}\right)$ in SI and pounds per square inch $\left(\mathrm{lb} / \mathrm{in}^{2}\right)$ or pounds per square foot $\left(\mathrm{lb} / \mathrm{ft}^{2}\right)$ in the BE system. As indicated in Table 6.8.1, one newton per square meter is called a pascal $(\mathrm{Pa})$. A pressure of 1 Pa is quite small ( $\left.1 \mathrm{~Pa}=1.45 \times 10^{-4} \mathrm{lb} / \mathrm{in}^{2}\right)$, so in countries using SI, tire pressure gauges are usually calibrated in kilopascals ( kPa ), which is 1000 pascals.

Table 6.8.1
UNITS OF FORCE AND PRESSURE

| SYSTEM | FORCE | $\div$ | AREA | $=$ |
| :--- | :--- | :--- | :--- | :--- |
| PRESSURE |  |  |  |  |
| SI | newton $(\mathrm{N})$ |  | square meter $\left(\mathrm{m}^{2}\right)$ | pascal $(\mathrm{Pa})$ |
| BE | pound $(\mathrm{lb})$ |  | square foot $\left(\mathrm{ft}^{2}\right)$ | $\mathrm{lb} / \mathrm{ft}^{2}$ |
| BE | pound $(\mathrm{lb})$ |  | square inch $\left(\mathrm{in}^{2}\right)$ | $\mathrm{lb} / \mathrm{in}^{2}(\mathrm{psi})$ |

CONVERSION FACTORS:
$1 \mathrm{~Pa} \approx 1.45 \times 10^{-4} \mathrm{lb} / \mathrm{in}^{2} \approx 2.09 \times 10^{-2} \mathrm{lb} / \mathrm{ft}^{2}$
$1 \mathrm{lb} / \mathrm{in}^{2} \approx 6.89 \times 10^{3} \mathrm{~Pa} \quad 1 \mathrm{lb} / \mathrm{ft}^{2} \approx 47.9 \mathrm{~Pa}$


Blaise Pascal (1623-1662) French mathematician and scientist. Pascal's mother died when he was three years old and his father, a highly educated magistrate, personally provided the boy's early education. Although Pascal showed an inclination for science and mathematics, his father refused to tutor him in those subjects until he mastered Latin and Greek. Pascal's sister and primary biographer claimed that he independently discovered the first thirty-two propositions of Euclid without ever reading a book on geometry. (However, it is generally agreed that the story is apocryphal.) Nevertheless, the precocious Pascal published a highly respected essay on conic sections by the time he was sixteen years old. Descartes, who read the essay, thought it so brilliant that he could not believe that it was written by such a young man. By age 18 his health began to fail and
until his death he was in frequent pain. However, his creativity was unimpaired.

Pascal's contributions to physics include the discovery that air pressure decreases with altitude and the principle of fluid pressure that bears his name. However, the originality of his work is questioned by some historians. Pascal made major contributions to a branch of mathematics called "projective geometry," and he helped to develop probability theory through a series of letters with Fermat.

In 1646, Pascal's health problems resulted in a deep emotional crisis that led him to become increasingly concerned with religious matters. Although born a Catholic, he converted to a religious doctrine called Jansenism and spent most of his final years writing on religion and philosophy.


Fluid forces always act perpendicular to the surface of a submerged object.

Figure 6.8.1

Table 6.8.2

| Weight densities |  |
| :--- | ---: |
| SI | $\mathrm{N} / \mathrm{m}^{3}$ |
| Machine oil | 4708 |
| Gasoline | 6602 |
| Fresh water | 9810 |
| Seawater | 10,045 |
| Mercury | 133,416 |
| BE SYSTEM | $1 \mathrm{~b} / \mathrm{ft}^{3}$ |
| Machine oil | 30.0 |
| Gasoline | 42.0 |
| Fresh water | 62.4 |
| Seawater | 64.0 |
| Mercury | 849.0 |

All densities are affected by variations in temperature and pressure. Weight densities are also affected by variations in $g$.

$\Delta$ Figure 6.8.2

In this section we will be interested in pressures and forces on objects submerged in fluids. Pressures themselves have no directional characteristics, but the forces that they create always act perpendicular to the face of the submerged object. Thus, in Figure 6.8.1 the water pressure creates horizontal forces on the sides of the tank, vertical forces on the bottom of the tank, and forces that vary in direction, so as to be perpendicular to the different parts of the swimmer's body.

- Example 1 Referring to Figure 6.8.1, suppose that the back of the swimmer's hand has a surface area of $8.4 \times 10^{-3} \mathrm{~m}^{2}$ and that the pressure acting on it is $5.1 \times 10^{4} \mathrm{~Pa}$ (a realistic value near the bottom of a deep diving pool). Find the force that acts on the swimmer's hand.

Solution. From (1), the force $F$ is

$$
F=P A=\left(5.1 \times 10^{4} \mathrm{~N} / \mathrm{m}^{2}\right)\left(8.4 \times 10^{-3} \mathrm{~m}^{2}\right) \approx 4.3 \times 10^{2} \mathrm{~N}
$$

This is quite a large force (nearly 100 lb in the BE system).

## FLUID DENSITY

Scuba divers know that the pressure and forces on their bodies increase with the depth they dive. This is caused by the weight of the water and air above-the deeper the diver goes, the greater the weight above and so the greater the pressure and force exerted on the diver.

To calculate pressures and forces on submerged objects, we need to know something about the characteristics of the fluids in which they are submerged. For simplicity, we will assume that the fluids under consideration are homogeneous, by which we mean that any two samples of the fluid with the same volume have the same mass. It follows from this assumption that the mass per unit volume is a constant $\delta$ that depends on the physical characteristics of the fluid but not on the size or location of the sample; we call

$$
\begin{equation*}
\delta=\frac{m}{V} \tag{2}
\end{equation*}
$$

the mass density of the fluid. Sometimes it is more convenient to work with weight per unit volume than with mass per unit volume. Thus, we define the weight density $\rho$ of a fluid to be

$$
\begin{equation*}
\rho=\frac{w}{V} \tag{3}
\end{equation*}
$$

where $w$ is the weight of a fluid sample of volume $V$. Thus, if the weight density of a fluid is known, then the weight $w$ of a fluid sample of volume $V$ can be computed from the formula $w=\rho V$. Table 6.8 .2 shows some typical weight densities.

## FLUID PRESSURE

To calculate fluid pressures and forces we will need to make use of an experimental observation. Suppose that a flat surface of area $A$ is submerged in a homogeneous fluid of weight density $\rho$ such that the entire surface lies between depths $h_{1}$ and $h_{2}$, where $h_{1} \leq h_{2}$ (Figure 6.8.2). Experiments show that on both sides of the surface, the fluid exerts a force that is perpendicular to the surface and whose magnitude $F$ satisfies the inequalities

$$
\begin{equation*}
\rho h_{1} A \leq F \leq \rho h_{2} A \tag{4}
\end{equation*}
$$

Thus, it follows from (1) that the pressure $P=F / A$ on a given side of the surface satisfies the inequalities

$$
\begin{equation*}
\rho h_{1} \leq P \leq \rho h_{2} \tag{5}
\end{equation*}
$$



The fluid force is the fluid pressure times the area.
$\Delta$ Figure 6.8.3

Note that it is now a straightforward matter to calculate fluid force and pressure on a flat surface that is submerged horizontally at depth $h$, for then $h=h_{1}=h_{2}$ and inequalities (4) and (5) become the equalities

$$
\begin{equation*}
F=\rho h A \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
P=\rho h \tag{7}
\end{equation*}
$$

- Example 2 Find the fluid pressure and force on the top of a flat circular plate of radius 2 m that is submerged horizontally in water at a depth of 6 m (Figure 6.8.3).

Solution. Since the weight density of water is $\rho=9810 \mathrm{~N} / \mathrm{m}^{3}$, it follows from (7) that the fluid pressure is

$$
P=\rho h=(9810)(6)=58,860 \mathrm{~Pa}
$$

and it follows from (6) that the fluid force is

$$
F=\rho h A=\rho h\left(\pi r^{2}\right)=(9810)(6)(4 \pi)=235,440 \pi \approx 739,700 \mathrm{~N}
$$

## FLUID FORCE ON A VERTICAL SURFACE

It was easy to calculate the fluid force on the horizontal plate in Example 2 because each point on the plate was at the same depth. The problem of finding the fluid force on a vertical surface is more complicated because the depth, and hence the pressure, is not constant over the surface. To find the fluid force on a vertical surface we will need calculus.
6.8.2 PROBLEM Suppose that a flat surface is immersed vertically in a fluid of weight density $\rho$ and that the submerged portion of the surface extends from $x=a$ to $x=b$ along an $x$-axis whose positive direction is down (Figure 6.8.4a). For $a \leq x \leq b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point $x$. Define what is meant by the fluid force $F$ on the surface, and find a formula for computing it.

