Calculus BC - 2022 AP Live Review Session 5

Working With and Manipulating Series on the AP Calculus BC Exam

What Do We Need to Know ?

Function	First Four Terms	General Term	Interval of Convergence
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$	$\frac{x^n}{n!}$	$-\infty<x<\infty$
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$	$\frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty<x<\infty$
$\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$	$\frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty<x<\infty$
$\frac{1}{1+x}$	$1-x+x^2-x^3+\ldots$	$(-1)^n x^n$	$-1<x<1$
The series for $\frac{1}{1+x}$ is useful when working with functions of the form $\frac{1}{1-x}$ and connecting them to the geometric series $\sum_{n=0}^{\infty} a\cdot r^n$.			

Definition of Taylor Series and Maclaurin Series

If f has a power series representation at $x=c$, then the series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots$$

is called the **Taylor series for f centered at c**.

If $c=0$, then

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

is also called the **Maclaurin series for f** .

Topic Name	Topic #
Finding Taylor Polynomial Approximations of Functions	10.11
Lagrange Error Bound	10.12
Radius and Interval of Convergence	10.13
Finding Taylor or Maclaurin Series for a Function	10.14
Representing Functions as Power Series	10.15

Finding Your Way Around AP Classroom

Open Response / Multiple Choice Practice

1. Level: AP3

Write the first four terms and the general term of the series expansion for the function $f(x)=e^{2x}$.

2. Level: AP3

Write the first three nonzero terms and the general term of the Maclaurin series for $x^2\cos(x)$.

3. Level: AP3

Given the infinite series $f(x)=x+x^2+\dfrac{x^3}{3}-\dfrac{x^5}{30}+\cdots$, which of the following represents the series $f(-3x^2)$?

(A) $-3x^2+3x^3-x^4+\dfrac{x^6}{10}+\cdots$

(B) $-3x^2-3x^4-x^6+\dfrac{x^{10}}{10}+\cdots$

(C) $-3x^2+9x^4-9x^6+\dfrac{81x^{10}}{10}+\cdots$

(D) $-3x^3-3x^4-x^5+\dfrac{x^7}{10}+\cdots$

4. Level: AP4

Given $f(x)=\sum_{n=0}^{\infty}\dfrac{5(2x-3)^n}{n!}$, find $f'(x)$.

(A) $\sum_{n=1}^{\infty}\dfrac{5(2x-3)^{n-1}}{(n-1)!}$

(B) $\sum_{n=0}^{\infty}\dfrac{10(2x-3)^{n-1}}{n!}$

(C) $\sum_{n=1}^{\infty}\dfrac{10(2x-3)^{n-1}}{(n-1)!}$

(D) $\sum_{n=0}^{\infty}\dfrac{10(2x-3)^{n+1}}{(n+1)!}$

5. Level: AP4

The Taylor series for $g(x)$ about $x=-2$ is given by $2+4(x+2)+4(x+2)^2+\frac{8}{3}(x+2)^3+\ldots+\frac{2^{n+1}}{n!}(x+2)^n+\ldots$.
Write the first three nonzero terms and the general term for $g'(x)$.

6. Level: AP4

Find the radius of convergence for $\sum_{n=1}^{\infty}\frac{(x+3)^n}{n\cdot 2^n}$.

7. Level: AP4

If the Taylor series for f about $x=2$ is $3+\frac{3}{4}(x-2)+\frac{3}{4^2}(x-2)^2+\frac{3}{4^3}(x-2)^3+\ldots$, then $f(1)=$

(A) $\frac{-3}{5}$

(B) $\frac{12}{5}$

(C) 4

(D) 12

8. Level: AP5

The power series $\sum_{n=0}^{\infty}b_n(x-5)^n$ is conditionally convergent at $x=1$. Which of the following *must* be true?

(A) The series converges absolutely at $x=7$.

(B) The series is conditionally convergent at $x=8$.

(C) The series is conditionally convergent at $x=9$.

(D) The series converges absolutely at $x=10$.

