Multiple Choice Practice

1. Given the series $\sum_{n=1}^{\infty} \frac{a_n 5^n}{n!}$, which of the following could be the correct set-up and conclusion for the ratio test?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}5^{n+1}}{(n+1)!}\cdot\frac{n!}{a_n 5^n}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}\cdot\frac{5a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}\right|\cdot\lim_{n\to\infty}\left|\frac{5a_{n+1}}{a_n}\right| = (0)\lim_{n\to\infty}\left|\frac{5a_{n+1}}{a_n}\right| = 0 < 1 \Rightarrow \text{converges}$$

(A) $\lim_{n\to\infty}\left|\frac{a_n 5^n}{n!}\cdot\frac{(n+1)!}{a_{n+1}5^{n+1}}\right| < 1$; the series converges. It is the reciprocal so wrong conclusion.

(B) $\lim_{n\to\infty}\left|\frac{a_{n+1}5^{n+1}}{(n+1)!}\cdot\frac{n!}{a_n 5^n}\right| < 1$; the series diverges. Correct set-up. Wrong conclusion.

(C) $\lim_{n\to\infty}\left|\frac{a_{n+1}5^{n+1}}{(n+1)!}\cdot\frac{n!}{a_n 5^n}\right| > 1$; the series converges. Correct set-up. Wrong conclusion.

(D) $\lim_{n\to\infty}\left|\frac{a_{n+1}5^{n+1}}{(n+1)!}\cdot\frac{n!}{a_n 5^n}\right| > 1$; the series diverges. Correct set-up and conclusion.

2. Level: AP2

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with $a_n = f(n)$. Which of the following conditions must be met to apply the integral test?

(A) $f(x)$ is differentiable Does not have to be differentiable, just continuous.

(B) $f(x)$ is positive

(C) a_n is continuous This a set of discrete values, cannot be continuous.

(D) a_n is decreasing This a set of discrete values need to be decreasing and going to zero.

3. Level: AP2

Given the series $\sum_{n=1}^{\infty}(-1)^n a_n$, which of the following conditions must be met in order to use the Alternating Series Test?

I. $a_{n+1} \le a_n$ II. $a_n > 0$ III. $\lim_{n\to\infty} a_n = 0$

(A) I only

(B) I and III only

(C) III only

(D) I, II, and III

$\sum_{n=1}^{\infty}(-1)^n a_n$ assumes that a_n is positive.

4. Level: AP2

Given the series $\sum_{n=1}^{\infty}\frac{n^2+3n-2}{n^3-n^2+1}$, which of the following series can be used as a comparison for the Limit Comparison test.

(A) $\sum_{n=1}^{\infty}\frac{1}{n^3}$

(B) $\sum_{n=1}^{\infty}\frac{1}{n^2}$

(C) $\sum_{n=1}^{\infty} n$

(D) $\sum_{n=1}^{\infty}\frac{1}{n}$

For large n, $\frac{n^2+3n-2}{n^3-n^2+1}$ behaves like $\frac{n^2}{n^3}=\frac{1}{n}$

5. Level: AP2

Which of the following infinite series diverges by the n-th term test?

(A) $\sum_{n=1}^{\infty}\ln(n)$ $\lim_{n\to\infty}\ln(n)=\infty$

(B) $\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ $\lim_{n\to\infty}\frac{\ln(n)}{n}=0$ comparing magnitudes. n dominates $\ln(n)$

(C) $\sum_{n=1}^{\infty}\frac{n^3}{2^n}$ $\lim_{n\to\infty}\frac{n^3}{2^n}=0$ comparing magnitudes. Exponential dominates power.

(D) $\sum_{n=1}^{\infty}\frac{3n^2+n-4}{n^3+2n}$ $\lim_{n\to\infty}\frac{3n^2+n-4}{n^3+2n}=0$ comparing magnitudes. n^3 dominates n^2

6. Level: AP3

Which of the following converge?