The basic idea for solving this problem is to divide the surface into horizontal strips whose areas may be approximated by areas of rectangles. These area approximations, along with inequalities (4), will allow us to create a Riemann sum that approximates the total force on the surface. By taking a limit of Riemann sums we will then obtain an integral for $F$.

To implement this idea, we divide the interval $[a, b]$ into $n$ subintervals by inserting the points $x_{1}, x_{2}, \ldots, x_{n-1}$ between $a=x_{0}$ and $b=x_{n}$. This has the effect of dividing the surface into $n$ strips of area $A_{k}, k=1,2, \ldots, n$ (Figure 6.8.4b). It follows from (4) that the force $F_{k}$ on the $k$ th strip satisfies the inequalities

$$
\rho h\left(x_{k-1}\right) A_{k} \leq F_{k} \leq \rho h\left(x_{k}\right) A_{k}
$$

or, equivalently,

$$
h\left(x_{k-1}\right) \leq \frac{F_{k}}{\rho A_{k}} \leq h\left(x_{k}\right)
$$

Since the depth function $h(x)$ increases linearly, there must exist a point $x_{k}^{*}$ between $x_{k-1}$ and $x_{k}$ such that

$$
h\left(x_{k}^{*}\right)=\frac{F_{k}}{\rho A_{k}}
$$

or, equivalently,

$$
F_{k}=\rho h\left(x_{k}^{*}\right) A_{k}
$$

We now approximate the area $A_{k}$ of the $k$ th strip of the surface by the area of a rectangle of width $w\left(x_{k}^{*}\right)$ and height $\Delta x_{k}=x_{k}-x_{k-1}$ (Figure 6.8.4c). It follows that $F_{k}$ may be approximated as

$$
F_{k}=\rho h\left(x_{k}^{*}\right) A_{k} \approx \rho h\left(x_{k}^{*}\right) \cdot \underbrace{w\left(x_{k}^{*}\right) \Delta x_{k}}_{\text {Area of rectangle }}
$$

Adding these approximations yields the following Riemann sum that approximates the total force $F$ on the surface:

$$
F=\sum_{k=1}^{n} F_{k} \approx \sum_{k=1}^{n} \rho h\left(x_{k}^{*}\right) w\left(x_{k}^{*}\right) \Delta x_{k}
$$

Taking the limit as $n$ increases and the widths of all the subintervals approach zero yields the definite integral

$$
F=\lim _{\max \Delta x_{k} \rightarrow 0} \sum_{k=1}^{n} \rho h\left(x_{k}^{*}\right) w\left(x_{k}^{*}\right) \Delta x_{k}=\int_{a}^{b} \rho h(x) w(x) d x
$$

In summary, we have the following result.
6.8.3 Definition Suppose that a flat surface is immersed vertically in a fluid of weight density $\rho$ and that the submerged portion of the surface extends from $x=a$ to $x=b$ along an $x$-axis whose positive direction is down (Figure 6.8.4a). For $a \leq x \leq b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point $x$. Then we define the fluid force $F$ on the surface to be

$$
\begin{equation*}
F=\int_{a}^{b} \rho h(x) w(x) d x \tag{8}
\end{equation*}
$$

- Example 3 The face of a dam is a vertical rectangle of height 100 ft and width 200 ft (Figure 6.8.5a). Find the total fluid force exerted on the face when the water surface is level with the top of the dam.

Solution. Introduce an $x$-axis with its origin at the water surface as shown in Figure 6.8.5b. At a point $x$ on this axis, the width of the dam in feet is $w(x)=200$ and the depth in feet is $h(x)=x$. Thus, from (8) with $\rho=62.4 \mathrm{lb} / \mathrm{ft}^{3}$ (the weight density of water) we obtain as the total force on the face

$$
\begin{aligned}
F=\int_{0}^{100}(62.4)(x)(200) d x & =12,480 \int_{0}^{100} x d x \\
& \left.=12,480 \frac{x^{2}}{2}\right]_{0}^{100}=62,400,000 \mathrm{lb}
\end{aligned}
$$

A Figure 6.8.6
(a)

(b)

(a)

(b)
$\triangle$ Figure 6.8.5

-

Thus, it follows from (8) that the force on the plate is

$$
\begin{aligned}
F & =\int_{a}^{b} \rho h(x) w(x) d x=\int_{0}^{4}(30)(3+x)\left(\frac{5}{2} x\right) d x \\
& =75 \int_{0}^{4}\left(3 x+x^{2}\right) d x=75\left[\frac{3 x^{2}}{2}+\frac{x^{3}}{3}\right]_{0}^{4}=3400 \mathrm{lb}
\end{aligned}
$$

## QUICK CHECK EXERCISES 6.8 (See page 473 for answers.)

1. The pressure unit equivalent to a newton per square meter ( $\mathrm{N} / \mathrm{m}^{2}$ ) is called a $\qquad$ The pressure unit psi stands for $\qquad$
2. Given that the weight density of water is $9810 \mathrm{~N} / \mathrm{m}^{3}$, the fluid pressure on a rectangular $2 \mathrm{~m} \times 3 \mathrm{~m}$ flat plate submerged horizontally in water at a depth of 10 m is $\qquad$ The fluid force on the plate is $\qquad$ —.
3. Suppose that a flat surface is immersed vertically in a fluid of weight density $\rho$ and that the submerged portion of the
surface extends from $x=a$ to $x=b$ along an $x$-axis whose positive direction is down. If, for $a \leq x \leq b$, the surface has width $w(x)$ and depth $h(x)$, then the fluid force on the surface is $F=$ $\qquad$
4. A rectangular plate 2 m wide and 3 m high is submerged vertically in water so that the top of the plate is 5 m below the water surface. An integral expression for the force of the water on the plate surface is $F=$ $\qquad$ —.
$\qquad$
[^3][^4]15. In any water tank with a flat base, no matter what the shape of the tank, the fluid force on the base is at most equal to the weight of water in the tank.

16-19 Formula (8) gives the fluid force on a flat surface immersed vertically in a fluid. More generally, if a flat surface is immersed so that it makes an angle of $0 \leq \theta<\pi / 2$ with the vertical, then the fluid force on the surface is given by

$$
F=\int_{a}^{b} \rho h(x) w(x) \sec \theta d x
$$

Use this formula in these exercises.
16. Derive the formula given above for the fluid force on a flat surface immersed at an angle in a fluid.
17. The accompanying figure shows a rectangular swimming pool whose bottom is an inclined plane. Find the fluid force on the bottom when the pool is filled to the top.


4 Figure Ex-17
18. By how many feet should the water in the pool of Exercise 17 be lowered in order for the force on the bottom to be reduced by a factor of $\frac{1}{2}$ ?
19. The accompanying figure shows a dam whose face is an inclined rectangle. Find the fluid force on the face when the water is level with the top of this dam.


4Figure Ex-19
20. An observation window on a submarine is a square with 2 ft sides. Using $\rho_{0}$ for the weight density of seawater, find
the fluid force on the window when the submarine has descended so that the window is vertical and its top is at a depth of $h$ feet.

## FOCUS ON CONCEPTS

21. (a) Show: If the submarine in Exercise 20 descends vertically at a constant rate, then the fluid force on the window increases at a constant rate.
(b) At what rate is the force on the window increasing if the submarine is descending vertically at $20 \mathrm{ft} / \mathrm{min}$ ?
22. (a) Let $D=D_{a}$ denote a disk of radius $a$ submerged in a fluid of weight density $\rho$ such that the center of $D$ is $h$ units below the surface of the fluid. For each value of $r$ in the interval $(0, a]$, let $D_{r}$ denote the disk of radius $r$ that is concentric with $D$. Select a side of the disk $D$ and define $P(r)$ to be the fluid pressure on the chosen side of $D_{r}$. Use (5) to prove that

$$
\lim _{r \rightarrow 0^{+}} P(r)=\rho h
$$

(b) Explain why the result in part (a) may be interpreted to mean that fluid pressure at a given depth is the same in all directions. (This statement is one version of a result known as Pascal's Principle .)
23. Writing Suppose that we model the Earth's atmosphere as a "fluid." Atmospheric pressure at sea level is $P=14.7$ $\mathrm{lb} / \mathrm{in}^{2}$ and the weight density of air at sea level is about $\rho=4.66 \times 10^{-5} \mathrm{lb} / \mathrm{in}^{3}$. With these numbers, what would Formula (7) yield as the height of the atmosphere above the Earth? Do you think this answer is reasonable? If not, explain how we might modify our assumptions to yield a more plausible answer.
24. Writing Suppose that the weight density $\rho$ of a fluid is a function $\rho=\rho(x)$ of the depth $x$ within the fluid. How do you think that Formula (7) for fluid pressure will need to be modified? Support your answer with plausible arguments.
2. $98,100 \mathrm{~Pa} ; 588,600 \mathrm{~N}$
3. $\int_{a}^{b} \rho h(x) w(x) d x \quad$ 4. $\int_{0}^{3} 9810[(5+x) 2] d x$

### 6.9 HYPERBOLIC FUNCTIONS AND HANGING CABLES

The terms "tanh," "sech," and "csch" are pronounced "tanch," "seech," and "coseech," respectively.

