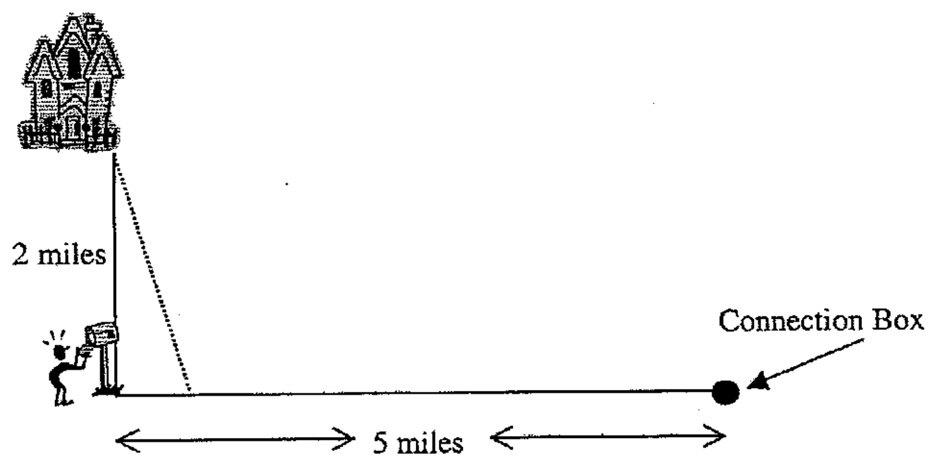
Your Notes

Activity Two:
The Least Expensive Cable

In this activity, you will be working with a partner who should be from a grade level or subject area different from your own.

Better Cable Company must provide service to a customer whose house is located 2 miles from the main highway. The nearest connection box for the cable is located 5 miles down the highway from the customer's driveway. The installation cost is $14 per mile for any cable that is laid from the house to the highway. (The cable may be laid along the driveway to the house or across the field.) The cost is $10 per mile when the cable is laid along the highway. Determine where the cable should be laid so that the installation cost is as low as possible.



1. How much will the customer have to pay if the cable is laid 5 miles along the highway and 2 miles along the drive to the house? Show your calculations. Do you think that this cable will be the least expensive possibility? Explain your reasoning.

2. Do you think a cable that runs directly from the house to the connection box will be the least expensive possibility? Explain your reasoning.

3. On an additional sheet of paper, model the situation with a sketch drawn to scale allowing one inch to represent one mile. Represent the house, the mailbox, and the connection box as points—H (house), M (mailbox), and C (connection box).