(A) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ — divergent p-series with $p = \frac{1}{2} < 1$

(B) $\sum_{n=1}^{\infty} \frac{-1}{n}$ — divergent Harmonic series p-series with $p = 1$

(C) $\sum_{n=1}^{\infty} \frac{n^{-2}}{\sqrt[3]{n}}$ — $\frac{n^{-2}}{\sqrt[3]{n}} = \frac{1}{n^2\sqrt[3]{n}} = \frac{1}{n^{\frac{7}{3}}}$ convergent p-series with $p > 1$

(D) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^4}}{6n^2}$ — $\frac{\sqrt[3]{n^4}}{n^2} = \frac{n^{\frac{4}{3}}}{n^2} = \frac{1}{n^{\frac{2}{3}}}$ divergent p-series with $p < 1$

7. Level: AP3

$\sum_{n=1}^{\infty} 4\left(\frac{2}{3}\right)^{n-1}$ is

(A) 6

(B) 8

(C) 12

(D) divergent

$$\sum_{n=1}^{\infty} 4\left(\frac{2}{3}\right)^{n-1} \Rightarrow a = \underbrace{4\left(\frac{2}{3}\right)^{1-1}}_{\text{first term}} = 4 \Rightarrow S = \frac{4}{1-\frac{2}{3}} = \frac{12}{3-2} = 12$$

convergent geometric series with $0 < r < 1$.

8. Level: AP3

Which of the following statements about the series $\sum_{n=3}^{\infty} \frac{n^2 - 2}{n^3 + 4n^2 + 1}$ is true?

(A) The series converges by the limit comparison test with $\sum_{n=3}^{\infty} \frac{1}{n}$.

(B) The series converges by the limit comparison test with $\sum_{n=3}^{\infty} \frac{1}{n^3}$.

(C) The series diverges by the limit comparison test with $\sum_{n=3}^{\infty} \frac{1}{n}$.

(D) The series diverges by the limit comparison test with $\sum_{n=3}^{\infty} \frac{1}{n^3}$.

(A) $\sum_{n=3}^{\infty} \frac{1}{n}$ is a divergent Harmonic series, shows divergence.

(B) & (D) $\lim_{n\to\infty} \frac{\frac{n^2 - 2}{n^3 + 4n^2 + 1}}{1/n^3} = \lim_{n\to\infty} \frac{(n^2 - 2)n^3}{n^3 + 4n^2 + 1} = \lim_{n\to\infty} \frac{n^5 - 2n^3}{n^3 + 4n^2 + 1} = \infty$

$\Rightarrow$ Cannot apply test.

(C) $\lim_{n\to\infty} \frac{\frac{n^2 - 2}{n^3 + 4n^2 + 1}}{1/n} = \lim_{n\to\infty} \frac{(n^2 - 2)n}{n^3 + 4n^2 + 1} = \lim_{n\to\infty} \frac{n^3 - 2n}{n^3 + 4n^2 + 1} = 1$

See (A) $\Rightarrow$ Both diverge.

9. Level: AP3

Which of the following statements describes the series $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+3}$ using the alternating series test?

(A) converges absolutely

(B) converges conditionally

(C) diverges

(D) alternating series test is inconclusive

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+3} = 0$ by comparing magnitudes, might converge.

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+3} = 0$ by comparing magnitudes, might converge.

$\sum_{n=1}^{\infty}\left|(-1)^n \frac{\sqrt{n}}{n+3}\right| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+3}$ does not converge because for large n it behaves like $\frac{\sqrt{n}}{n} = n^{\frac{1}{2}}$,

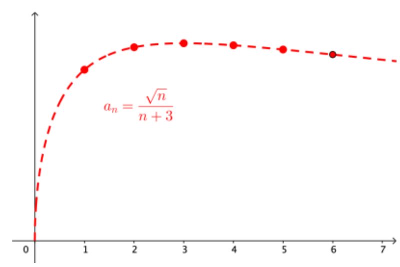
because for large n it behaves like $\frac{\sqrt{n}}{n} = n^{\frac{1}{2}}$, the p-series with $\Rightarrow p = \frac{1}{2} \Rightarrow$ Not absolutely convergent