In this section we will study certain combinations of $e^{x}$ and $e^{-x}$, called "hyperbolic functions." These functions, which arise in various engineering applications, have many properties in common with the trigonometric functions. This similarity is somewhat surprising, since there is little on the surface to suggest that there should be any relationship between exponential and trigonometric functions. This is because the relationship occurs within the context of complex numbers, a topic which we will leave for more advanced courses.

## DEFINITIONS OF HYPERBOLIC FUNCTIONS

To introduce the hyperbolic functions, observe from Exercise 61 in Section 0.2 that the function $e^{x}$ can be expressed in the following way as the sum of an even function and an odd function:

$$
e^{x}=\underbrace{\frac{e^{x}+e^{-x}}{2}}_{\text {Even }}+\underbrace{\frac{e^{x}-e^{-x}}{2}}_{\text {Odd }}
$$

These functions are sufficiently important that there are names and notation associated with them: the odd function is called the hyperbolic sine of $x$ and the even function is called the hyperbolic cosine of $x$. They are denoted by

$$
\sinh x=\frac{e^{x}-e^{-x}}{2} \quad \text { and } \quad \cosh x=\frac{e^{x}+e^{-x}}{2}
$$

where sinh is pronounced "cinch" and cosh rhymes with "gosh." From these two building blocks we can create four more functions to produce the following set of six hyperbolic functions.

### 6.9.1 DEFINITION

$$
\begin{array}{ll}
\text { Hyperbolic sine } & \sinh x=\frac{e^{x}-e^{-x}}{2} \\
\text { Hyperbolic cosine } & \cosh x=\frac{e^{x}+e^{-x}}{2} \\
\text { Hyperbolic tangent } & \tanh x=\frac{\sinh x}{\cosh x}=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \\
\text { Hyperbolic cotangent } & \operatorname{coth} x=\frac{\cosh x}{\sinh x}=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}} \\
\text { Hyperbolic secant } & \operatorname{sech} x=\frac{1}{\cosh x}=\frac{2}{e^{x}+e^{-x}} \\
\text { Hyperbolic cosecant } & \operatorname{csch} x=\frac{1}{\sinh x}=\frac{2}{e^{x}-e^{-x}}
\end{array}
$$

## - Example 1

$$
\begin{aligned}
& \sinh 0=\frac{e^{0}-e^{-0}}{2}=\frac{1-1}{2}=0 \\
& \cosh 0=\frac{e^{0}+e^{-0}}{2}=\frac{1+1}{2}=1 \\
& \sinh 2=\frac{e^{2}-e^{-2}}{2} \approx 3.6269
\end{aligned}
$$

## GRAPHS OF THE HYPERBOLIC FUNCTIONS

The graphs of the hyperbolic functions, which are shown in Figure 6.9.1, can be generated with a graphing utility, but it is worthwhile to observe that the general shape of the graph of $y=\cosh x$ can be obtained by sketching the graphs of $y=\frac{1}{2} e^{x}$ and $y=\frac{1}{2} e^{-x}$ separately and adding the corresponding $y$-coordinates [see part (a) of the figure]. Similarly, the general shape of the graph of $y=\sinh x$ can be obtained by sketching the graphs of $y=\frac{1}{2} e^{x}$ and $y=-\frac{1}{2} e^{-x}$ separately and adding corresponding $y$-coordinates [see part $(b)$ of the figure].

$\Delta$ Figure 6.9.1

Observe that $\sinh x$ has a domain of $(-\infty,+\infty)$ and a range of $(-\infty,+\infty)$, whereas $\cosh x$ has a domain of $(-\infty,+\infty)$ and a range of $[1,+\infty)$. Observe also that $y=\frac{1}{2} e^{x}$ and $y=\frac{1}{2} e^{-x}$ are curvilinear asymptotes for $y=\cosh x$ in the sense that the graph of $y=\cosh x$ gets closer and closer to the graph of $y=\frac{1}{2} e^{x}$ as $x \rightarrow+\infty$ and gets closer and closer to the graph of $y=\frac{1}{2} e^{-x}$ as $x \rightarrow-\infty$. (See Section 4.3.) Similarly, $y=\frac{1}{2} e^{x}$ is a curvilinear asymptote for $y=\sinh x$ as $x \rightarrow+\infty$ and $y=-\frac{1}{2} e^{-x}$ is a curvilinear asymptote as $x \rightarrow-\infty$. Other properties of the hyperbolic functions are explored in the exercises.

## HANGING CABLES AND OTHER APPLICATIONS

Hyperbolic functions arise in vibratory motions inside elastic solids and more generally in many problems where mechanical energy is gradually absorbed by a surrounding medium. They also occur when a homogeneous, flexible cable is suspended between two points, as with a telephone line hanging between two poles. Such a cable forms a curve, called a catenary (from the Latin catena, meaning "chain"). If, as in Figure 6.9.2, a coordinate system is introduced so that the low point of the cable lies on the $y$-axis, then it can be shown using principles of physics that the cable has an equation of the form

$$
y=a \cosh \left(\frac{x}{a}\right)+c
$$



Figure 6.9.2


Larry Auippy/Mira.com/Digital Railroad, Inc.
A flexible cable suspended between two poles forms a catenary.

(a)

(b)
$\Delta$ Figure 6.9.3
where the parameters $a$ and $c$ are determined by the distance between the poles and the composition of the cable.

## HYPERBOLIC IDENTITIES

The hyperbolic functions satisfy various identities that are similar to identities for trigonometric functions. The most fundamental of these is

$$
\begin{equation*}
\cosh ^{2} x-\sinh ^{2} x=1 \tag{1}
\end{equation*}
$$

which can be proved by writing

$$
\begin{aligned}
\cosh ^{2} x-\sinh ^{2} x & =(\cosh x+\sinh x)(\cosh x-\sinh x) \\
& =\left(\frac{e^{x}+e^{-x}}{2}+\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}-\frac{e^{x}-e^{-x}}{2}\right) \\
& =e^{x} \cdot e^{-x}=1
\end{aligned}
$$

Other hyperbolic identities can be derived in a similar manner or, alternatively, by performing algebraic operations on known identities. For example, if we divide (1) by $\cosh ^{2} x$, we obtain

$$
1-\tanh ^{2} x=\operatorname{sech}^{2} x
$$

and if we divide (1) by $\sinh ^{2} x$, we obtain

$$
\operatorname{coth}^{2} x-1=\operatorname{csch}^{2} x
$$

The following theorem summarizes some of the more useful hyperbolic identities. The proofs of those not already obtained are left as exercises.

### 6.9.2 THEOREM

$$
\begin{array}{ll}
\cosh x+\sinh x=e^{x} & \sinh (x+y)=\sinh x \cosh y+\cosh x \sinh y \\
\cosh x-\sinh x=e^{-x} & \cosh (x+y)=\cosh x \cosh y+\sinh x \sinh y \\
\cosh ^{2} x-\sinh ^{2} x=1 & \sinh (x-y)=\sinh x \cosh y-\cosh x \sinh y \\
1-\tanh ^{2} x=\operatorname{sech}^{2} x & \cosh (x-y)=\cosh x \cosh y-\sinh x \sinh y \\
\operatorname{coth}^{2} x-1=\operatorname{csch}^{2} x & \sinh 2 x=2 \sinh x \cosh x \\
\cosh (-x)=\cosh x & \cosh 2 x=\cosh ^{2} x+\sinh ^{2} x \\
\sinh (-x)=-\sinh x & \cosh 2 x=2 \sinh ^{2} x+1=2 \cosh ^{2} x-1
\end{array}
$$

## WHY THEY ARE CALLED HYPERBOLIC FUNCTIONS

Recall that the parametric equations

$$
x=\cos t, \quad y=\sin t \quad(0 \leq t \leq 2 \pi)
$$

represent the unit circle $x^{2}+y^{2}=1$ (Figure 6.9.3a), as may be seen by writing

$$
x^{2}+y^{2}=\cos ^{2} t+\sin ^{2} t=1
$$

If $0 \leq t \leq 2 \pi$, then the parameter $t$ can be interpreted as the angle in radians from the positive $x$-axis to the point $(\cos t, \sin t)$ or, alternatively, as twice the shaded area of the sector in Figure 6.9.3a (verify). Analogously, the parametric equations

$$
x=\cosh t, \quad y=\sinh t \quad(-\infty<t<+\infty)
$$

represent a portion of the curve $x^{2}-y^{2}=1$, as may be seen by writing

$$
x^{2}-y^{2}=\cosh ^{2} t-\sinh ^{2} t=1
$$

and observing that $x=\cosh t>0$. This curve, which is shown in Figure 6.9.3b, is the right half of a larger curve called the unit hyperbola; this is the reason why the functions in this section are called hyperbolic functions. It can be shown that if $t \geq 0$, then the parameter $t$ can be interpreted as twice the shaded area in Figure 6.9.3b. (We omit the details.)