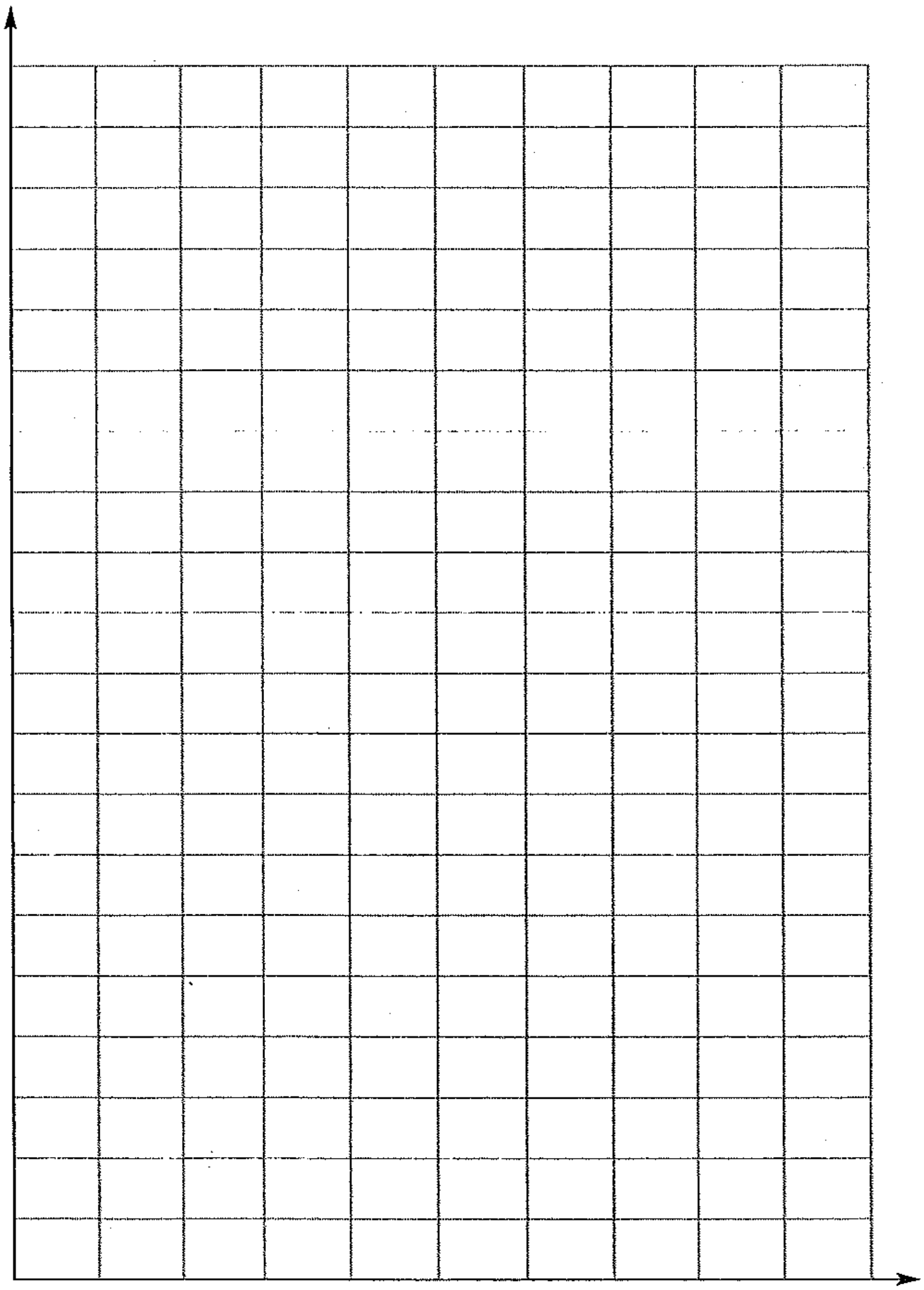
4. On your drawing, locate a point P on the highway that will be $\frac{1}{2}$ mile from the mailbox. The cable company is going to lay the cable so that it follows the highway from the connection box to this point P and then crosses the field to point H where the cable connects to the house. Carefully use your ruler to sketch the cable on your scale drawing.

5. As accurately as you can, measure the length of cable that follows the highway. Convert your fractional measurement to decimal form. Calculate the cost of this portion of the cable. (Remember that the cable costs $10 per mile when it is installed along the highway.) Measure the length of the cable that crosses the field and calculate its cost ($14 per mile). Enter your numbers in the chart provided on the next page and calculate the total cost.

6. Repeat the process from question 5 for the other distances and complete the table.

Distance from mailbox to point P (miles)	Length of cable from house to highway (fractional form, miles)	Length of cable from house to highway (decimal form, miles)	Cost of cable from house to highway (dollars)	Length of cable along highway to connection box (miles)	Cost of cable along highway to connection box (dollars)	Total cost of cable installation (dollars)
0.5 mile						
1.0 mile						
1.5 miles						
2.0 miles						
2.5 miles						
3.0 miles						
3.5 miles						
4.0 miles						
4.5 miles						
5.0 miles						

7. You should discuss with your partner how to label and scale the graph below. After you have done so, plot the total cost of the cable installation as a function of the distance from the mailbox. (The first cost that you calculated for the chart was 0.5 miles from the mailbox.) Include the calculation from question 1 as the point 0 miles from the mailbox.



8. Using a red mark, locate the point on your graph that represents the cost of the least expensive cable installation. In order to provide the least expensive cable, how far from the mailbox should the cable be located? What would be the approximate cost of the least expensive cable?

Teacher Reflection: What mathematics from your particular class or grade level was covered in the activity that you just completed? When you cover that content, skill, or strategy, determine if you are introducing it, teaching it, reviewing it, or expecting students to master it.

