The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function $F(x) = \int_a^x f(t)\, dt$ has a derivative at every point $x \in [a, b]$, and $\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\, dt = f(x)$.

Proof:

First, we will apply the definition of the derivative directly to the function F :

$$\frac{dF}{dx} = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.$$

by substituting our initial conjecture for F, we get:

$$\frac{dF}{dx} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt}{h}.$$

Using a property of definite integrals ($\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$), we can rewrite the integral as follows:

$$\frac{dF}{dx} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\, dt + \int_x^a f(t)\, dt}{h}.$$

In order for the next step to be easier to see, we will commute the order of the numerator:

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{\int_x^a f(t)\, dt + \int_a^{x+h} f(t)\, dt}{h}$$; students should notice something . . . maybe a property of definite integrals that could be incorporated.

By using the Additivity property (i.e., $\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$), we can state:

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\, dt}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt$$. At this point, we are hoping some of the students see $\frac{1}{h} \int_x^{x+h} f(t)\, dt$ is the same as $\frac{1}{b-a} \int_a^b f(t)\, dt$ - - - the MVT for the definite integral!

By applying the concept of the MVT for the definite integral, we can say:

$$\exists\, c \in [x,\, x+h] \ni \frac{1}{h} \int_x^{x+h} f(t)\, dt = f(c)$$.

Thus we can make the following substitution:

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c), \text{where } c\in[x, x+h].$$

Now this is the cool part . . . as $h\to 0$, $c\to x$ by the Squeeze Theorem. So:

$$\lim_{h\to 0} f(c) = f(x).$$

$$\boxed{\textbf{Therefore, } \frac{dF}{dx} = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c) = f(x) \quad \blacksquare}$$

The Fundamental Theorem of Calculus, Part 2

If f is continuous at every point on $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

By definition, $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) * \Delta x$, where $\Delta x = \frac{b-a}{n}$ and c_i is chosen arbitrarily in the i^{th} subinterval (i.e., $c_i \in [x_i,\ x_{i+1}]$ where $x_1 = a$ and $x_{n+1} = b$)

Hence,

$\int_a^b f(x)\,dx = f(c_1)*\Delta x + f(c_2)*\Delta x + f(c_3)*\Delta x + \ldots + f(c_{n-1})*\Delta x + f(c_n)*\Delta x$ as $n \to \infty$.

Since F is any antiderivative of f on $[a, b]$, $F'(x) = f(x)$ for all $x \in [a, b]$. Therefore, the previous equation can be rewritten as:

$\int_a^b f(x)\,dx = F'(c_1)*\Delta x + F'(c_2)*\Delta x + F'(c_3)*\Delta x + \ldots + F'(c_{n-1})*\Delta x + F'(c_n)*\Delta x$

as $n \to \infty$.

The Mean Value Theorem for Derivatives (MVT) states if a function is continuous on a closed interval and is differentiable over the open interval, the function has a point whose derivative value is equal to the slope of the secant line that connects the endpoints of the interval. Thus, there exists a point $c_i \in [x_i,\ x_{i+1}]$ that satisfies the conditions for the MVT, then we can state $F'(c_i) = \frac{F(x_{i+1}) - F(x_i)}{\Delta x}$. It is important to note the randomness of the nature of c_i is not violated, since $n \to \infty$. Applying the Squeeze Theorem, as the intervals become infinitely small, all c_i values will approach the conditions that satisfy the MVT.

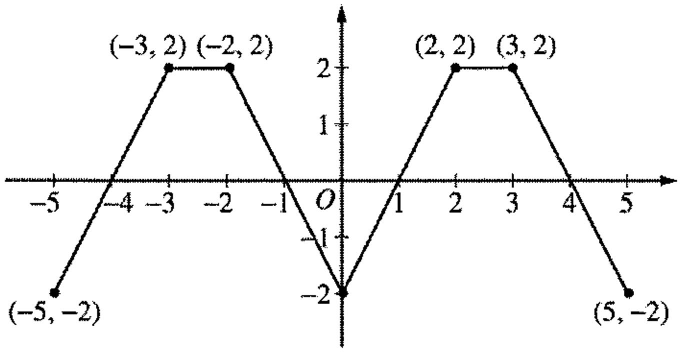
Applying the MVT to the previous equation,

$$\int_a^b f(x)\,dx = F'(c_1)*\Delta x + F'(c_2)*\Delta x + F'(c_3)*\Delta x + \ldots + F'(c_n)*\Delta x$$

$$= \frac{F(x_2)-F(a)}{\Delta x}*\Delta x + \frac{F(x_3)-F(x_2)}{\Delta x}*\Delta x + \frac{F(x_4)-F(x_3)}{\Delta x}*\Delta x + . . .$$

$$+ \frac{F(x_n)-F(x_{n-1})}{\Delta x}*\Delta x + \frac{F(b)-F(x_n)}{\Delta x}*\Delta x = F(b)-F(a).$$

Therefore, $\int_a^b f(x)\,dx = F(b)-F(a)$ ■



Graph of f

3. The graph of the function f shown above consists of six line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

   (a) Find $g(4)$, $g'(4)$, and $g''(4)$.

   (b) Does g have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.

   (c) Suppose that f is defined for all real numbers x and is periodic with a period of length 5. The graph above shows two periods of f. Given that $g(5) = 2$, find $g(10)$ and write an equation for the line tangent to the graph of g at $x = 108$.

WRITE ALL WORK IN THE PINK EXAM BOOKLET.

END OF PART A OF SECTION II

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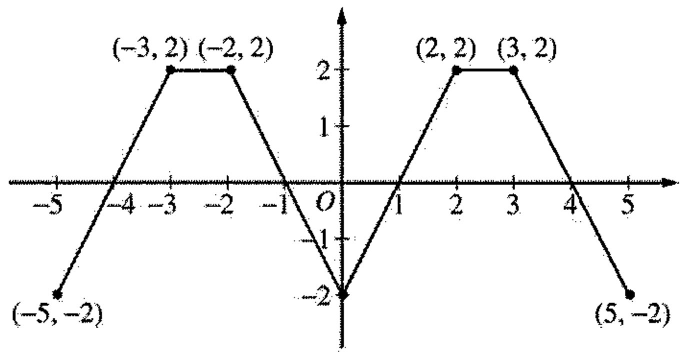
Question 3

The graph of the function f shown above consists of six line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.