## DERIVATIVE AND INTEGRAL FORMULAS

Derivative formulas for $\sinh x$ and $\cosh x$ can be obtained by expressing these functions in terms of $e^{x}$ and $e^{-x}$ :

$$
\begin{aligned}
& \frac{d}{d x}[\sinh x]=\frac{d}{d x}\left[\frac{e^{x}-e^{-x}}{2}\right]=\frac{e^{x}+e^{-x}}{2}=\cosh x \\
& \frac{d}{d x}[\cosh x]=\frac{d}{d x}\left[\frac{e^{x}+e^{-x}}{2}\right]=\frac{e^{x}-e^{-x}}{2}=\sinh x
\end{aligned}
$$

Derivatives of the remaining hyperbolic functions can be obtained by expressing them in terms of sinh and cosh and applying appropriate identities. For example,

$$
\begin{aligned}
\frac{d}{d x}[\tanh x] & =\frac{d}{d x}\left[\frac{\sinh x}{\cosh x}\right]=\frac{\cosh x \frac{d}{d x}[\sinh x]-\sinh x \frac{d}{d x}[\cosh x]}{\cosh ^{2} x} \\
& =\frac{\cosh ^{2} x-\sinh ^{2} x}{\cosh ^{2} x}=\frac{1}{\cosh ^{2} x}=\operatorname{sech}^{2} x
\end{aligned}
$$

The following theorem provides a complete list of the generalized derivative formulas and corresponding integration formulas for the hyperbolic functions.

### 6.9.3 THEOREM

$$
\begin{array}{rlrl}
\frac{d}{d x}[\sinh u] & =\cosh u \frac{d u}{d x} & & \int \cosh u d u=\sinh u+C \\
\frac{d}{d x}[\cosh u] & =\sinh u \frac{d u}{d x} & & \int \sinh u d u=\cosh u+C \\
\frac{d}{d x}[\tanh u]=\operatorname{sech}^{2} u \frac{d u}{d x} & & \int \operatorname{sech}^{2} u d u=\tanh u+C \\
\frac{d}{d x}[\operatorname{coth} u] & =-\operatorname{csch}^{2} u \frac{d u}{d x} & & \int \operatorname{csch}^{2} u d u=-\operatorname{coth} u+C \\
\frac{d}{d x}[\operatorname{sech} u]=-\operatorname{sech} u \tanh u \frac{d u}{d x} & & \int \operatorname{sech} u \tanh u d u=-\operatorname{sech} u+C \\
\frac{d}{d x}[\operatorname{csch} u]=-\operatorname{csch} u \operatorname{coth} u \frac{d u}{d x} & & \int \operatorname{csch} u \operatorname{coth} u d u=-\operatorname{csch} u+C
\end{array}
$$

## Example 2

$$
\begin{aligned}
& \frac{d}{d x}\left[\cosh \left(x^{3}\right)\right]=\sinh \left(x^{3}\right) \cdot \frac{d}{d x}\left[x^{3}\right]=3 x^{2} \sinh \left(x^{3}\right) \\
& \frac{d}{d x}[\ln (\tanh x)]=\frac{1}{\tanh x} \cdot \frac{d}{d x}[\tanh x]=\frac{\operatorname{sech}^{2} x}{\tanh x}
\end{aligned}
$$


$\Delta$ Figure 6.9.4

$$
\begin{aligned}
& \int \sinh ^{5} x \cosh x d x=\frac{1}{6} \sinh ^{6} x+C \quad \begin{array}{c}
u=\sinh x \\
d u=\cosh x d x
\end{array} \\
& \begin{aligned}
\int \tanh x d x & =\int \frac{\sinh x}{\cosh x} d x \\
& =\ln |\cosh x|+C \\
& =\ln (\cosh x)+C
\end{aligned}
\end{aligned}
$$

We were justified in dropping the absolute value signs since $\cosh x>0$ for all $x$.

- Example 4 A 100 ft wire is attached at its ends to the tops of two 50 ft poles that are positioned 90 ft apart. How high above the ground is the middle of the wire?

Solution. From above, the wire forms a catenary curve with equation

$$
y=a \cosh \left(\frac{x}{a}\right)+c
$$

where the origin is on the ground midway between the poles. Using Formula (4) of Section 6.4 for the length of the catenary, we have

$$
\begin{aligned}
100 & =\int_{-45}^{45} \sqrt{1+\left(\frac{d y}{d x}\right)^{2}} d x \\
& =2 \int_{0}^{45} \sqrt{1+\left(\frac{d y}{d x}\right)^{2}} d x \quad \begin{array}{l}
\text { By symmetry } \\
\text { about the } y \text {-axis }
\end{array} \\
& =2 \int_{0}^{45} \sqrt{1+\sinh ^{2}\left(\frac{x}{a}\right)} d x \\
& =2 \int_{0}^{45} \cosh \left(\frac{x}{a}\right) d x \quad \begin{array}{l}
\text { By }(1) \text { and the fact } \\
\text { that cosh } x>0
\end{array} \\
& \left.=2 a \sinh \left(\frac{x}{a}\right)\right]_{0}^{45}=2 a \sinh \left(\frac{45}{a}\right)
\end{aligned}
$$

Using a calculating utility's numeric solver to solve

$$
100=2 a \sinh \left(\frac{45}{a}\right)
$$

for $a$ gives $a \approx 56.01$. Then

$$
50=y(45)=56.01 \cosh \left(\frac{45}{56.01}\right)+c \approx 75.08+c
$$

so $c \approx-25.08$. Thus, the middle of the wire is $y(0) \approx 56.01-25.08=30.93 \mathrm{ft}$ above the ground (Figure 6.9.4).

## INVERSES OF HYPERBOLIC FUNCTIONS

Referring to Figure 6.9.1, it is evident that the graphs of $\sinh x, \tanh x, \operatorname{coth} x$, and $\operatorname{csch} x$ pass the horizontal line test, but the graphs of $\cosh x$ and $\operatorname{sech} x$ do not. In the latter case, restricting $x$ to be nonnegative makes the functions invertible (Figure 6.9.5). The graphs of the six inverse hyperbolic functions in Figure 6.9 .6 were obtained by reflecting the graphs of the hyperbolic functions (with the appropriate restrictions) about the line $y=x$.


With the restriction that $x \geq 0$, the curves $y=\cosh x$ and $y=\operatorname{sech} x$ pass the horizontal line test.
$\Delta$ Figure 6.9.5

Figure 6.9.6

Table 6.9.1 summarizes the basic properties of the inverse hyperbolic functions. You should confirm that the domains and ranges listed in this table agree with the graphs in Figure 6.9.6.