$\frac{\sqrt{n}}{n+3} > 0 \qquad \frac{d}{dx}\left(\frac{\sqrt{n}}{n+3}\right) = \frac{(n+3)\left(\frac{1}{2\sqrt{n}}\right) - (1)\sqrt{n}}{(n+3)^2} = \frac{(n+3) - 2\left(\sqrt{n}\right)^2}{2\sqrt{n}(n+3)^2} = \frac{(n+3) - 2n}{2\sqrt{n}(n+3)^2}$

$= \frac{3-n}{2\sqrt{n}(n+3)^2} < 0$ when $n > 3 \Rightarrow \frac{\sqrt{n}}{n+3}$ is decreasing or $\frac{\sqrt{n+1}}{n+4} < \frac{\sqrt{n}}{n+3}$.

$\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+3}$ converges by the AST.

$\frac{\sqrt{n}}{n+3}$ is decreasing for $n > 3$ Look at the graph.



10. Level: AP3

Which of the following series converge?

I. $\sum_{n=0}^{\infty} \frac{3}{2^n}$ II. $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ III. $\sum_{n=1}^{\infty} (-1)^n$

(A) I only

(B) II only

(C) I and II only

(D) I, II, and III

I. $\sum_{n=0}^{\infty} \frac{3}{2^n} = 3\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \Rightarrow$ convergent geometric series $r = \frac{1}{2} < 1$

II. $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}} \Rightarrow$ convergent p-series $p = \sqrt{2} > 1$

III. $\sum_{n=1}^{\infty} (-1)^n \Rightarrow$ divergent by nth term test.

11. Level: AP3

Let $\sum_{n=1}^{\infty} a_n$ be a series whose n^{th} partial sum is $S_n = \dfrac{2n+1}{5n-3}$. Which of the following statements is true?

(A) The series $\sum_{n=1}^{\infty} a_n$ diverges by the nth term test. We need a_n to do the nth term test.

(B) The series $\sum_{n=1}^{\infty} a_n$ diverges by the p-series test. We need a_n to do the p-series test.

(C) The series $\sum_{n=1}^{\infty} a_n$ converges because $\lim_{n\to\infty} S_n \neq 0$. We need a_n to do the nth term test, not S_n.

(D) The series $\sum_{n=1}^{\infty} a_n$ converges to $\dfrac{2}{5}$. $\lim_{n\to\infty} \dfrac{2n+1}{5n-3} = \dfrac{2}{5}$ $S = \lim_{n\to\infty} S_n$

12. Level: AP4

For which of the following values of x does the series $\sum_{n=1}^{\infty} \left(\dfrac{x-2}{3}\right)^{n-1}$ converge?

(A) $x = -1$

(B) $x = 1$

(C) $x = -5$

(D) $x = 5$

$\sum_{n=1}^{\infty} \left(\dfrac{x-2}{3}\right)^{n-1}$ is geometric with $r = \dfrac{x-2}{3}$

$|r| = \dfrac{1}{3}|x-2| < 1 \Rightarrow |x-2| < 3 \Rightarrow \boxed{-1 < x < 5}$

$x = 1$ is in this interval

13. Level: AP4

When applied to the series $\sum_{n=1}^{\infty} a_n$, the ratio test yields the result $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L$. Which of the following *could* be true?

(A) $L = \dfrac{1}{2}$ and $\sum_{n=1}^{\infty} a_n$ converges conditionally. $L = \dfrac{1}{2} < 1$ converges by the Ratio Test

(B) $L = \dfrac{2}{3}$ and $\sum_{n=1}^{\infty} a_n$ diverges. $L = \dfrac{2}{3} < 1$ converges by the Ratio Test

(C) $L = 1$ and $\sum_{n=1}^{\infty} a_n$ converges conditionally. $L = 1$ Ratio Test inconclusive

(D) $L = \dfrac{3}{2}$ and $\sum_{n=1}^{\infty} a_n$ converges absolutely. $L = \dfrac{3}{2} > 1$ diverges by the Ratio Test

14. Level: AP4

Which of the following series converge conditionally?