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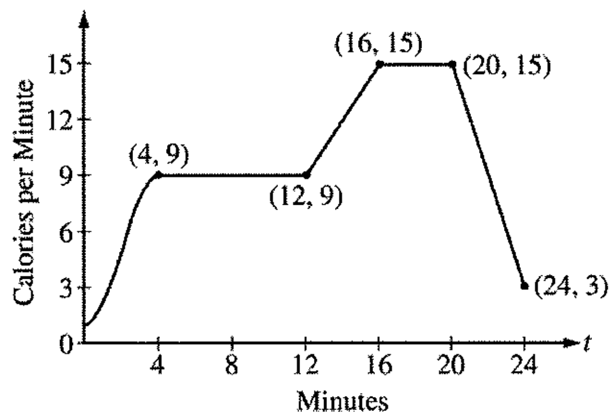
(a) $g(4) = \int_0^4 f(t)\,dt = 3$ $g'(4) = f(4) = 0$ $g''(4) = f'(4) = -2$	$3 : \begin{cases} 1 : g(4) \\ 1 : g'(4) \\ 1 : g''(4) \end{cases}$
(b) g has a relative minimum at $x = 1$ because $g' = f$ changes from negative to positive at $x = 1$.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$
(c) $g(0) = 0$ and the function values of g increase by 2 for every increase of 5 in x. $g(10) = 2g(5) = 4$ $g(108) = \int_0^{105} f(t)\,dt + \int_{105}^{108} f(t)\,dt$ $= 21g(5) + g(3) = 44$ $g'(108) = f(108) = f(3) = 2$ An equation for the line tangent to the graph of g at $x = 108$ is $y - 44 = 2(x - 108)$.	$4 : \begin{cases} 1 : g(10) \\ 3 : \begin{cases} 1 : g(108) \\ 1 : g'(108) \\ 1 : \text{equation of tangent line} \end{cases} \end{cases}$

CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.



4. The rate, in calories per minute, at which a person using an exercise machine burns calories is modeled by the function f. In the figure above, $f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$ for $0 \le t \le 4$ and f is piecewise linear for $4 \le t \le 24$.

   (a) Find $f'(22)$. Indicate units of measure.

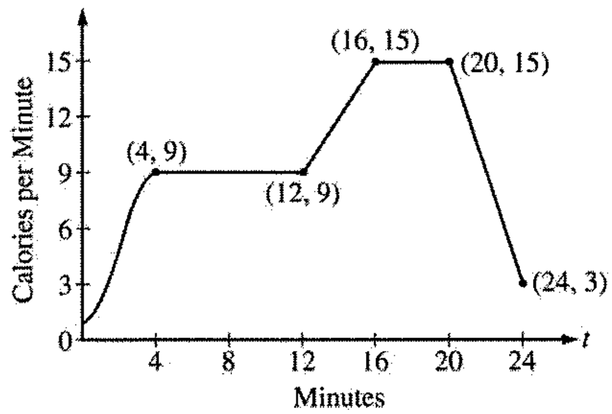
   (b) For the time interval $0 \le t \le 24$, at what time t is f increasing at its greatest rate? Show the reasoning that supports your answer.

   (c) Find the total number of calories burned over the time interval $6 \le t \le 18$ minutes.

   (d) The setting on the machine is now changed so that the person burns $f(t) + c$ calories per minute. For this setting, find c so that an average of 15 calories per minute is burned during the time interval $6 \le t \le 18$.

WRITE ALL WORK IN THE EXAM BOOKLET.

Question 4



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(d) The setting on the machine is now changed so that the person burns $f(t) + c$ calories per minute. For this setting, find c so that an average of 15 calories per minute is burned during the time interval $6 \le t \le 18$.

(a) $f'(22) = \frac{15-3}{20-24} = -3$ calories/min/min

1 : $f'(22)$ and units

(b) f is increasing on $[0, 4]$ and on $[12, 16]$.

On $(12, 16)$, $f'(t) = \frac{15-9}{16-12} = \frac{3}{2}$ since f has constant slope on this interval.

On $(0, 4)$, $f'(t) = -\frac{3}{4}t^2 + 3t$ and

$f''(t) = -\frac{3}{2}t + 3 = 0$ when $t = 2$. This is where f' has a maximum on $[0, 4]$ since $f'' > 0$ on $(0, 2)$ and $f'' < 0$ on $(2, 4)$.

On $[0, 24]$, f is increasing at its greatest rate when $t = 2$ because $f'(2) = 3 > \frac{3}{2}$.

$$4 : \begin{cases} 1 : f' \text{ on } (0, 4) \\ 1 : \text{shows } f' \text{ has a max at } t = 2 \text{ on } (0, 4) \\ 1 : \text{shows for } 12 < t < 16,\ f'(t) < f'(2) \\ 1 : \text{answer} \end{cases}$$

(c) $\int_6^{18} f(t)\,dt = 6(9) + \frac{1}{2}(4)(9+15) + 2(15)$
$= 132$ calories

$$2 : \begin{cases} 1 : \text{method} \\ 1 : \text{answer} \end{cases}$$

(d) We want $\frac{1}{12}\int_6^{18} (f(t) + c)\,dt = 15$.

This means $132 + 12c = 15(12)$. So, $c = 4$.

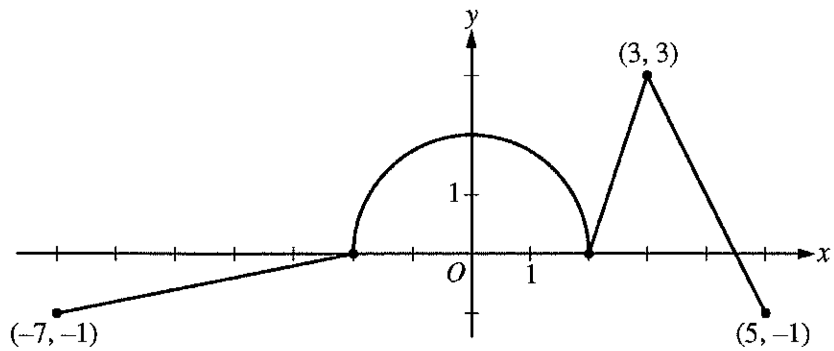
OR

Currently, the average is $\frac{132}{12} = 11$ calories/min.

Adding c to $f(t)$ will shift the average by c.

So $c = 4$ to get an average of 15 calories/min.

$$2 : \begin{cases} 1 : \text{setup} \\ 1 : \text{value of } c \end{cases}$$



Graph of g'

5. The function g is defined and differentiable on the closed interval $[-7, 5]$ and satisfies $g(0) = 5$. The graph of $y = g'(x)$, the derivative of g, consists of a semicircle and three line segments, as shown in the figure above.

   (a) Find $g(3)$ and $g(-2)$.

   (b) Find the x-coordinate of each point of inflection of the graph of $y = g(x)$ on the interval $-7 < x < 5$. Explain your reasoning.

   (c) The function h is defined by $h(x) = g(x) - \frac{1}{2}x^2$. Find the x-coordinate of each critical point of h, where $-7 < x < 5$, and classify each critical point as the location of a relative minimum, relative maximum, or neither a minimum nor a maximum. Explain your reasoning.