$y=\sinh ^{-1} x$


$$
y=\operatorname{coth}^{-1} x
$$




$y=\operatorname{sech}^{-1} x$


Table 6.9.1
PROPERTIES OF INVERSE HYPERBOLIC FUNCTIONS

| FUNCTION | DOMAIN | RANGE | BASIC RELATIONSHIPS |
| :---: | :---: | :---: | :---: |
| $\sinh ^{-1} x$ | $(-\infty,+\infty)$ | $(-\infty,+\infty)$ | $\begin{array}{lll} \sinh ^{-1}(\sinh x)=x & \text { if } & -\infty<x<+\infty \\ \sinh \left(\sinh ^{-1} x\right)=x & \text { if } & -\infty<x<+\infty \end{array}$ |
| $\cosh ^{-1} x$ | $[1,+\infty)$ | $[0,+\infty)$ | $\begin{array}{lll} \cosh ^{-1}(\cosh x)=x & \text { if } & x \geq 0 \\ \cosh \left(\cosh ^{-1} x\right)=x & \text { if } & x \geq 1 \end{array}$ |
| $\tanh ^{-1} x$ | $(-1,1)$ | $(-\infty,+\infty)$ | $\begin{array}{lll} \tanh ^{-1}(\tanh x)=x & \text { if } & -\infty<x<+\infty \\ \tanh \left(\tanh ^{-1} x\right)=x & \text { if } & -1<x<1 \end{array}$ |
| $\operatorname{coth}^{-1} x$ | $(-\infty,-1) \cup(1,+\infty)$ | $(-\infty, 0) \cup(0,+\infty)$ | $\begin{array}{lll} \operatorname{coth}^{-1}(\operatorname{coth} x)=x & \text { if } & x<0 \text { or } x>0 \\ \operatorname{coth}\left(\operatorname{coth}^{-1} x\right)=x & \text { if } & x<-1 \text { or } x>1 \end{array}$ |
| $\operatorname{sech}^{-1} x$ | (0, 1] | $[0,+\infty)$ | $\begin{array}{lll} \operatorname{sech}^{-1}(\operatorname{sech} x)=x & \text { if } & x \geq 0 \\ \operatorname{sech}\left(\operatorname{sech}^{-1} x\right)=x & \text { if } & 0<x \leq 1 \end{array}$ |
| $\operatorname{csch}^{-1} x$ | $(-\infty, 0) \cup(0,+\infty)$ | $(-\infty, 0) \cup(0,+\infty)$ | $\begin{array}{lll} \operatorname{csch}^{-1}(\operatorname{csch} x)=x & \text { if } & x<0 \text { or } x>0 \\ \operatorname{csch}\left(\operatorname{csch}^{-1} x\right)=x & \text { if } & x<0 \text { or } x>0 \end{array}$ |

## LOGARITHMIC FORMS OF INVERSE HYPERBOLIC FUNCTIONS

Because the hyperbolic functions are expressible in terms of $e^{x}$, it should not be surprising that the inverse hyperbolic functions are expressible in terms of natural logarithms; the next theorem shows that this is so.
6.9.4 THEOREM The following relationships hold for all $x$ in the domains of the stated inverse hyperbolic functions:

$$
\begin{array}{ll}
\sinh ^{-1} x=\ln \left(x+\sqrt{x^{2}+1}\right) & \cosh ^{-1} x=\ln \left(x+\sqrt{x^{2}-1}\right) \\
\tanh ^{-1} x=\frac{1}{2} \ln \left(\frac{1+x}{1-x}\right) & \operatorname{coth}^{-1} x=\frac{1}{2} \ln \left(\frac{x+1}{x-1}\right) \\
\operatorname{sech}^{-1} x=\ln \left(\frac{1+\sqrt{1-x^{2}}}{x}\right) & \operatorname{csch}^{-1} x=\ln \left(\frac{1}{x}+\frac{\sqrt{1+x^{2}}}{|x|}\right)
\end{array}
$$

We will show how to derive the first formula in this theorem and leave the rest as exercises. The basic idea is to write the equation $x=\sinh y$ in terms of exponential functions and solve this equation for $y$ as a function of $x$. This will produce the equation $y=\sinh ^{-1} x$ with $\sinh ^{-1} x$ expressed in terms of natural logarithms. Expressing $x=\sinh y$ in terms of exponentials yields

$$
x=\sinh y=\frac{e^{y}-e^{-y}}{2}
$$

which can be rewritten as

$$
e^{y}-2 x-e^{-y}=0
$$

Multiplying this equation through by $e^{y}$ we obtain

$$
e^{2 y}-2 x e^{y}-1=0
$$

and applying the quadratic formula yields

$$
e^{y}=\frac{2 x \pm \sqrt{4 x^{2}+4}}{2}=x \pm \sqrt{x^{2}+1}
$$

Since $e^{y}>0$, the solution involving the minus sign is extraneous and must be discarded. Thus,

$$
e^{y}=x+\sqrt{x^{2}+1}
$$

Taking natural logarithms yields

$$
y=\ln \left(x+\sqrt{x^{2}+1}\right) \quad \text { or } \quad \sinh ^{-1} x=\ln \left(x+\sqrt{x^{2}+1}\right)
$$

## - Example 5

$$
\begin{aligned}
& \sinh ^{-1} 1=\ln \left(1+\sqrt{1^{2}+1}\right)=\ln (1+\sqrt{2}) \approx 0.8814 \\
& \tanh ^{-1}\left(\frac{1}{2}\right)=\frac{1}{2} \ln \left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)=\frac{1}{2} \ln 3 \approx 0.5493
\end{aligned}
$$

## DERIVATIVES AND INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

Formulas for the derivatives of the inverse hyperbolic functions can be obtained from Theorem 6.9.4. For example,

$$
\begin{aligned}
\frac{d}{d x}\left[\sinh ^{-1} x\right] & =\frac{d}{d x}\left[\ln \left(x+\sqrt{x^{2}+1}\right)\right]=\frac{1}{x+\sqrt{x^{2}+1}}\left(1+\frac{x}{\sqrt{x^{2}+1}}\right) \\
& =\frac{\sqrt{x^{2}+1}+x}{\left(x+\sqrt{x^{2}+1}\right)\left(\sqrt{x^{2}+1}\right)}=\frac{1}{\sqrt{x^{2}+1}}
\end{aligned}
$$

This computation leads to two integral formulas, a formula that involves $\sinh ^{-1} x$ and an equivalent formula that involves logarithms:

$$
\int \frac{d x}{\sqrt{x^{2}+1}}=\sinh ^{-1} x+C=\ln \left(x+\sqrt{x^{2}+1}\right)+C
$$

The following two theorems list the generalized derivative formulas and corresponding integration formulas for the inverse hyperbolic functions. Some of the proofs appear as exercises.

### 6.9.5 THEOREM

$$
\begin{aligned}
\frac{d}{d x}\left(\sinh ^{-1} u\right) & =\frac{1}{\sqrt{1+u^{2}}} \frac{d u}{d x} & \frac{d}{d x}\left(\operatorname{coth}^{-1} u\right)=\frac{1}{1-u^{2}} \frac{d u}{d x}, \quad|u|>1 \\
\frac{d}{d x}\left(\cosh ^{-1} u\right) & =\frac{1}{\sqrt{u^{2}-1}} \frac{d u}{d x}, \quad u>1 & \frac{d}{d x}\left(\operatorname{sech}^{-1} u\right)=-\frac{1}{u \sqrt{1-u^{2}}} \frac{d u}{d x}, \quad 0<u<1 \\
\frac{d}{d x}\left(\tanh ^{-1} u\right) & =\frac{1}{1-u^{2}} \frac{d u}{d x}, \quad|u|<1 & \frac{d}{d x}\left(\operatorname{csch}^{-1} u\right)=-\frac{1}{|u| \sqrt{1+u^{2}}} \frac{d u}{d x}, \quad u \neq 0
\end{aligned}
$$

$$
\begin{aligned}
& \text { 6.9.6 THEOREM If } a>0 \text {, then } \\
& \int \frac{d u}{\sqrt{a^{2}+u^{2}}}=\sinh ^{-1}\left(\frac{u}{a}\right)+C \text { or } \ln \left(u+\sqrt{u^{2}+a^{2}}\right)+C \\
& \int \frac{d u}{\sqrt{u^{2}-a^{2}}}=\cosh ^{-1}\left(\frac{u}{a}\right)+C \text { or } \ln \left(u+\sqrt{u^{2}-a^{2}}\right)+C, \quad u>a \\
& \int \frac{d u}{a^{2}-u^{2}}=\left\{\begin{array}{l}
\frac{1}{a} \tanh ^{-1}\left(\frac{u}{a}\right)+C, \quad|u|<a \\
\frac{1}{a} \operatorname{coth}^{-1}\left(\frac{u}{a}\right)+C, \quad|u|>a \\
\int \frac{d u}{u \sqrt{a^{2}-u^{2}}}=-\frac{1}{a} \operatorname{sech}^{-1}\left|\frac{1}{a}\right|+C \text { or }-\frac{1}{a} \ln \left|\frac{a+u}{a-u}\right|+C, \quad|u| \neq a \\
\int \frac{d u}{u \sqrt{a^{2}+u^{2}}}=-\frac{1}{a} \operatorname{csch}^{-1}\left|\frac{u}{a}\right|+C \text { or }-\frac{1}{a} \ln \left(\frac{a+\sqrt{a^{2}-u^{2}}}{|u|}\right)+C, \quad 0<|u|<a \\
|u|
\end{array}\right)+C, \quad u \neq 0
\end{aligned}
$$