I. $\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n$ II. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)^{3/2}$ III. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2}{n}\right)$

(A) I and II only

(B) I and III only

(C) III only

(D) I, II, and III

I. $\sum_{n=0}^{\infty}\left|(-1)^n\left(\frac{2}{3}\right)^n\right| = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n \Rightarrow$ geometric series $|r| = \frac{2}{3} < 1$ absolutely convergent

II. $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\left(\frac{1}{n}\right)^{3/2}\right| = \sum_{n=1}^{\infty}\left(\frac{1}{n^{3/2}}\right) \Rightarrow$ p-series $p = \frac{3}{2} > 1 \Rightarrow$ absolutely convergent

III. $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\left(\frac{2}{n}\right)\right| = \sum_{n=1}^{\infty}\left(\frac{2}{n}\right) \Rightarrow$ divergent Harmonic series

$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2}{n}\right) \Rightarrow$ convergent by AST $\Rightarrow$ conditionally convergent

15. Level: AP3

Which of the following geometric series converge?

I. $\sum_{n=0}^{\infty}\left(\frac{\pi}{3}\right)^n$ II. $\sum_{n=0}^{\infty}(0.99)^n$ III. $\sum_{n=1}^{\infty}2\left(\frac{\pi}{e+1}\right)^n$

(A) II only

(B) I and II only

(C) II and III only

(D) I, II, and III

I. $\sum_{n=0}^{\infty}\left(\frac{\pi}{3}\right)^n \Rightarrow r = \frac{\pi}{3} > 1$ divergent

II. $\sum_{n=0}^{\infty}(0.99)^n \Rightarrow r = 0.99 < 1$ convergent

III. $\sum_{n=1}^{\infty}2\left(\frac{\pi}{e+1}\right)^n \Rightarrow r = \frac{\pi}{e+1} \approx \frac{3.14}{2.7+1} < 1$ convergent

16. Level: AP3

Which of the following series converges?

(A) $\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n}$ $= \frac{(-1)^1}{1} + \frac{(-1)^3}{2} + \frac{(-1)^5}{3} \Rightarrow$ divergent Harmonic series

(B) $\sum_{n=1}^{\infty}(-1)^n \cdot \frac{n+1}{3n-2}$ $\lim_{n\to\infty}\frac{n+1}{3n-2} = \frac{1}{3}$ divergent by nth term test

(C) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt[3]{n}}$ $= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1/3}}$ convergent by AST

(D) $\sum_{n=1}^{\infty}(-1)^n \cdot \frac{2^n}{n^{99}}$ $\lim_{n\to\infty}\frac{2^n}{n^{99}} \to \infty$ The exponential dominates the power, divergent by nth term test.

17. Level: AP3

Which of the following series converge?

I. $\sum_{k=4}^{\infty} \frac{k^2+k}{\sqrt{k-3}}$ II. $\sum_{k=1}^{\infty} \frac{k+3}{(k-2)^2+1}$ III. $\sum_{k=1}^{\infty} \frac{6k+3}{k^3+k}$

(A) I and III only

(B) I and II only

(C) III only

(D) II and III only

I. $\sum_{k=4}^{\infty} \frac{k^2+k}{\sqrt{k-3}}$ $\lim_{k\to\infty} \frac{k^2+k}{\sqrt{k-3}} = \lim_{k\to\infty} \frac{k^2}{k^{1/2}} = \infty$ divergent

II. $\sum_{k=1}^{\infty} \frac{k+3}{(k-2)^2+1} \approx \sum_{k=1}^{\infty} \frac{k}{(k)^2} \approx \sum_{k=1}^{\infty} \frac{1}{k}$ like Harmonic, divergent

$\lim_{k\to\infty} \frac{\frac{k+3}{(k-2)^2+1}}{1/k} = \lim_{k\to\infty} \frac{(k+3)k}{(k-2)^2+1} = \lim_{k\to\infty} \frac{(k)k}{(k)^2} = 1$ Both diverge by Limit Comparison Test

III. $\sum_{k=1}^{\infty} \frac{6k+3}{k^3+k} \approx \sum_{k=1}^{\infty} \frac{6k}{k^3} \approx \sum_{k=1}^{\infty} \frac{6}{k^2}$ like convergent p-series $p = 2 > 1$

$\lim_{k\to\infty} \frac{\frac{6k+3}{k^3+k}}{6/k^2} = \lim_{k\to\infty} \frac{(6k+3)k^2}{(k^3+k)6} = \lim_{k\to\infty} \frac{(6k^3)}{(6k^3)} = 1$ Both converge by Limit Comparison Test

18. Level: AP4

Let $a_n = \sqrt{n}$. Which of the following series converges?