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6. Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

   (a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

   (b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

   (c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

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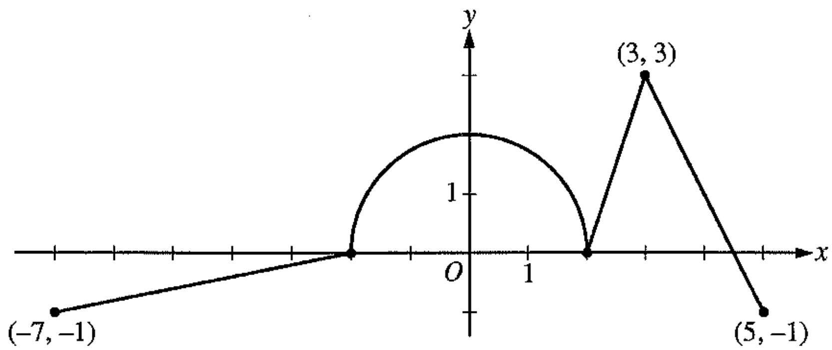
END OF EXAM

Question 5



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(c) The function h is defined by $h(x) = g(x) - \frac{1}{2}x^2$. Find the x-coordinate of each critical point of h, where $-7 < x < 5$, and classify each critical point as the location of a relative minimum, relative maximum, or neither a minimum nor a maximum. Explain your reasoning.

(a) $g(3) = 5 + \int_0^3 g'(x)\,dx = 5 + \frac{\pi \cdot 2^2}{4} + \frac{3}{2} = \frac{13}{2} + \pi$

$g(-2) = 5 + \int_0^{-2} g'(x)\,dx = 5 - \pi$

$3 : \begin{cases} 1 : \text{uses } g(0) = 5 \\ 1 : g(3) \\ 1 : g(-2) \end{cases}$

(b) The graph of $y = g(x)$ has points of inflection at $x = 0$, $x = 2$, and $x = 3$ because g' changes from increasing to decreasing at $x = 0$ and $x = 3$, and g' changes from decreasing to increasing at $x = 2$.

$2 : \begin{cases} 1 : \text{identifies } x = 0, 2, 3 \\ 1 : \text{explanation} \end{cases}$

(c) $h'(x) = g'(x) - x = 0 \Rightarrow g'(x) = x$

On the interval $-2 \le x \le 2$, $g'(x) = \sqrt{4 - x^2}$.

On this interval, $g'(x) = x$ when $x = \sqrt{2}$.

The only other solution to $g'(x) = x$ is $x = 3$.

$h'(x) = g'(x) - x > 0$ for $0 \le x < \sqrt{2}$

$h'(x) = g'(x) - x \le 0$ for $\sqrt{2} < x \le 5$

Therefore h has a relative maximum at $x = \sqrt{2}$, and h has neither a minimum nor a maximum at $x = 3$.

$4 : \begin{cases} 1 : h'(x) \\ 1 : \text{identifies } x = \sqrt{2}, 3 \\ 1 : \text{answer for } \sqrt{2} \text{ with analysis} \\ 1 : \text{answer for 3 with analysis} \end{cases}$

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Question 6

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

(c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

(a) $f'(1) = \left.\frac{dy}{dx}\right|_{(1,\,2)} = 8$

An equation of the tangent line is $y = 2 + 8(x - 1)$.

$2 : \begin{cases} 1 : f'(1) \\ 1 : \text{answer} \end{cases}$

(b) $f(1.1) \approx 2.8$

Since $y = f(x) > 0$ on the interval $1 \le x < 1.1$,

$\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right) > 0$ on this interval.

Therefore on the interval $1 < x < 1.1$, the line tangent to the graph of $y = f(x)$ at $x = 1$ lies below the curve and the approximation 2.8 is less than $f(1.1)$.

$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{conclusion with explanation} \end{cases}$

(c) $\frac{dy}{dx} = xy^3$

$\int \frac{1}{y^3}\,dy = \int x\,dx$

$-\frac{1}{2y^2} = \frac{x^2}{2} + C$

$-\frac{1}{2 \cdot 2^2} = \frac{1^2}{2} + C \Rightarrow C = -\frac{5}{8}$

$y^2 = \frac{1}{\frac{5}{4} - x^2}$

$f(x) = \frac{2}{\sqrt{5 - 4x^2}}, \quad \frac{-\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$

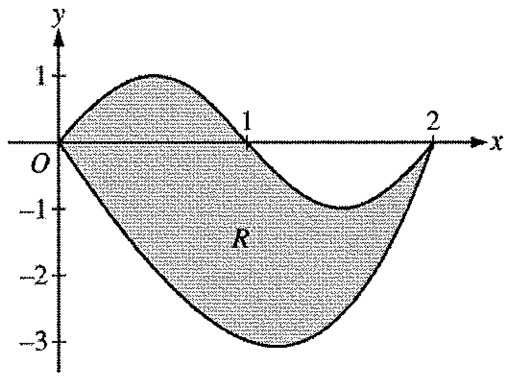
$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

   (a) Find the area of R.

   (b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

   (d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

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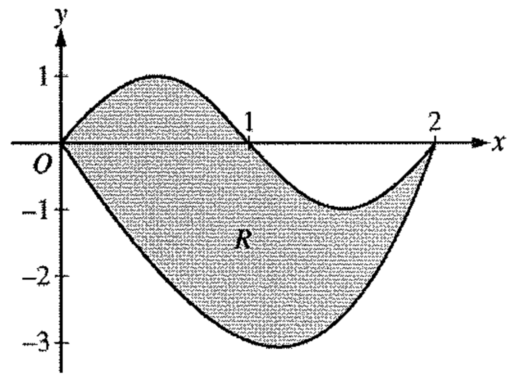
Question 1



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(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$ Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$	$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$ The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$	$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \end{cases}$
(c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or 8.370	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

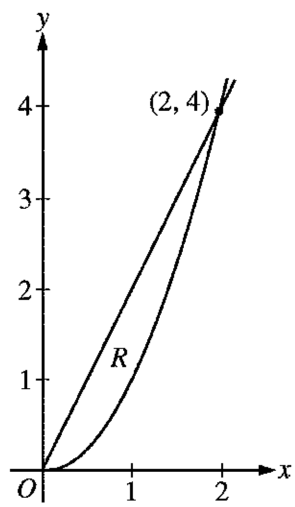
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



4. Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.

   (a) Find the area of R.

   (b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

   (c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

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GO ON TO THE NEXT PAGE.