Example 6 Evaluate $\int \frac{d x}{\sqrt{4 x^{2}-9}}, x>\frac{3}{2}$.
Solution. Let $u=2 x$. Thus, $d u=2 d x$ and

$$
\begin{aligned}
\int \frac{d x}{\sqrt{4 x^{2}-9}} & =\frac{1}{2} \int \frac{2 d x}{\sqrt{4 x^{2}-9}}=\frac{1}{2} \int \frac{d u}{\sqrt{u^{2}-3^{2}}} \\
& =\frac{1}{2} \cosh ^{-1}\left(\frac{u}{3}\right)+C=\frac{1}{2} \cosh ^{-1}\left(\frac{2 x}{3}\right)+C
\end{aligned}
$$

Alternatively, we can use the logarithmic equivalent of $\cosh ^{-1}(2 x / 3)$,

$$
\cosh ^{-1}\left(\frac{2 x}{3}\right)=\ln \left(2 x+\sqrt{4 x^{2}-9}\right)-\ln 3
$$

(verify), and express the answer as

$$
\int \frac{d x}{\sqrt{4 x^{2}-9}}=\frac{1}{2} \ln \left(2 x+\sqrt{4 x^{2}-9}\right)+C
$$

## QUICK CHECK EXERCISES 6.9 (See page 485 for answers.)

1. $\cosh x=$ $\qquad$ $\sinh x=$ $\qquad$ $\tanh x=$ $\qquad$
2. Complete the table.

|  | $\cosh x$ | $\sinh x$ | $\tanh x$ | $\operatorname{coth} x$ | $\operatorname{sech} x$ | $\operatorname{csch} x$ |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- |
| DOMAIN |  |  |  |  |  |  |
| RANGE |  |  |  |  |  |  |

3. The parametric equations

$$
x=\cosh t, \quad y=\sinh t \quad(-\infty<t<+\infty)
$$

represent the right half of the curve called a $\qquad$ Eliminating the parameter, the equation of this curve is $\qquad$
4. $\frac{d}{d x}[\cosh x]=$ $\qquad$ $\frac{d}{d x}[\tanh x]=$ $\qquad$ $\frac{d}{d x}[\sinh x]=$ $\qquad$
5. $\int \cosh x d x=\int \sinh x d x=$ $\qquad$ $\int \tanh x d x=$
6. $\frac{d}{d x}\left[\cosh ^{-1} x\right]=\square \frac{d}{d x}\left[\sinh ^{-1} x\right]=$ $\qquad$ $\frac{d}{d x}\left[\tanh ^{-1} x\right]=$

## EXERCISE SET 6.9 ~ Graphing Utility

1-2 Approximate the expression to four decimal places.

1. (a) $\sinh 3$
(b) $\cosh (-2)$
(c) $\tanh (\ln 4)$
(d) $\sinh ^{-1}(-2)$
(e) $\cosh ^{-1} 3$
(f) $\tanh ^{-1} \frac{3}{4}$
2. (a) $\operatorname{csch}(-1)$
(b) $\operatorname{sech}(\ln 2)$
(c) coth 1
(d) $\operatorname{sech}^{-1} \frac{1}{2}$
(e) $\operatorname{coth}^{-1} 3$
(f) $\operatorname{csch}^{-1}(-\sqrt{3})$
3. Find the exact numerical value of each expression.
(a) $\sinh (\ln 3)$
(b) $\cosh (-\ln 2)$
(c) $\tanh (2 \ln 5)$
(d) $\sinh (-3 \ln 2)$
4. In each part, rewrite the expression as a ratio of polynomials.
(a) $\cosh (\ln x)$
(b) $\sinh (\ln x)$
(c) $\tanh (2 \ln x)$
(d) $\cosh (-\ln x)$
5. In each part, a value for one of the hyperbolic functions is given at an unspecified positive number $x_{0}$. Use appropri-
ate identities to find the exact values of the remaining five hyperbolic functions at $x_{0}$.
(a) $\sinh x_{0}=2$
(b) $\cosh x_{0}=\frac{5}{4}$
(c) $\tanh x_{0}=\frac{4}{5}$
6. Obtain the derivative formulas for $\operatorname{csch} x$, $\operatorname{sech} x$, and $\operatorname{coth} x$ from the derivative formulas for $\sinh x, \cosh x$, and $\tanh x$.
7. Find the derivatives of $\cosh ^{-1} x$ and $\tanh ^{-1} x$ by differentiating the formulas in Theorem 6.9.4.
8. Find the derivatives of $\sinh ^{-1} x, \cosh ^{-1} x$, and $\tanh ^{-1} x$ by differentiating the equations $x=\sinh y, x=\cosh y$, and $x=\tanh y$ implicitly.

9-28 Find $d y / d x$.
9. $y=\sinh (4 x-8)$
10. $y=\cosh \left(x^{4}\right)$
11. $y=\operatorname{coth}(\ln x)$
12. $y=\ln (\tanh 2 x)$
13. $y=\operatorname{csch}(1 / x)$
14. $y=\operatorname{sech}\left(e^{2 x}\right)$
15. $y=\sqrt{4 x+\cosh ^{2}(5 x)}$
16. $y=\sinh ^{3}(2 x)$
17. $y=x^{3} \tanh ^{2}(\sqrt{x})$
19. $y=\sinh ^{-1}\left(\frac{1}{3} x\right)$
18. $y=\sinh (\cos 3 x)$
21. $y=\ln \left(\cosh ^{-1} x\right)$
20. $y=\sinh ^{-1}(1 / x)$
21. $y=\ln \left(\cosh ^{-1} x\right)$
22. $y=\cosh ^{-1}\left(\sinh ^{-1} x\right)$
23. $y=\frac{1}{\tanh ^{-1} x}$
24. $y=\left(\operatorname{coth}^{-1} x\right)^{2}$
25. $y=\cosh ^{-1}(\cosh x)$
26. $y=\sinh ^{-1}(\tanh x)$
27. $y=e^{x} \operatorname{sech}^{-1} \sqrt{x}$
28. $y=\left(1+x \operatorname{csch}^{-1} x\right)^{10}$

29-44 Evaluate the integrals.
29. $\int \sinh ^{6} x \cosh x d x$
30. $\int \cosh (2 x-3) d x$
31. $\int \sqrt{\tanh x} \operatorname{sech}^{2} x d x$
32. $\int \operatorname{csch}^{2}(3 x) d x$
33. $\int \tanh 2 x d x$
34. $\int \operatorname{coth}^{2} x \operatorname{csch}^{2} x d x$
35. $\int_{\ln 2}^{\ln 3} \tanh x \operatorname{sech}^{3} x d x$
36. $\int_{0}^{\ln 3} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} d x$
37. $\int \frac{d x}{\sqrt{1+9 x^{2}}}$
38. $\int \frac{d x}{\sqrt{x^{2}-2}}(x>\sqrt{2})$
39. $\int \frac{d x}{\sqrt{1-e^{2 x}}} \quad(x<0)$
40. $\int \frac{\sin \theta d \theta}{\sqrt{1+\cos ^{2} \theta}}$
41. $\int \frac{d x}{x \sqrt{1+4 x^{2}}}$
42. $\int \frac{d x}{\sqrt{9 x^{2}-25}}(x>5 / 3)$
43. $\int_{0}^{1 / 2} \frac{d x}{1-x^{2}}$
44. $\int_{0}^{\sqrt{3}} \frac{d t}{\sqrt{t^{2}+1}}$

45-48 True-False Determine whether the statement is true or false. Explain your answer.
45. The equation $\cosh x=\sinh x$ has no solutions.
46. Exactly two of the hyperbolic functions are bounded.
47. There is exactly one hyperbolic function $f(x)$ such that for all real numbers $a$, the equation $f(x)=a$ has a unique solution $x$.
48. The identities in Theorem 6.9.2 may be obtained from the corresponding trigonometric identities by replacing each trigonometric function with its hyperbolic analogue.
49. Find the area enclosed by $y=\sinh 2 x, y=0$, and $x=\ln 3$.
50. Find the volume of the solid that is generated when the region enclosed by $y=\operatorname{sech} x, y=0, x=0$, and $x=\ln 2$ is revolved about the $x$-axis.
51. Find the volume of the solid that is generated when the region enclosed by $y=\cosh 2 x, y=\sinh 2 x, x=0$, and $x=5$ is revolved about the $x$-axis.
52. Approximate the positive value of the constant $a$ such that the area enclosed by $y=\cosh a x, y=0, x=0$, and $x=1$
is 2 square units. Express your answer to at least five decimal places.
53. Find the arc length of the catenary $y=\cosh x$ between $x=0$ and $x=\ln 2$.
54. Find the arc length of the catenary $y=a \cosh (x / a)$ between $x=0$ and $x=x_{1}\left(x_{1}>0\right)$.
55. In parts (a)-(f) find the limits, and confirm that they are consistent with the graphs in Figures 6.9.1 and 6.9.6.
(a) $\lim _{x \rightarrow+\infty} \sinh x$
(b) $\lim _{x \rightarrow-\infty} \sinh x$
(c) $\lim _{x \rightarrow+\infty} \tanh x$
(d) $\lim _{x \rightarrow-\infty} \tanh x$
(e) $\lim _{x \rightarrow+\infty} \sinh ^{-1} x$
(f) $\lim _{x \rightarrow 1^{-}} \tanh ^{-1} x$

## FOCUS ON CONCEPTS

56. Explain how to obtain the asymptotes for $y=\tanh x$ from the curvilinear asymptotes for $y=\cosh x$ and $y=\sinh x$.
57. Prove that $\sinh x$ is an odd function of $x$ and that $\cosh x$ is an even function of $x$, and check that this is consistent with the graphs in Figure 6.9.1.