(A) $\sum_{n=1}^{\infty} \frac{(5na_n+3)^4}{2n^6+3n^5}$ $\lim_{n\to\infty} \frac{(5n\sqrt{n}+3)^4}{2n^6+3n^5} = \lim_{n\to\infty} \frac{(5n^{3/2})^4}{2n^6} = \lim_{n\to\infty} \frac{(5^4 n^6)}{2n^6} = \frac{5^4}{2} \neq 0$ Divergent by nth term test.

(B) $\sum_{n=1}^{\infty} \frac{n!}{2^n \cdot a_n}$ $\lim_{n\to\infty} \frac{n!}{2^n \cdot \sqrt{n}} = \infty$ because factorial dominates. Divergent by nth term test.

(C) $\sum_{n=1}^{\infty} \frac{5n-1}{n \cdot a_n}$ $\lim_{n\to\infty} \frac{5n-1}{n\cdot\sqrt{n}} = \lim_{n\to\infty} \frac{5n}{n\cdot\sqrt{n}} = \lim_{n\to\infty} \frac{5}{\sqrt{n}} = 0$ $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$ is a divergent p-series.

$\lim_{n\to\infty} \frac{\frac{5n-1}{n^{3/2}}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{(5n)\sqrt{n}}{n^{3/2}} = \lim_{n\to\infty} \frac{5n^{3/2}}{n^{3/2}} = 5$ Both converge by Limit Comparison Test

(D) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot a_n}{n}$ $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Convergent by AST

19. Level: AP4

Which of the following series can be shown to converge by the alternating series test?

I. $\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$ II. $\sum_{n=0}^{\infty}\frac{\cos(\pi n)}{\sqrt{n}}$ III. $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{\sin(n)}{n}$

(A) I only

(B) II only

(C) I and III only

(D) II and III only

I. Not an Alternating Series

II. $\sum_{n=0}^{\infty}\frac{\cos(\pi n)}{\sqrt{n}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ and convergent by AST

III. $\lim_{n\to\infty}\left|\frac{\sin(n)}{n}\right|=0$ because n dominates sin(n), but the terms do not strictly decrease.

$\frac{d}{dx}\left(\frac{\sin(n)}{n}\right)=\frac{n\cos(n)-\sin(n)}{n^2}\Rightarrow$ $n>1$ then $n\cos(n)$ dominates but $-1\le\cos(n)\le 1$

so the derivative is not always negative.

20. Level: AP4

Which of the following series can be shown to converge by the alternating series test?

I. $\frac{1}{2}-2+\frac{27}{8}-4+\frac{125}{32}+\ldots+a_n+\ldots$, where $a_n=(-1)^{n-1}\cdot\frac{n^3}{2^n}$.

II. $1-\frac{1}{8}-\frac{1}{18}+\frac{1}{16}-\frac{1}{50}-\frac{1}{72}+\frac{1}{49}-\frac{1}{128}-\frac{1}{162}+\ldots+a_n+\ldots$, where $a_n=\frac{\cos\left(\frac{2\pi n}{3}\right)}{(n+1)^2}$.

III. $\frac{1}{2}-\frac{\sqrt[3]{2}}{5}+\frac{\sqrt[3]{3}}{8}-\frac{\sqrt[3]{4}}{11}+\frac{\sqrt[3]{5}}{14}-\frac{\sqrt[3]{6}}{17}+\ldots+a_n+\ldots$, where $a_n=(-1)^{n+1}\frac{\sqrt[3]{n}}{3n-1}$.