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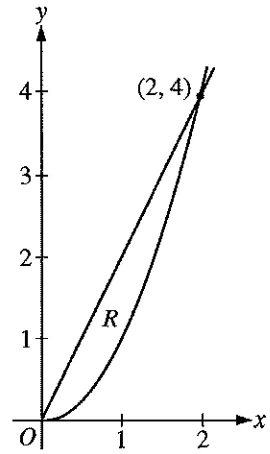
Question 4

Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.



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(c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

(a) Area $= \int_0^2 \left(2x - x^2\right) dx$

$$= x^2 - \frac{1}{3}x^3 \Big|_{x=0}^{x=2}$$

$$= \frac{4}{3}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \int_0^2 \sin\left(\frac{\pi}{2}x\right) dx$

$$= -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big|_{x=0}^{x=2}$$

$$= \frac{4}{\pi}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right)^2 dy$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \end{cases}$

Logistic Differentiation

Question: Find the particular solution to $\frac{dP}{dt} = P(2 - \frac{P}{5000})$ if $P(0) = 3{,}000$. Use the solution to find $\lim_{t \to \infty} P(t)$.

$\frac{dP}{dt} = P(2 - \frac{P}{5000})$ can be written in another form that is more "user-friendly." Try to get the expression in the form: $$\frac{dP}{dt} = \frac{k}{M} P(M - P)$$.

This can easily be converted to $$P = \frac{M}{1 + Ae^{-kt}}$$ by solving the differential equation using partial fractions. Use this simple substitution from the above equations instead of using partial fractions.

So $\frac{dP}{dt} = P(2 - \frac{P}{5000}) = \frac{1}{5000} P(10{,}000 - P) = \frac{2}{10{,}000} P(10{,}000 - P)$.

Thus, $P = \frac{10{,}000}{1 + Ae^{-2t}}$.

Using the initial condition $P(0) = 3{,}000$, you can find $A = \frac{7}{3}$. It is interesting to note the initial condition is not needed once the equation is in the above form. In this case, as $t \to \infty$, $P(t) \to M$, since $Ae^{-2t} \to 0$. Therefore, the value of A is of no consequence. This could be an important teaching point because it could save students time if they recognize what happens to the denominator as $t \to \infty$.

Formally:

So $\frac{dP}{dt} = P(2 - \frac{P}{5000}) = \frac{1}{5000}P(10{,}000 - P) = \frac{2}{10{,}000}P(10{,}000 - P)$.

Therefore, $P = \frac{10{,}000}{1 + Ae^{-2t}}$. Using $P(0) = 3{,}000$, $3{,}000 = \frac{10{,}000}{1 + Ae^{-2\bullet 0}} = \frac{10{,}000}{1 + Ae^{0}} = \frac{10{,}000}{1 + A}$.

So $\frac{3{,}000}{10{,}000} = \frac{3}{10} = \frac{1}{1 + A}$. Thus $3(1 + A) = 10$, and $A = \frac{7}{3}$. Hence, $P = \frac{10{,}000}{1 + \frac{7}{3}e^{-2t}}$.

Accordingly, $\lim_{t \to \infty} P(t) = \frac{10{,}000}{1 + 0} = 10{,}000$ since $\lim_{t \to \infty} \frac{7}{3}e^{-2t} = 0$.

CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

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4. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\dfrac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P ? Justify your answer.

---

5. A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim\limits_{t \to \infty} P(t)$?
If $P(0) = 20$, what is $\lim\limits_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim\limits_{t \to \infty} Y(t)$?

GO ON TO THE NEXT PAGE.

Question 5

A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim_{t \to \infty} Y(t)$?

(a) For this logistic differential equation, the carrying capacity is 12. If $P(0) = 3$, $\lim_{t \to \infty} P(t) = 12$. If $P(0) = 20$, $\lim_{t \to \infty} P(t) = 12$.	$2: \begin{cases} 1: \text{answer} \\ 1: \text{answer} \end{cases}$
(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6$.	1 : answer
(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$ $\ln\lvert Y\rvert = \frac{t}{5} - \frac{t^2}{120} + C$ $Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$ $K = 3$ $Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$	$5: \begin{cases} 1: \text{separates variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables
(d) $\lim_{t \to \infty} Y(t) = 0$	1 : answer $0/1$ if Y is not exponential

AP®

AP® Calculus AB/BC
2012 Scoring Guidelines

The College Board

The College Board is a mission-driven not-for-profit organization that connects students to college success and opportunity. Founded in 1900, the College Board was created to expand access to higher education. Today, the membership association is made up of more than 5,900 of the world's leading educational institutions and is dedicated to promoting excellence and equity in education. Each year, the College Board helps more than seven million students prepare for a successful transition to college through programs and services in college readiness and college success — including the SAT® and the Advanced Placement Program®. The organization also serves the education community through research and advocacy on behalf of students, educators, and schools. The College Board is committed to the principles of excellence and equity, and that commitment is embodied in all of its programs, services, activities, and concerns.

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CollegeBoard

Question 1

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

(a) $$W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = \frac{67.9 - 61.8}{6} = 1.017 \text{ (or } 1.016)$$

The water temperature is increasing at a rate of approximately 1.017 °F per minute at time $t = 12$ minutes.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$

(b) $$\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16$$

The water has warmed by 16 °F over the interval from $t = 0$ to $t = 20$ minutes.

$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{interpretation with units} \end{cases}$

(c) $$\begin{aligned} \frac{1}{20}\int_0^{20} W(t)\,dt &\approx \frac{1}{20}(4 \cdot W(0) + 5 \cdot W(4) + 6 \cdot W(9) + 5 \cdot W(15)) \\ &= \frac{1}{20}(4 \cdot 55.0 + 5 \cdot 57.1 + 6 \cdot 61.8 + 5 \cdot 67.9) \\ &= \frac{1}{20} \cdot 1215.8 = 60.79 \end{aligned}$$

This approximation is an underestimate, because a left Riemann sum is used and the function W is strictly increasing.

$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \\ 1 : \text{underestimate with reason} \end{cases}$

(d) $$\begin{aligned} W(25) &= 71.0 + \int_{20}^{25} W'(t)\,dt \\ &= 71.0 + 2.043155 = 73.043 \end{aligned}$$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

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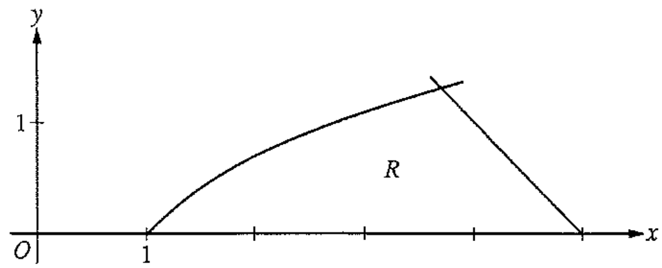
Question 2

Let R be the region in the first quadrant bounded by the x-axis and the graphs of $y = \ln x$ and $y = 5 - x$, as shown in the figure above.