Unit $\frac{6!}{5!*2!}$ Exam – B^2 Version

December 20, 2012

Use the table of values to answer the questions below; show all the process steps that lead you to your answer:

x	f (x)	g (x)	f '(x)	g '(x)
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

A. Let $h(x) = f(g(x))$; find $h'(1)$

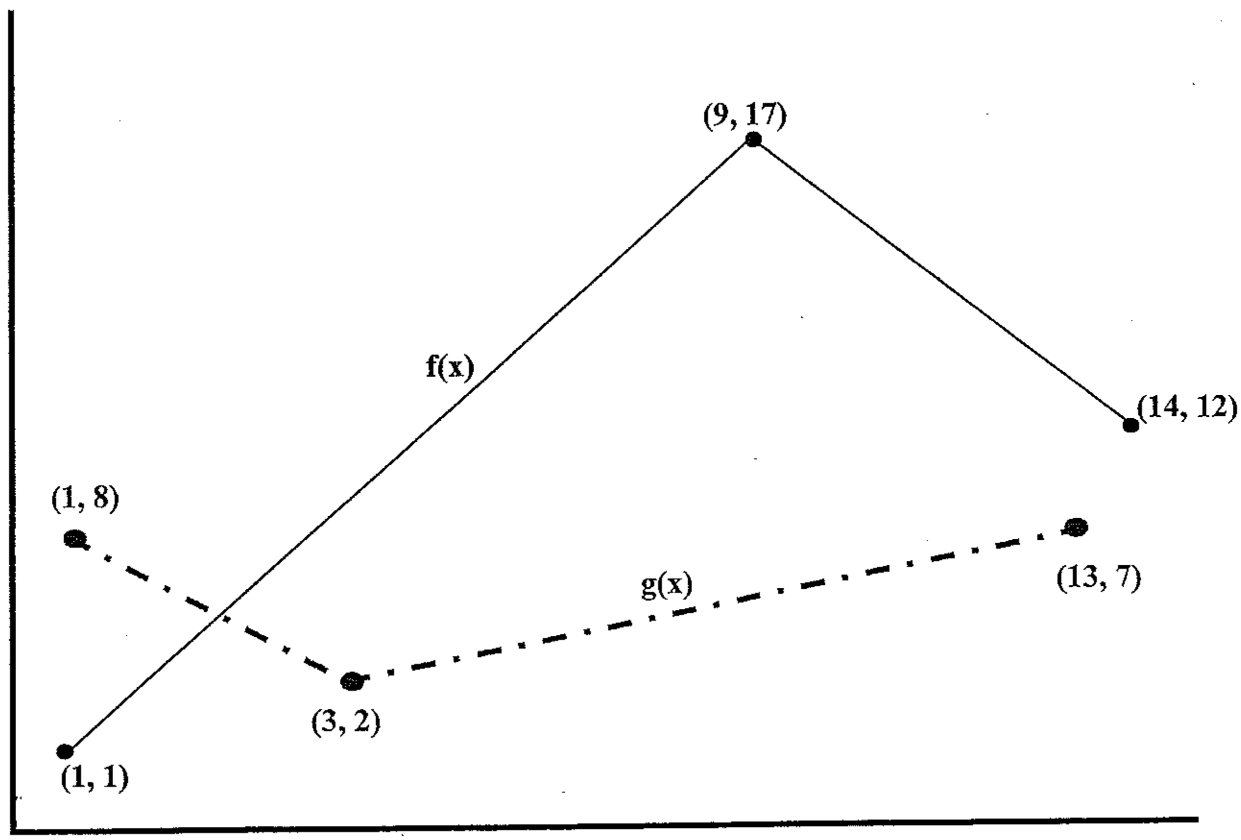
B Let $k(x) = g^2(f(x))$; find $k'(1)$

C. Let $j(x) = \sqrt{f(x) + g(x)}$; find $j'(2)$

D. Find the $1,031^{st}$ derivative for $x \cdot e^{-x}$

E. Find the interval of convergence for $\sum_{n=0}^{\infty} 4\left(\frac{x+1}{3}\right)^n$

Use the graphs provided to solve the problems below:
NOTE: GRAPHS NOT DRAWN TO SCALE



F. Let $K(x) = g(2f(x))$. Find $K'(3)$

G. Let $Q(x) = f(g(3x))$. Find $Q'(2)$

H. Let $f(x) = x^4 - 2x^3 + x^2 - 2x + 1$. Approximate $f(x)$ with a 3rd order Taylor Polynomial centered at x = 2.

I. Let $f(x) = 4x^3 + 3x - 5$. Approximate $f(x)$ with a 2nd order Taylor Polynomial centered at x = 1.

Differentials and their implications

J. Let $\frac{dy}{dx} = x + 2y$ be the differential for the $y = f(x)$ equation with the point $y(2) = 1$; use this differential and this point for all parts of this problem

i. Using Euler's Method, approximate $y(3)$ using $\Delta x = \frac{1}{2}$ and $y(2) = 1$.