(A) II only (B) III only **(C)** I and III only (D) II and III only

I. $\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}\frac{n^3}{2^n}=0$ because the exponential will eventually dominate the power. $\frac{(n+1)^3}{2^{n+1}}<\frac{n^3}{2^n}\Rightarrow n\ge 4$

Therefore this series does converge by AST.

II. This is not an alternating series because $\cos\left(\frac{2\pi n}{3}\right)\neq\pm1$

III. $\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}\frac{\sqrt[3]{n}}{3n-1}=0$ because n eventually dominates $n^{1/3}$.

$\frac{\sqrt[3]{n}}{3n-1}\approx\frac{n^{1/3}}{3n}=\frac{1}{3n^{2/3}}\Rightarrow$ Eventually $|a_n|$ behaves like $\frac{1}{3n^{2/3}}$ which is strictly decreasing. Therefore this series does converge by AST.

21. Level: AP

Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{3^n n^{100}}{n!}$ II. $\sum_{n=1}^{\infty} \frac{3n-1}{7n}$ III. $\sum_{n=1}^{\infty} \left(\frac{\pi-1}{e}\right)^n$

(A) I only (B) III only **(C)** I and III only (D) I, II, and III

I. $\lim_{n\to\infty} \left| \frac{3^{n+1}(n+1)^{100}}{(n+1)!} \cdot \frac{n!}{3^n n^{100}} \right| = \lim_{n\to\infty} \left| \frac{3(n+1)^{100}}{(n+1)n^{100}} \right| = 3\lim_{n\to\infty} \left| \frac{(n+1)^{99}}{n^{100}} \right| = 0 < 1 \Rightarrow$ converges by ratio test

II. $\lim_{n\to\infty} \frac{3n-1}{7n} = \frac{3}{7} \neq 0 \Rightarrow$ diverges by nth term test.

III. Geometric with $r = \frac{\pi - 1}{e} \approx \frac{3.14-1}{2.72} = \frac{2.14}{2.72} < 1 \Rightarrow$ converges by geometric test.

22. Level: AP4

What is the sum of the series $\frac{3}{\pi} - \frac{3}{2\pi} + \frac{3}{4\pi} - \frac{3}{8\pi} + \ldots + \left(\frac{-1}{2}\right)^n \frac{3}{\pi} + \ldots$?

(A) $\frac{2}{\pi}$

Geometric with $a = \frac{3}{\pi}, r = -\frac{1}{2} \Rightarrow S = \frac{3/\pi}{1-(-1/2)} = \frac{3/\pi}{3/2} \cdot \frac{2\pi}{2\pi} = \frac{6}{3\pi} = \boxed{\frac{2}{\pi}}$

(B) $\frac{6}{\pi}$

(C) $\frac{6}{2\pi+1}$

(D) The series diverges.

23. Level: AP4

Consider the series $\sum_{n=2}^{\infty} a_n$ where $f(n) = a_n$. Which of the following statements about $f(x)$ and $\sum_{n=2}^{\infty} a_n$ could be true?

(A) $f(x) = \frac{1}{x}$ and $\sum_{n=2}^{\infty} a_n$ converges by the integral test.

$$\int_2^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_2^b f(x)\,dx = \lim_{b\to\infty}\left[\ln x\right]_2^b = \lim_{b\to\infty}\left[\ln b - \ln 2\right] = \infty \Rightarrow \text{ diverges}$$

(B) $f(x) = \frac{1}{2^x}$ and $\sum_{n=2}^{\infty} a_n$ converges by the integral test.

$$\int_2^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_2^b 2^{-x}\,dx = \lim_{b\to\infty}\left[-2^{-x}\ln 2\right]_2^b = \lim_{b\to\infty}\left[\left(-2^{-b}\ln 2\right) - \left(-2^{-2}\ln 2\right)\right] = 0 + \frac{\ln 2}{4} \Rightarrow \text{ converges}$$

(C) $f(x) = xe^x$ and $\sum_{n=2}^{\infty} a_n$ diverges by the integral test.