(a) Find the area of R.

(b) Region R is the base of a solid. For the solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(c) The horizontal line $y = k$ divides R into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

$\ln x = 5 - x \Rightarrow x = 3.69344$

Therefore, the graphs of $y = \ln x$ and $y = 5 - x$ intersect in the first quadrant at the point $(A, B) = (3.69344, 1.30656)$.

(a) Area $= \int_0^B \left(5 - y - e^y\right) dy$ $= 2.986$ (or 2.985) OR Area $= \int_1^A \ln x\, dx + \int_A^5 (5 - x)\, dx$ $= 2.986$ (or 2.985)	$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$
(b) Volume $= \int_1^A (\ln x)^2\, dx + \int_A^5 (5 - x)^2\, dx$	$3 : \begin{cases} 2 : \text{integrands} \\ 1 : \text{expression for total volume} \end{cases}$
(c) $\int_0^k \left(5 - y - e^y\right) dy = \frac{1}{2} \cdot 2.986 \left(\text{or } \frac{1}{2} \cdot 2.985\right)$	$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{equation} \end{cases}$

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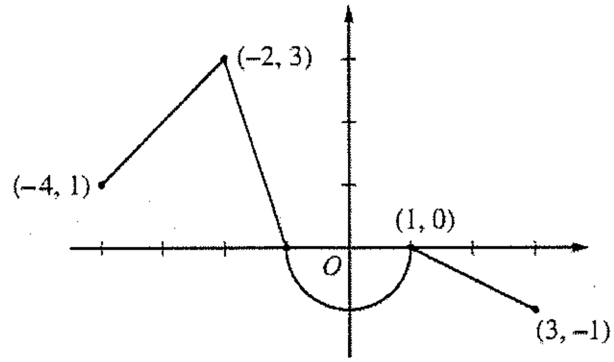
Question 3



Graph of f

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$ $g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^1 f(t)\,dt$ $= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$	$2 : \begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$
(b) $g'(x) = f(x) \Rightarrow g'(-3) = f(-3) = 2$ $g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$	$2 : \begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$
(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$. $g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, g has a relative maximum at $x = -1$. $g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.	$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$
(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

Question 4

The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x + 7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

(a) $f'(x) = \frac{1}{2}\left(25 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{25 - x^2}}, \quad -5 < x < 5$	$2 : f'(x)$
(b) $f'(-3) = \frac{3}{\sqrt{25 - 9}} = \frac{3}{4}$ $f(-3) = \sqrt{25 - 9} = 4$ An equation for the tangent line is $y = 4 + \frac{3}{4}(x + 3)$.	$2 : \begin{cases} 1 : f'(-3) \\ 1 : \text{answer} \end{cases}$
(c) $\lim\limits_{x \to -3^-} g(x) = \lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^-} \sqrt{25 - x^2} = 4$ $\lim\limits_{x \to -3^+} g(x) = \lim\limits_{x \to -3^+} (x + 7) = 4$ Therefore, $\lim\limits_{x \to -3} g(x) = 4$. $g(-3) = f(-3) = 4$ So, $\lim\limits_{x \to -3} g(x) = g(-3)$. Therefore, g is continuous at $x = -3$.	$2 : \begin{cases} 1 : \text{considers one-sided limits} \\ 1 : \text{answer with explanation} \end{cases}$
(d) Let $u = 25 - x^2 \Rightarrow du = -2x\,dx$ $\int_0^5 x\sqrt{25 - x^2}\, dx = -\frac{1}{2}\int_{25}^{0} \sqrt{u}\, du$ $= \left[-\frac{1}{2} \cdot \frac{2}{3} u^{3/2}\right]_{u=25}^{u=0}$ $= -\frac{1}{3}(0 - 125) = \frac{125}{3}$	$3 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

Question 5

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

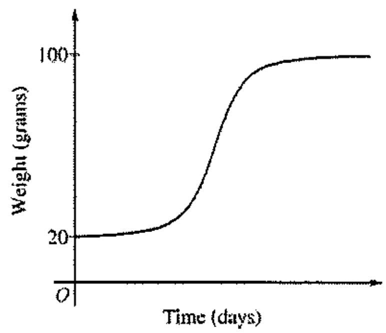
$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.



(a) $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(60) = 12$

$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(30) = 6$

Because $\left.\frac{dB}{dt}\right|_{B=40} > \left.\frac{dB}{dt}\right|_{B=70}$, the bird is gaining weight faster when it weighs 40 grams.

$2: \begin{cases} 1: \text{uses } \frac{dB}{dt} \\ 1: \text{answer with reason} \end{cases}$

(b) $\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt} = -\frac{1}{5}\cdot\frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$

Therefore, the graph of B is concave down for $20 \le B < 100$. A portion of the given graph is concave up.

$2: \begin{cases} 1: \frac{d^2B}{dt^2} \text{ in terms of } B \\ 1: \text{explanation} \end{cases}$

(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$\int \frac{1}{100 - B}\, dB = \int \frac{1}{5}\, dt$

$-\ln|100 - B| = \frac{1}{5}t + C$

Because $20 \le B < 100$, $|100 - B| = 100 - B$.

$-\ln(100 - 20) = \frac{1}{5}(0) + C \Rightarrow -\ln(80) = C$

$100 - B = 80e^{-t/5}$

$B(t) = 100 - 80e^{-t/5}, \quad t \ge 0$

$5: \begin{cases} 1: \text{separation of variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } B \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Question 6

For $0 \le t \le 12$, a particle moves along the x-axis. The velocity of the particle at time t is given by $v(t) = \cos\left(\frac{\pi}{6}t\right)$. The particle is at position $x = -2$ at time $t = 0$.

(a) For $0 \le t \le 12$, when is the particle moving to the left?

(b) Write, but do not evaluate, an integral expression that gives the total distance traveled by the particle from time $t = 0$ to time $t = 6$.

(c) Find the acceleration of the particle at time t. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.

(d) Find the position of the particle at time $t = 4$.

(a) $v(t) = \cos\left(\frac{\pi}{6}t\right) = 0 \;\Rightarrow\; t = 3,\ 9$

The particle is moving to the left when $v(t) < 0$.
This occurs when $3 < t < 9$.

$2 : \begin{cases} 1 : \text{considers } v(t) = 0 \\ 1 : \text{interval} \end{cases}$

(b) $\int_0^6 |v(t)|\, dt$

1 : answer

(c) $a(t) = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right)$

$a(4) = -\frac{\pi}{6}\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}\pi}{12} < 0$

$v(4) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} < 0$

The speed is increasing at time $t = 4$, because velocity and acceleration have the same sign.