58-59 Prove the identities.
58. (a) $\cosh x+\sinh x=e^{x}$
(b) $\cosh x-\sinh x=e^{-x}$
(c) $\sinh (x+y)=\sinh x \cosh y+\cosh x \sinh y$
(d) $\sinh 2 x=2 \sinh x \cosh x$
(e) $\cosh (x+y)=\cosh x \cosh y+\sinh x \sinh y$
(f) $\cosh 2 x=\cosh ^{2} x+\sinh ^{2} x$
(g) $\cosh 2 x=2 \sinh ^{2} x+1$
(h) $\cosh 2 x=2 \cosh ^{2} x-1$
59. (a) $1-\tanh ^{2} x=\operatorname{sech}^{2} x$
(b) $\tanh (x+y)=\frac{\tanh x+\tanh y}{1+\tanh x \tanh y}$
(c) $\tanh 2 x=\frac{2 \tanh x}{1+\tanh ^{2} x}$
60. Prove:
(a) $\cosh ^{-1} x=\ln \left(x+\sqrt{x^{2}-1}\right), \quad x \geq 1$
(b) $\tanh ^{-1} x=\frac{1}{2} \ln \left(\frac{1+x}{1-x}\right), \quad-1<x<1$.
61. Use Exercise 60 to obtain the derivative formulas for $\cosh ^{-1} x$ and $\tanh ^{-1} x$.
62. Prove:

$$
\begin{array}{ll}
\operatorname{sech}^{-1} x=\cosh ^{-1}(1 / x), & 0<x \leq 1 \\
\operatorname{coth}^{-1} x=\tanh ^{-1}(1 / x), & |x|>1 \\
\operatorname{csch}^{-1} x=\sinh ^{-1}(1 / x), & x \neq 0
\end{array}
$$

63. Use Exercise 62 to express the integral

$$
\int \frac{d u}{1-u^{2}}
$$

entirely in terms of $\tanh ^{-1}$.
64. Show that
(a) $\frac{d}{d x}\left[\operatorname{sech}^{-1}|x|\right]=-\frac{1}{x \sqrt{1-x^{2}}}$
(b) $\frac{d}{d x}\left[\operatorname{csch}^{-1}|x|\right]=-\frac{1}{x \sqrt{1+x^{2}}}$.
65. In each part, find the limit.
(a) $\lim _{x \rightarrow+\infty}\left(\cosh ^{-1} x-\ln x\right)$
(b) $\lim _{x \rightarrow+\infty} \frac{\cosh x}{e^{x}}$
66. Use the first and second derivatives to show that the graph of $y=\tanh ^{-1} x$ is always increasing and has an inflection point at the origin.
67. The integration formulas for $1 / \sqrt{u^{2}-a^{2}}$ in Theorem 6.9.6 are valid for $u>a$. Show that the following formula is valid for $u<-a$ :
$\int \frac{d u}{\sqrt{u^{2}-a^{2}}}=-\cosh ^{-1}\left(-\frac{u}{a}\right)+C \quad$ or $\quad \ln \left|u+\sqrt{u^{2}-a^{2}}\right|+C$
68. Show that $(\sinh x+\cosh x)^{n}=\sinh n x+\cosh n x$.
69. Show that

$$
\int_{-a}^{a} e^{t x} d x=\frac{2 \sinh a t}{t}
$$

70. A cable is suspended between two poles as shown in Figure 6.9.2. Assume that the equation of the curve formed by the cable is $y=a \cosh (x / a)$, where $a$ is a positive constant. Suppose that the $x$-coordinates of the points of support are $x=-b$ and $x=b$, where $b>0$.
(a) Show that the length $L$ of the cable is given by

$$
L=2 a \sinh \frac{b}{a}
$$

(b) Show that the sag $S$ (the vertical distance between the highest and lowest points on the cable) is given by

$$
S=a \cosh \frac{b}{a}-a
$$

71-72 These exercises refer to the hanging cable described in Exercise 70.
71. Assuming that the poles are 400 ft apart and the sag in the cable is 30 ft , approximate the length of the cable by approximating $a$. Express your final answer to the nearest tenth of a foot. [Hint: First let $u=200 / a$.]72. Assuming that the cable is 120 ft long and the poles are 100 ft apart, approximate the sag in the cable by approximating $a$. Express your final answer to the nearest tenth of a foot. [Hint: First let $u=50 / a$.]
73. The design of the Gateway Arch in St. Louis, Missouri, by architect Eero Saarinan was implemented using equations provided by Dr. Hannskarl Badel. The equation used for the centerline of the arch was

$$
y=693.8597-68.7672 \cosh (0.0100333 x) \mathrm{ft}
$$

for $x$ between -299.2239 and 299.2239.
(a) Use a graphing utility to graph the centerline of the arch.
(b) Find the length of the centerline to four decimal places.
(c) For what values of $x$ is the height of the arch 100 ft ? Round your answers to four decimal places.
(d) Approximate, to the nearest degree, the acute angle that the tangent line to the centerline makes with the ground at the ends of the arch.
74. Suppose that a hollow tube rotates with a constant angular velocity of $\omega \mathrm{rad} / \mathrm{s}$ about a horizontal axis at one end of the tube, as shown in the accompanying figure. Assume that an object is free to slide without friction in the tube while the tube is rotating. Let $r$ be the distance from the object to the pivot point at time $t \geq 0$, and assume that the object is at rest and $r=0$ when $t=0$. It can be shown that if the tube is horizontal at time $t=0$ and rotating as shown in the figure, then

$$
r=\frac{g}{2 \omega^{2}}[\sinh (\omega t)-\sin (\omega t)]
$$

during the period that the object is in the tube. Assume that $t$ is in seconds and $r$ is in meters, and use $g=9.8 \mathrm{~m} / \mathrm{s}^{2}$ and $\omega=2 \mathrm{rad} / \mathrm{s}$.
(a) Graph $r$ versus $t$ for $0 \leq t \leq 1$.
(b) Assuming that the tube has a length of 1 m , approximately how long does it take for the object to reach the end of the tube?
(c) Use the result of part (b) to approximate $d r / d t$ at the instant that the object reaches the end of the tube.


4Figure Ex-74
75. The accompanying figure (on the next page) shows a person pulling a boat by holding a rope of length $a$ attached to the bow and walking along the edge of a dock. If we assume that the rope is always tangent to the curve traced by the bow of the boat, then this curve, which is called a tractrix, has the property that the segment of the tangent line between the curve and the $y$-axis has a constant length $a$. It can be proved that the equation of this tractrix is

$$
y=a \operatorname{sech}^{-1} \frac{x}{a}-\sqrt{a^{2}-x^{2}}
$$

(a) Show that to move the bow of the boat to a point $(x, y)$, the person must walk a distance

$$
D=a \operatorname{sech}^{-1} \frac{x}{a}
$$

from the origin.
(b) If the rope has a length of 15 m , how far must the person walk from the origin to bring the boat 10 m from the dock? Round your answer to two decimal places.
(c) Find the distance traveled by the bow along the tractrix as it moves from its initial position to the point where it is 5 m from the dock.

76. Writing Suppose that, by analogy with the trigonometric functions, we define $\cosh t$ and $\sinh t$ geometrically using Figure 6.9.3b:
"For any real number $t$, define $x=\cosh t$ and $y=\sinh t$ to be the unique values of $x$ and $y$ such that
(i) $P(x, y)$ is on the right branch of the unit hyperbola $x^{2}-y^{2}=1$
(ii) $t$ and $y$ have the same sign (or are both 0 );
(iii) the area of the region bounded by the $x$-axis, the right branch of the unit hyperbola, and the segment from the origin to $P$ is $|t| / 2$."
Discuss what properties would first need to be verified in order for this to be a legitimate definition.
77. Writing Investigate what properties of $\cosh t$ and $\sinh t$ can be proved directly from the geometric definition in Exercise 76. Write a short description of the results of your investigation.