$f(x)$ is not decreasing so the integral test cannot be applied. $\lim_{n\to\infty} a_n \neq 0$

(D) $f(x) = \frac{1}{x(\ln(x))^2}$ and $\sum_{n=2}^{\infty} a_n$ diverges by the integral test.

$$\int_2^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x(\ln(x))^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{\ln x}\right]_2^b = \lim_{b\to\infty}\left[\left(-\frac{1}{\ln b}\right) - \left(-\frac{1}{\ln 2}\right)\right] = 0 + \frac{1}{\ln 2} \Rightarrow \text{ converges}$$

$$\int \frac{1}{x(\ln(x))^2}\,dx = \int \frac{1}{(\underbrace{\ln(x)}_{u})^2}\left(\frac{1}{x}dx\right) = \int u^{-2}\,du = -u^{-1} = -\frac{1}{\ln(x)}$$

24. Level: AP4

If $0 < a_n < b_n$ where a_n and b_n are both decreasing sequences for $n \geq 0$, which of the following must be true?

(A) If $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} b_n$ converges.

(B) If $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ converges.

(C) If $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

$\lim_{n\to\infty} b_n = 0$ and $0 < a_n < b_n$ then $\lim_{n\to\infty} a_n = 0$

$\Rightarrow \sum_{n=0}^{\infty} (-1)^n a_n$ converges by AST.

(D) If $\sum_{n=0}^{\infty} (-1)^n b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

25. Level: AP4

Given $\sum_{n=1}^{\infty}(-1)^n\frac{n^p}{\sqrt{n^2-3}}$, for what value(s) of p will the series be conditionally convergent?

(A) $p \geq 1$

(B) $p = 1$

(C) $\frac{1}{2} < p < \frac{3}{2}$

(D) $0 < p < 1$

$\lim_{n\to\infty}\frac{n^p}{\sqrt{n^2-3}} = \lim_{n\to\infty}\frac{n^p}{\sqrt{n^2}} = \lim_{n\to\infty}\frac{n^p}{n} = 0 \Rightarrow 0 < p < 1$

numerator power less than denominator power

$\sum_{n=1}^{\infty}(-1)^n\frac{n^p}{\sqrt{n^2-3}}$ converges by AST.

$\sum_{n=1}^{\infty}\left|(-1)^n\frac{n^p}{\sqrt{n^2-3}}\right| = \sum_{n=1}^{\infty}\frac{n^p}{\sqrt{n^2-3}} \approx \sum_{n=1}^{\infty}\frac{n^p}{n} = \sum_{n=1}^{\infty}\frac{1}{n^{1-p}}$ divergent p-series

because $1-p<1$ on the interval $0<p<1 \Rightarrow$ conditionally convergent

26. Level: AP4

Consider the series $\sum_{n=1}^{\infty}a_n$ where $\lim_{n\to\infty}a_n = \frac{1}{2}$. Which of the following must be true?

(A) The series $\sum_{n=1}^{\infty}a_n$ diverges by the nth term test.

(B) The series $\sum_{n=1}^{\infty}a_n$ converges by the nth term test.

(C) The series $\sum_{n=1}^{\infty}a_n$ converges by the ratio test.

(D) The series $\sum_{n=1}^{\infty}a_n$ could converge or diverge.

27. Level: AP5

The sum of the geometric series $\sum_{n=1}^{\infty}4\left(\frac{-2}{k}\right)^n$ is -1 where k is a constant. What is the value of k?

(A) $k = -\frac{2}{5}$ (B) $k = 6$ (C) $k = 8$ (D) $k = 10$

$n=1 \Rightarrow a = 4\left(\frac{-2}{k}\right)^1$

$S = \frac{4\left(\frac{-2}{k}\right)}{1-\left(\frac{-2}{k}\right)} = \frac{-8}{k+2} = -1 \Rightarrow -(k+2) = -8 \Rightarrow k+2 = 8 \Rightarrow \boxed{k=6}$

28. Level: AP5

Which of the following is true when the integral test is applied to the series $\sum_{n=1}^{\infty} ne^{-n}$ where $f(x) = xe^{-x}$?