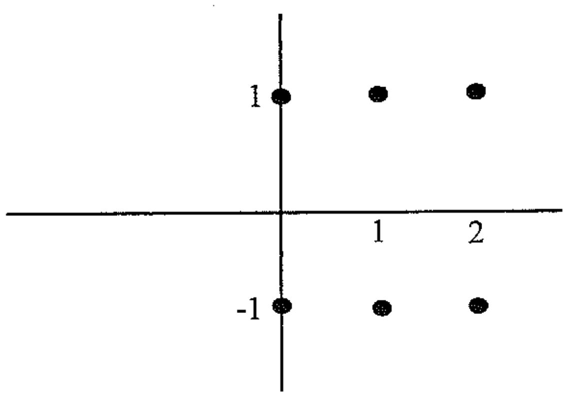
ii. Write the equation of the tangent to the curve at the point (2, 1), and use this tangent to approximate $y(3)$.

iii. Write a second-order Taylor Polynomial that approximates $y = f(x)$, centered at the point (2, 1); use this Taylor Polynomial to approximate $y(3)$.

More Differentials and their implications

K. Let $\frac{dy}{dx} = xy$ be the separable differential equation for the $y = f(x)$ with the point $y(1) = 3$; use this differential and this point for all parts of this problem

i. Sketch a slope field using the differential at the six points indicated



ii. Find the particular $y = f(x)$ solution for the separable differential equation $\frac{dy}{dx} = xy$ containing the point (1, 3)

Position $s(t)$, Velocity $v(t)$, and Acceleration $a(t)$

L. Let $s(t) = e^{3t} + 5t^2$; find $v(0)$ and $a(0)$

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function $F(x) = \int_a^x f(t)\, dt$ has a derivative at every point $x \in [a, b]$, and $\dfrac{dF}{dx} = \dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Proof:

First, we will apply the definition of the derivative directly to the function F:

$$\frac{dF}{dx} = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.$$

by substituting our initial conjecture for F, we get:

$$\frac{dF}{dx} = \mathop{Lim}_{h \to 0} \frac{\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt}{h}.$$

Using a property of definite integrals ($\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$), we can rewrite the integral as follows:

$$\frac{dF}{dx} = \mathop{Lim}_{h \to 0} \frac{\int_a^{x+h} f(t)\, dt + \int_x^a f(t)\, dt}{h}.$$

In order for the next step to be easier to see, we will commute the order of the numerator:

$$\frac{dF}{dx} = \underset{h\to 0}{Lim}\frac{\int_x^a f(t)\,dt + \int_a^{x+h} f(t)\,dt}{h}$$; students should notice something . . . maybe a property of definite integrals that could be incorporated.

By using the Additivity property (i.e., $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$), we can state:

$$\frac{dF}{dx} = \underset{h\to 0}{Lim}\frac{\int_x^{x+h} f(t)\,dt}{h} = \underset{h\to 0}{Lim}\frac{1}{h}\int_x^{x+h} f(t)\,dt$$. At this point, we are hoping some of the students see $\frac{1}{h}\int_x^{x+h} f(t)\,dt$ is the same as $\frac{1}{b-a}\int_a^b f(t)\,dt$ - - - the MVT for the definite integral!

By applying the concept of the MVT for the definite integral, we can say:

$$\exists\, c\in[x,\, x+h] \ni \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c)$$.