$3 : \begin{cases} 1 : a(t) \\ 2 : \text{conclusion with reason} \end{cases}$

(d)
$$\begin{aligned} x(4) &= -2 + \int_0^4 \cos\left(\frac{\pi}{6}t\right) dt \\ &= -2 + \left[\frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)\right]_0^4 \\ &= -2 + \frac{6}{\pi}\left[\sin\left(\frac{2\pi}{3}\right) - 0\right] \\ &= -2 + \frac{6}{\pi}\cdot\frac{\sqrt{3}}{2} = -2 + \frac{3\sqrt{3}}{\pi} \end{aligned}$$

$3 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

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Question 2

For $t \geq 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 2$, the particle is at position $(1, 5)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+2}}{e^t}$ and $\frac{dy}{dt} = \sin^2 t$.

(a) Is the horizontal movement of the particle to the left or to the right at time $t = 2$? Explain your answer. Find the slope of the path of the particle at time $t = 2$.

(b) Find the x-coordinate of the particle's position at time $t = 4$.

(c) Find the speed of the particle at time $t = 4$. Find the acceleration vector of the particle at time $t = 4$.

(d) Find the distance traveled by the particle from time $t = 2$ to $t = 4$.

(a) $\left.\frac{dx}{dt}\right\rvert_{t=2} = \frac{2}{e^2}$ Because $\left.\frac{dx}{dt}\right\rvert_{t=2} > 0$, the particle is moving to the right at time $t = 2$. $\left.\frac{dy}{dx}\right\rvert_{t=2} = \frac{\left. dy/dt\right\rvert_{t=2}}{\left. dx/dt\right\rvert_{t=2}} = 3.055$ (or 3.054)	$3: \begin{cases} 1: \text{moving to the right with reason} \\ 1: \text{considers } \frac{dy/dt}{dx/dt} \\ 1: \text{slope at } t = 2 \end{cases}$
(b) $x(4) = 1 + \int_2^4 \frac{\sqrt{t+2}}{e^t}\, dt = 1.253$ (or 1.252)	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(c) Speed $= \sqrt{(x'(4))^2 + (y'(4))^2} = 0.575$ (or 0.574) Acceleration $= \langle x''(4), y''(4) \rangle$ $= \langle -0.041, 0.989 \rangle$	$2: \begin{cases} 1: \text{speed} \\ 1: \text{acceleration} \end{cases}$
(d) Distance $= \int_2^4 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ $= 0.651$ (or 0.650)	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

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Question 4

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

(a) $f(1) = 15,\ f'(1) = 8$

An equation for the tangent line is
$y = 15 + 8(x-1)$.

$f(1.4) \approx 15 + 8(1.4 - 1) = 18.2$

$2: \begin{cases} 1: \text{tangent line} \\ 1: \text{approximation} \end{cases}$

(b) $\int_1^{1.4} f'(x)\,dx \approx (0.2)(10) + (0.2)(13) = 4.6$

$f(1.4) = f(1) + \int_1^{1.4} f'(x)\,dx$

$f(1.4) \approx 15 + 4.6 = 19.6$

$3: \begin{cases} 1: \text{midpoint Riemann sum} \\ 1: \text{Fundamental Theorem of Calculus} \\ 1: \text{answer} \end{cases}$

(c) $f(1.2) \approx f(1) + (0.2)(8) = 16.6$

$f(1.4) \approx 16.6 + (0.2)(12) = 19.0$

$2: \begin{cases} 1: \text{Euler's method with two steps} \\ 1: \text{answer} \end{cases}$

(d) $T_2(x) = 15 + 8(x-1) + \frac{20}{2!}(x-1)^2$
$= 15 + 8(x-1) + 10(x-1)^2$

$f(1.4) \approx 15 + 8(1.4-1) + 10(1.4-1)^2 = 19.8$

$2: \begin{cases} 1: \text{Taylor polynomial} \\ 1: \text{approximation} \end{cases}$

Question 6

The function g has derivatives of all orders, and the Maclaurin series for g is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots$.

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$

$\lim_{n\to\infty}\left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$

$x^2 < 1 \Rightarrow -1 < x < 1$
The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$
This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$
This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

(b) $\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

2 :
- 1 : uses the third term as an error bound
- 1 : error bound

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$

2 :
- 1 : first three terms
- 1 : general term

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2013 Scoring Guidelines

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2013 SCORING GUIDELINES

Question 1

On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

(a) Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

(a) $G'(5) = -24.588$ (or -24.587)

The rate at which gravel is arriving is decreasing by 24.588 (or 24.587) tons per hour per hour at time $t = 5$ hours.

$2: \begin{cases} 1: G'(5) \\ 1: \text{interpretation with units} \end{cases}$

(b) $\int_0^8 G(t)\,dt = 825.551$ tons

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(c) $G(5) = 98.140764 < 100$

At time $t = 5$, the rate at which unprocessed gravel is arriving is less than the rate at which it is being processed. Therefore, the amount of unprocessed gravel at the plant is decreasing at time $t = 5$.

$2: \begin{cases} 1: \text{compares } G(5) \text{ to } 100 \\ 1: \text{conclusion} \end{cases}$

(d) The amount of unprocessed gravel at time t is given by $A(t) = 500 + \int_0^t (G(s) - 100)\,ds.$

$A'(t) = G(t) - 100 = 0 \Rightarrow t = 4.923480$

t	$A(t)$
0	500
4.92348	635.376123
8	525.551089

The maximum amount of unprocessed gravel at the plant during this workday is 635.376 tons.

$3: \begin{cases} 1: \text{considers } A'(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

Question 2

A particle moves along a straight line. For $0 \le t \le 5$, the velocity of the particle is given by $v(t) = -2 + \left(t^2 + 3t\right)^{6/5} - t^3$, and the position of the particle is given by $s(t)$. It is known that $s(0) = 10$.

(a) Find all values of t in the interval $2 \le t \le 4$ for which the speed of the particle is 2.

(b) Write an expression involving an integral that gives the position $s(t)$. Use this expression to find the position of the particle at time $t = 5$.

(c) Find all times t in the interval $0 \le t \le 5$ at which the particle changes direction. Justify your answer.

(d) Is the speed of the particle increasing or decreasing at time $t = 4$? Give a reason for your answer.