(A) $\int_1^{\infty} f(x)\,dx = \frac{2}{e}$ and $\sum_{n=1}^{\infty} ne^{-n}$ converges.

(B) $\int_1^{\infty} f(x)\,dx$ converges and $\sum_{n=1}^{\infty} ne^{-n} = \int_1^{\infty} f(x)\,dx$.

$\sum_{n=1}^{\infty} ne^{-n}$ is a Riemann Sum $\neq \int_1^{\infty} f(x)\,dx$ is the area under the curve.

(C) $\int_1^{\infty} f(x)\,dx$ diverges and the series $\sum_{n=1}^{\infty} ne^{-n}$ diverges. The integral converges.

(D) The conditions for the integral test are not met and the integral test cannot be applied.

$u = x \Rightarrow du = dx$

$dv = e^{-x}dx \Rightarrow v = -e^{-x}$

$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} = -e^{-x}(x+1)$

$\int_1^{\infty} xe^{-x}\,dx = \lim_{b\to\infty} \int_1^{b} xe^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}(x+1)\right]_1^{b} = -\lim_{b\to\infty}\left[e^{-b}(b+1) - (2e^{-1})\right]$

$= -\lim_{b\to\infty}\left[\left(\frac{b+1}{e^{b}}\right) - \left(\frac{2}{e}\right)\right] = -\left[(0) - \left(\frac{2}{e}\right)\right] = \boxed{\frac{2}{e}} \Rightarrow$ integral converges

29. Level: AP5

Which of the following could be a valid conclusion of the integral test for the series $\sum_{n=1}^{\infty} a_n$ where $f(n) = a_n$ and $f(n)$ is known to be positive and decreasing ?

(A) The series $\sum_{n=1}^{\infty} a_n$ diverges because $f(x)$ is not continuous over the interval $[1,\infty)$.

If $f(x)$ is not continuous then you cannot apply the integral test.

(B) The series $\sum_{n=1}^{\infty} a_n$ diverges because $\int_1^{\infty} f(x)dx \neq 0$. Diverges if the integral is not a finite, positive value.

(C) The series $\sum_{n=1}^{\infty} a_n$ converges to 2 because $f(x)$ is differentiable over the interval $[1,\infty)$ and $\int_1^{\infty} f(x)dx = 2$. The sum is not equal to the integral.

(D) The series $\sum_{n=1}^{\infty} a_n$ converges because $f(x)$ is continuous over the interval $[1,\infty)$ and $\int_1^{\infty} f(x)dx = 8$.

The integral is a finite, positive value and all conditions of the integral test are met.

30. Level: AP5

For which value(s) of k do both $\sum_{n=1}^{\infty} \frac{n^3}{n^{k^2}}$ and $\sum_{n=1}^{\infty} \left(\frac{k}{2}+1\right)^{-n}$ converge?

(A) $-\infty < k < -2$

(B) $-4 < k < -\sqrt{3}$

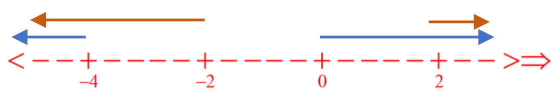
(C) $k < -4$ or $k > 2$

(D) $-4 < k < 0$ and $k > \sqrt{3}$

$\sum_{n=1}^{\infty} \frac{n^3}{n^{k^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{k^2-3}} \Rightarrow$ convergent p-series if $k^2 - 3 > 1 \Rightarrow k^2 > 4 \Rightarrow k < -2$ or $k > 2$

$\sum_{n=1}^{\infty} \left(\frac{k}{2}+1\right)^{-n} = \sum_{n=1}^{\infty} \left(\frac{k+2}{2}\right)^{-n} = \sum_{n=1}^{\infty} \left(\frac{2}{k+2}\right)^{n} \Rightarrow$ convergent geometric series if $\left|\frac{2}{k+2}\right| < 1$

$|k+2| > 2 \Rightarrow k < -4$ or $k > 0$



$\Rightarrow$ $\boxed{k < -4 \text{ or } k > 2}$