Free Response Practice

1. The Taylor series for a function f about $x=1$ is given by $f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)\cdot 3^{n-1}}(x-1)^n$.

(a) Use the ratio test to show that the radius of convergence of $f(x)$ is 3.

(b) Write an equation of the line tangent to the graph of $f(x)$ at $x=1$.

(c) Determine if $f(x)$ converges absolutely, converges conditionally, or diverges at each $x=-2$ and $x=4$. Justify your answer.

(d) Let $g(x)=\int_1^x f(t)dt$. Find $T_3(x)$, the third-degree Taylor polynomial to $g(x)$ about $x=1$.

(e) Show that $\left|g\left(\frac{3}{2}\right)-T_3\left(\frac{3}{2}\right)\right|\le\frac{1}{3000}$.

Multiple Choice Practice

9. Level: AP5

The power series $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots + \frac{(-1)^n}{(2n+1)!}x^{4n+2} + \ldots$ converges to which of the following?

(A) $x\sin(x)$

(B) $\sin(x^2)$

(C) $x^2\cos(x)$

(D) $\cos(x^3) + x^2 - 1$

10. Level: AP2

Which of the following is a power series for $f(x) = \sin(x^2)$?

(A) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(B) $\sum_{n=0}^{\infty} \frac{x^{4n+2}}{(2n+1)!}$

(C) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$

(D) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$

11. Level: AP4

Which of the following is the coefficient of the x^7 term in the Maclaurin series for $3 - x\cos(x)$?

(A) $\frac{1}{6!}$

(B) $-\frac{1}{6!}$

(C) $-\frac{1}{7!}$

(D) $\frac{1}{7!}$

12. Level: AP3

Find the interval of convergence for the series $h(x)=\frac{x-3}{2}+\frac{(x-3)^2}{2\cdot 2^2}+\frac{(x-3)^3}{3\cdot 2^3}+\frac{(x-3)^4}{4\cdot 2^4}+\cdots$

(A) $2\le x\le 4$

(B) $1<x<5$

(C) $1\le x<5$

(D) $2\le x<4$

13. Level: AP3

Let h be a function where h and its derivatives are positive for all values of x. The Taylor series for h centered at $x=-3$ is $3+(x+3)+\frac{7}{2!}(x+3)^2+\frac{3}{7\cdot 3!}(x+3)^3+\frac{5}{6\cdot 4!}(x+3)^4+\frac{1}{5!}(x+3)^5+\ldots$, and let $T(x)$ be the Taylor polynomial for h, centered at $x=-3$, using the first four terms of the series for h. The Lagrange error bound states that $\left|h(-1)-T(-1)\right|\le K$, where K is a constant. What is the value of K ?

(A) $K=\frac{5}{9}$

(B) $K=\frac{4}{15}$

(C) $K=\frac{5}{144}$

(D) $K=\frac{1}{120}$

14. Level: AP3

Which of the following is $\int_0^x \cos\left(\sqrt{t}\right)dt$?

(A) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n)!}$

(B) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)(2n)!}$

(C) $\sum_{n=0}^{\infty}\frac{(-1)^n \left(\sqrt{x}\right)^{2n+1}}{(2n+1)(2n+1)!}$

(D) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(n+1)(2n)!}$

15. Level: AP3

For $-2 < x < 2$, the function h is defined by $h(x) = \frac{-3}{4+2x}$. Which of the following series could be $h(x)$?

(A) $-3 - 6x - 12x^2 - 24x^3 - 48x^4 - \ldots$

(B) $-3 + \frac{3x}{2} - \frac{3x^2}{4} + \frac{3x^3}{8} - \frac{3x^4}{16} + \ldots$

(C) $-\frac{3}{4} + \frac{3x}{8} - \frac{3x^2}{16} + \frac{3x^3}{32} - \frac{3x^4}{64} + \ldots$

(D) $-\frac{3}{4} - \frac{3x}{8} - \frac{3x^2}{16} - \frac{3x^3}{32} - \frac{3x^4}{64} - \ldots$