(a) Solve $\lvert v(t) \rvert = 2$ on $2 \le t \le 4$. $t = 3.128$ (or 3.127) and $t = 3.473$	$2 : \begin{cases} 1 : \text{considers } \lvert v(t) \rvert = 2 \\ 1 : \text{answer} \end{cases}$
(b) $s(t) = 10 + \int_0^t v(x)\,dx$ $s(5) = 10 + \int_0^5 v(x)\,dx = -9.207$	$2 : \begin{cases} 1 : s(t) \\ 1 : s(5) \end{cases}$
(c) $v(t) = 0$ when $t = 0.536033, 3.317756$ $v(t)$ changes sign from negative to positive at time $t = 0.536033$. $v(t)$ changes sign from positive to negative at time $t = 3.317756$. Therefore, the particle changes direction at time $t = 0.536$ and time $t = 3.318$ (or 3.317).	$3 : \begin{cases} 1 : \text{considers } v(t) = 0 \\ 2 : \text{answers with justification} \end{cases}$
(d) $v(4) = -11.475758 < 0$, $a(4) = v'(4) = -22.295714 < 0$ The speed is increasing at time $t = 4$ because velocity and acceleration have the same sign.	2 : conclusion with reason

Question 3

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

(a) $C'(3.5) \approx \dfrac{C(4) - C(3)}{4 - 3} = \dfrac{12.8 - 11.2}{1} = 1.6$ ounces/min

$$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{units} \end{cases}$$

(b) C is differentiable $\Rightarrow$ C is continuous (on the closed interval)

$\dfrac{C(4) - C(2)}{4 - 2} = \dfrac{12.8 - 8.8}{2} = 2$

Therefore, by the Mean Value Theorem, there is at least one time t, $2 < t < 4$, for which $C'(t) = 2$.

$$2 : \begin{cases} 1 : \dfrac{C(4) - C(2)}{4 - 2} \\ 1 : \text{conclusion, using MVT} \end{cases}$$

(c)
$$\frac{1}{6}\int_0^6 C(t)\,dt \approx \frac{1}{6}[2 \cdot C(1) + 2 \cdot C(3) + 2 \cdot C(5)]$$
$$= \frac{1}{6}(2 \cdot 5.3 + 2 \cdot 11.2 + 2 \cdot 13.8)$$
$$= \frac{1}{6}(60.6) = 10.1 \text{ ounces}$$

$\frac{1}{6}\int_0^6 C(t)\,dt$ is the average amount of coffee in the cup, in ounces, over the time interval $0 \le t \le 6$ minutes.

$$3 : \begin{cases} 1 : \text{midpoint sum} \\ 1 : \text{approximation} \\ 1 : \text{interpretation} \end{cases}$$

(d) $B'(t) = -16(-0.4)e^{-0.4t} = 6.4e^{-0.4t}$

$B'(5) = 6.4e^{-0.4(5)} = \dfrac{6.4}{e^2}$ ounces/min

$$2 : \begin{cases} 1 : B'(t) \\ 1 : B'(5) \end{cases}$$

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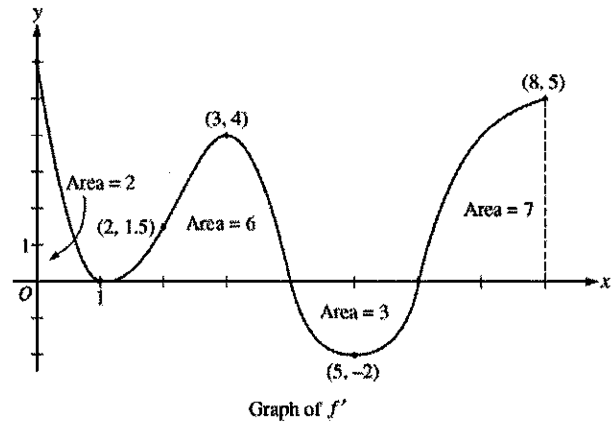
Question 4



Graph of f'

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

(a) $x = 6$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a local minimum at $x = 6$.

1 : answer with justification

(b) From part (a), the absolute minimum occurs either at $x = 6$ or at an endpoint.

$$f(0) = f(8) + \int_8^0 f'(x)\,dx$$
$$= f(8) - \int_0^8 f'(x)\,dx = 4 - 12 = -8$$
$$f(6) = f(8) + \int_8^6 f'(x)\,dx$$
$$= f(8) - \int_6^8 f'(x)\,dx = 4 - 7 = -3$$
$$f(8) = 4$$

The absolute minimum value of f on the closed interval $[0, 8]$ is -8.

$3 : \begin{cases} 1 : \text{considers } x = 0 \text{ and } x = 6 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(c) The graph of f is concave down and increasing on $0 < x < 1$ and $3 < x < 4$, because f' is decreasing and positive on these intervals.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

(d) $g'(x) = 3[f(x)]^2 \cdot f'(x)$

$g'(3) = 3[f(3)]^2 \cdot f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot 4 = 75$

$3 : \begin{cases} 2 : g'(x) \\ 1 : \text{answer} \end{cases}$

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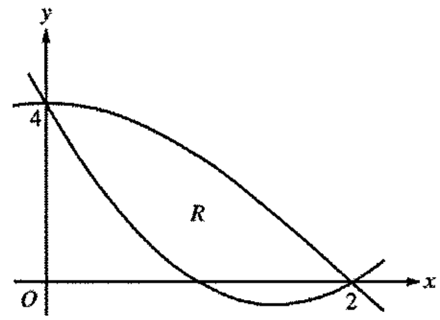
Question 5

Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos\left(\frac{1}{4}\pi x\right)$. Let R be the region bounded by the graphs of f and g, as shown in the figure above.



(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(a)
$$\begin{aligned}\text{Area} &= \int_0^2 [g(x) - f(x)]\,dx \\ &= \int_0^2 \left[4\cos\left(\frac{\pi}{4}x\right) - \left(2x^2 - 6x + 4\right)\right] dx \\ &= \left[4\cdot\frac{4}{\pi}\sin\left(\frac{\pi}{4}x\right) - \left(\frac{2x^3}{3} - 3x^2 + 4x\right)\right]_0^2 \\ &= \frac{16}{\pi} - \left(\frac{16}{3} - 12 + 8\right) = \frac{16}{\pi} - \frac{4}{3}\end{aligned}$$

$$4 : \begin{cases} 1 : \text{integrand} \\ 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

(b)
$$\begin{aligned}\text{Volume} &= \pi\int_0^2 \left[(4 - f(x))^2 - (4 - g(x))^2\right] dx \\ &= \pi\int_0^2 \left[\left(4 - \left(2x^2 - 6x + 4\right)\right)^2 - \left(4 - 4\cos\left(\frac{\pi}{4}x\right)\right)^2\right] dx\end{aligned}$$

$$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$$

(c)
$$\begin{aligned}\text{Volume} &= \int_0^2 [g(x) - f(x)]^2\,dx \\ &= \int_0^2 \left[4\cos\left(\frac{\pi}{4}x\right) - \left(2x^2 - 6x + 4\right)\right]^2 dx\end{aligned}$$

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$$

Question 6

Consider the differential equation $\frac{dy}{dx} = e^y\left(3x^2 - 6x\right)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through $(1, 0)$.