## QUICK CHECK ANSWERS 6.9

1. $\frac{e^{x}+e^{-x}}{2} ; \frac{e^{x}-e^{-x}}{2} ; \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$
2. 

|  | $\cosh x$ | $\sinh x$ | $\tanh x$ | $\operatorname{coth} x$ | $\operatorname{sech} x$ | $\operatorname{csch} x$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| DOMAIN | $(-\infty,+\infty)$ | $(-\infty,+\infty)$ | $(-\infty,+\infty)$ | $(-\infty, 0) \cup(0,+\infty)$ | $(-\infty,+\infty)$ | $(-\infty, 0) \cup(0,+\infty)$ |
| RANGE | $[1,+\infty)$ | $(-\infty,+\infty)$ | $(-1,1)$ | $(-\infty,-1) \cup(1,+\infty)$ | $(0,1]$ | $(-\infty, 0) \cup(0,+\infty)$ |

3. unit hyperbola; $x^{2}-y^{2}=1 \quad$ 4. $\sinh x ; \cosh x ; \operatorname{sech}^{2} x \quad$ 5. $\sinh x+C ; \cosh x+C ; \ln (\cosh x)+C$
4. $\frac{1}{\sqrt{x^{2}-1}} ; \frac{1}{\sqrt{1+x^{2}}} ; \frac{1}{1-x^{2}}$

## CHAPTER 6 REVIEW EXERCISES

1. Describe the method of slicing for finding volumes, and use that method to derive an integral formula for finding volumes by the method of disks.
2. State an integral formula for finding a volume by the method of cylindrical shells, and use Riemann sums to derive the formula.
3. State an integral formula for finding the arc length of a smooth curve $y=f(x)$ over an interval $[a, b]$, and use Riemann sums to derive the formula.
4. State an integral formula for the work $W$ done by a variable force $F(x)$ applied in the direction of motion to an object moving from $x=a$ to $x=b$, and use Riemann sums to derive the formula.
5. State an integral formula for the fluid force $F$ exerted on a vertical flat surface immersed in a fluid of weight density $\rho$, and use Riemann sums to derive the formula.
6. Let $R$ be the region in the first quadrant enclosed by $y=x^{2}$, $y=2+x$, and $x=0$. In each part, set up, but do not eval-
uate, an integral or a sum of integrals that will solve the problem.
(a) Find the area of $R$ by integrating with respect to $x$.
(b) Find the area of $R$ by integrating with respect to $y$.
(c) Find the volume of the solid generated by revolving $R$ about the $x$-axis by integrating with respect to $x$.
(d) Find the volume of the solid generated by revolving $R$ about the $x$-axis by integrating with respect to $y$.
(e) Find the volume of the solid generated by revolving $R$ about the $y$-axis by integrating with respect to $x$.
(f) Find the volume of the solid generated by revolving $R$ about the $y$-axis by integrating with respect to $y$.
(g) Find the volume of the solid generated by revolving $R$ about the line $y=-3$ by integrating with respect to $x$.
(h) Find the volume of the solid generated by revolving $R$ about the line $x=5$ by integrating with respect to $x$.
7. (a) Set up a sum of definite integrals that represents the total shaded area between the curves $y=f(x)$ and $y=g(x)$ in the accompanying figure on the next page. (cont.)
(b) Find the total area enclosed between $y=x^{3}$ and $y=x$ over the interval $[-1,2]$.


4Figure Ex-7
8. The accompanying figure shows velocity versus time curves for two cars that move along a straight track, accelerating from rest at a common starting line.
(a) How far apart are the cars after 60 seconds?
(b) How far apart are the cars after $T$ seconds, where $0 \leq T \leq 60$ ?


4 Figure Ex-8
9. Let $R$ be the region enclosed by the curves $y=x^{2}+4$, $y=x^{3}$, and the $y$-axis. Find and evaluate a definite integral that represents the volume of the solid generated by revolving $R$ about the $x$-axis.
10. A football has the shape of the solid generated by revolving the region bounded between the $x$-axis and the parabola $y=4 R\left(x^{2}-\frac{1}{4} L^{2}\right) / L^{2}$ about the $x$-axis. Find its volume.
11. Find the volume of the solid whose base is the region bounded between the curves $y=\sqrt{x}$ and $y=1 / \sqrt{x}$ for $1 \leq x \leq 4$ and whose cross sections perpendicular to the $x$-axis are squares.
12. Consider the region enclosed by $y=\sin ^{-1} x, y=0$, and $x=1$. Set up, but do not evaluate, an integral that represents the volume of the solid generated by revolving the region about the $x$-axis using
(a) disks
(b) cylindrical shells.
13. Find the arc length in the second quadrant of the curve $x^{2 / 3}+y^{2 / 3}=4$ from $x=-8$ to $x=-1$.
14. Let $C$ be the curve $y=e^{x}$ between $x=0$ and $x=\ln 10$. In each part, set up, but do not evaluate, an integral that solves the problem.
(a) Find the arc length of $C$ by integrating with respect to $x$.
(b) Find the arc length of $C$ by integrating with respect to $y$.
15. Find the area of the surface generated by revolving the curve $y=\sqrt{25-x}, 9 \leq x \leq 16$, about the $x$-axis.
16. Let $C$ be the curve $27 x-y^{3}=0$ between $y=0$ and $y=2$. In each part, set up, but do not evaluate, an integral or a sum of integrals that solves the problem.
(a) Find the area of the surface generated by revolving $C$ about the $x$-axis by integrating with respect to $x$.
(b) Find the area of the surface generated by revolving $C$ about the $y$-axis by integrating with respect to $y$.
(c) Find the area of the surface generated by revolving $C$ about the line $y=-2$ by integrating with respect to $y$.
17. (a) A spring exerts a force of 0.5 N when stretched 0.25 m beyond its natural length. Assuming that Hooke's law applies, how much work was performed in stretching the spring to this length?
(b) How far beyond its natural length can the spring be stretched with 25 J of work?
18. A boat is anchored so that the anchor is 150 ft below the surface of the water. In the water, the anchor weighs 2000 lb and the chain weighs $30 \mathrm{lb} / \mathrm{ft}$. How much work is required to raise the anchor to the surface?

19-20 Find the centroid of the region.
19. The region bounded by $y^{2}=4 x$ and $y^{2}=8(x-2)$.
20. The upper half of the ellipse $(x / a)^{2}+(y / b)^{2}=1$.
21. In each part, set up, but do not evaluate, an integral that solves the problem.
(a) Find the fluid force exerted on a side of a box that has a 3 m square base and is filled to a depth of 1 m with a liquid of weight density $\rho \mathrm{N} / \mathrm{m}^{3}$.
(b) Find the fluid force exerted by a liquid of weight density $\rho \mathrm{lb} / \mathrm{ft}^{3}$ on a face of the vertical plate shown in part (a) of the accompanying figure.
(c) Find the fluid force exerted on the parabolic dam in part (b) of the accompanying figure by water that extends to the top of the dam.

(a)

(b)

## $\triangle$ Figure Ex-21

22. Show that for any constant $a$, the function $y=\sinh (a x)$ satisfies the equation $y^{\prime \prime}=a^{2} y$.
23. In each part, prove the identity.
(a) $\cosh 3 x=4 \cosh ^{3} x-3 \cosh x$
(b) $\cosh \frac{1}{2} x=\sqrt{\frac{1}{2}(\cosh x+1)}$
(c) $\sinh \frac{1}{2} x= \pm \sqrt{\frac{1}{2}(\cosh x-1)}$

[^0]:    *This notation was devised by Leibniz. In his early papers Leibniz used the notation "omn." (an abbreviation for the Latin word "omnes") to denote integration. Then on October 29, 1675 he wrote, "It will be useful to write $\int$ for omn., thus $\int l$ for omn. $l \ldots$." Two or three weeks later he refined the notation further and wrote $\int[] d x$ rather than $\int$ alone. This notation is so useful and so powerful that its development by Leibniz must be regarded as a major milestone in the history of mathematics and science.

[^1]:    *Strictly speaking, the constant $g$ varies with the latitude and the distance from the Earth's center. However, for motion at a fixed latitude and near the surface of the Earth, the assumption of a constant $g$ is satisfactory for many applications.

[^2]:    A Figure 5.7.8

[^3]:    

[^4]:    