(a) Write an equation for the line tangent to the graph of f at the point $(1, 0)$. Use the tangent line to approximate $f(1.2)$.

(b) Find $y = f(x)$, the particular solution to the differential equation that passes through $(1, 0)$.

(a) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(1,\,0)} = e^0\left(3\cdot 1^2 - 6\cdot 1\right) = -3$

An equation for the tangent line is $y = -3(x-1)$.

$f(1.2) \approx -3(1.2 - 1) = -0.6$

$$3 : \begin{cases} 1 : \frac{dy}{dx} \text{ at the point } (x, y) = (1, 0) \\ 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$$

(b)
$$\frac{dy}{e^y} = \left(3x^2 - 6x\right) dx$$
$$\int \frac{dy}{e^y} = \int \left(3x^2 - 6x\right) dx$$
$$-e^{-y} = x^3 - 3x^2 + C$$
$$-e^{-0} = 1^3 - 3\cdot 1^2 + C \Rightarrow C = 1$$
$$-e^{-y} = x^3 - 3x^2 + 1$$
$$e^{-y} = -x^3 + 3x^2 - 1$$
$$-y = \ln\left(-x^3 + 3x^2 - 1\right)$$
$$y = -\ln\left(-x^3 + 3x^2 - 1\right)$$

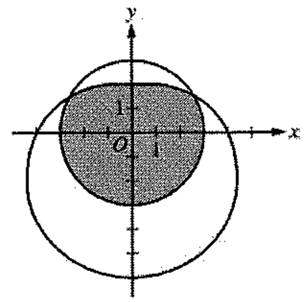
Note: This solution is valid on an interval containing $x = 1$ for which $-x^3 + 3x^2 - 1 > 0$.

$$6 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

Question 2



The graphs of the polar curves $r = 3$ and $r = 4 - 2\sin\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

(a) Let S be the shaded region that is inside the graph of $r = 3$ and also inside the graph of $r = 4 - 2\sin\theta$. Find the area of S.

(b) A particle moves along the polar curve $r = 4 - 2\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

(c) For the particle described in part (b), find the position vector in terms of t. Find the velocity vector at time $t = 1.5$.

(a) Area $= 6\pi + \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4 - 2\sin\theta)^2\, d\theta = 24.709$ (or 24.708)

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$

(b) $x = r\cos\theta \Rightarrow x(\theta) = (4 - 2\sin\theta)\cos\theta$

$x(t) = \left(4 - 2\sin\left(t^2\right)\right)\cos\left(t^2\right)$

$x(t) = -1$ when $t = 1.428$ (or 1.427)

$3 : \begin{cases} 1 : x(\theta) \text{ or } x(t) \\ 1 : x(\theta) = -1 \text{ or } x(t) = -1 \\ 1 : \text{answer} \end{cases}$

(c) $y = r\sin\theta \Rightarrow y(\theta) = (4 - 2\sin\theta)\sin\theta$

$y(t) = \left(4 - 2\sin\left(t^2\right)\right)\sin\left(t^2\right)$

Position vector $= \langle x(t), y(t) \rangle$

$= \left\langle \left(4 - 2\sin\left(t^2\right)\right)\cos\left(t^2\right), \left(4 - 2\sin\left(t^2\right)\right)\sin\left(t^2\right) \right\rangle$

$v(1.5) = \langle x'(1.5), y'(1.5) \rangle$

$= \langle -8.072, -1.673 \rangle$ (or $\langle -8.072, -1.672 \rangle$)

$3 : \begin{cases} 2 : \text{position vector} \\ 1 : \text{velocity vector} \end{cases}$

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Question 5

Consider the differential equation $\frac{dy}{dx} = y^2(2x + 2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x\to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) $\lim_{x\to 0}(f(x)+1) = -1+1 = 0$ and $\lim_{x\to 0}\sin x = 0$

Using L'Hospital's Rule,

$$\lim_{x\to 0}\frac{f(x)+1}{\sin x} = \lim_{x\to 0}\frac{f'(x)}{\cos x} = \frac{f'(0)}{\cos 0} = \frac{(-1)^2\cdot 2}{1} = 2$$

$2 : \begin{cases} 1 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(b) $f\left(\frac{1}{4}\right) \approx f(0) + f'(0)\left(\frac{1}{4}\right)$

$= -1 + (2)\left(\frac{1}{4}\right) = -\frac{1}{2}$

$f\left(\frac{1}{2}\right) \approx f\left(\frac{1}{4}\right) + f'\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$

$= -\frac{1}{2} + \left(-\frac{1}{2}\right)^2\left(2\cdot\frac{1}{4}+2\right)\left(\frac{1}{4}\right) = -\frac{11}{32}$

$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dy}{dx} = y^2(2x+2)$

$\frac{dy}{y^2} = (2x+2)\,dx$

$\int \frac{dy}{y^2} = \int (2x+2)\,dx$

$-\frac{1}{y} = x^2 + 2x + C$

$-\frac{1}{-1} = 0^2 + 2\cdot 0 + C \Rightarrow C = 1$

$-\frac{1}{y} = x^2 + 2x + 1$

$y = -\frac{1}{x^2+2x+1} = -\frac{1}{(x+1)^2}$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Question 5

Note: This solution is valid for $x > -1$.

Question 6

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$

$$P_1\left(\frac{1}{2}\right) = -4 + f'(0)\cdot\frac{1}{2} = -3$$
$$f'(0)\cdot\frac{1}{2} = 1$$
$$f'(0) = 2$$

$2 : \begin{cases} 1 : \text{uses } P_1(x) \\ 1 : \text{verifies } f'(0) = 2 \end{cases}$

(b) $P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right)\cdot\frac{x^2}{2!} + \frac{1}{3}\cdot\frac{x^3}{3!}$

$$= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$$

$3 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \end{cases}$

(c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$.

$$h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$$
$$Q_3(x) = -4x + 4\cdot\frac{x^2}{2} - \frac{4}{3}\cdot\frac{x^3}{3} + C;\ C = Q_3(0) = h(0) = 7$$
$$Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

OR

$$h'(x) = f(2x),\ h''(x) = 2f'(2x),\ h'''(x) = 4f''(2x)$$
$$h'(0) = f(0) = -4,\ h''(0) = 2f'(0) = 4,\ h'''(0) = 4f''(0) = -\frac{8}{3}$$
$$Q_3(x) = 7 - 4x + 4\cdot\frac{x^2}{2!} - \frac{8}{3}\cdot\frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

$4 : \begin{cases} 2 : \text{applies } h'(x) = f(2x) \\ 1 : \text{constant term} \\ 1 : \text{remaining terms} \end{cases}$