Thus we can make the following substitution:

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt = \lim_{h \to 0} f(c), \textit{where } c \in [x, x+h].$$

Now this is the cool part . . . as $h \to 0$, $c \to x$ by the Squeeze Theorem. So:

$$\lim_{h \to 0} f(c) = f(x).$$

Therefore, $$\frac{dF}{dx} = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\, dt}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt = \lim_{h \to 0} f(c) = f(x) \quad \blacksquare$$

The Fundamental Theorem of Calculus, Part 2

If f is continuous at every point on $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

By definition, $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) * \Delta x$, where $\Delta x = \dfrac{b-a}{n}$ and c_i is chosen arbitrarily in the i^{th} subinterval (i.e., $c_i \in [x_i, x_{i+1}]$ where $x_1 = a$ and $x_{n+1} = b$)

Hence,

$$\int_a^b f(x)\,dx = f(c_1)*\Delta x + f(c_2)*\Delta x + f(c_3)*\Delta x + \ldots + f(c_{n-1})*\Delta x + f(c_n)*\Delta x$$ as $n \to \infty$.

Since F is any antiderivative of f on $[a, b]$, $F'(x) = f(x)$ for all $x \in [a, b]$. Therefore, the previous equation can be rewritten as:

$$\int_a^b f(x)\,dx = F'(c_1)*\Delta x + F'(c_2)*\Delta x + F'(c_3)*\Delta x + \ldots + F'(c_{n-1})*\Delta x + F'(c_n)*\Delta x$$

as $n \to \infty$.

The Mean Value Theorem for Derivatives (MVT) states if a function is continuous on a closed interval and is differentiable over the open interval, the function has a point whose derivative value is equal to the slope of the secant line that connects the endpoints of the interval. Thus, there exists a point $c_i \in [x_i, x_{i+1}]$ that satisfies the conditions for the MVT, then we can state $F'(c_i) = \dfrac{F(x_{i+1}) - F(x_i)}{\Delta x}$. It is important to note the randomness of the nature of c_i is not violated, since $n \to \infty$. Applying the Squeeze Theorem, as the intervals become infinitely small, all c_i values will approach the conditions that satisfy the MVT.

Applying the MVT to the previous equation,

$$\int_a^b f(x)\,dx = F'(c_1)*\Delta x + F'(c_2)*\Delta x + F'(c_3)*\Delta x + \ldots + F'(c_n)*\Delta x$$

$$= \frac{F(x_2)-F(a)}{\Delta x}*\Delta x + \frac{F(x_3)-F(x_2)}{\Delta x}*\Delta x + \frac{F(x_4)-F(x_3)}{\Delta x}*\Delta x + \ldots$$

$$+ \frac{F(x_n)-F(x_{n-1})}{\Delta x}*\Delta x + \frac{F(b)-F(x_n)}{\Delta x}*\Delta x = F(b)-F(a).$$

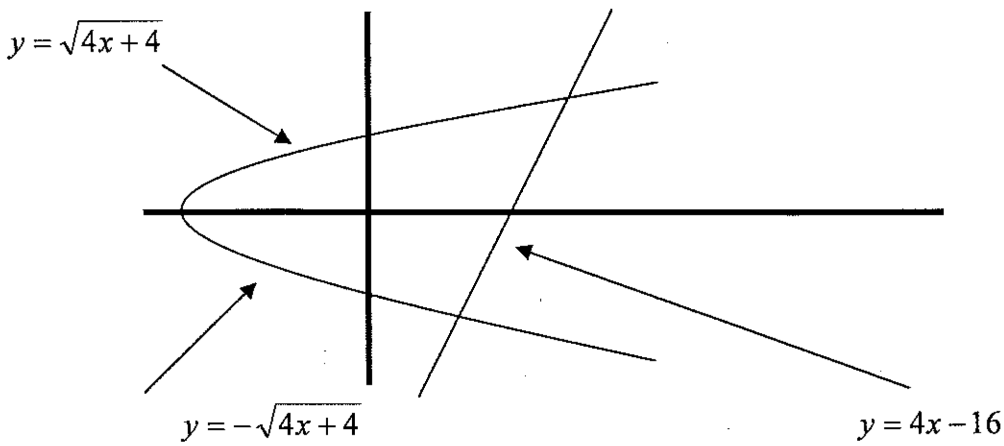
Therefore, $\int_a^b f(x)\,dx = F(b)-F(a)$ ■

B^2 Secrets Revealed
Volume of solids by rotation: Disk/ Washer & Shell Methods

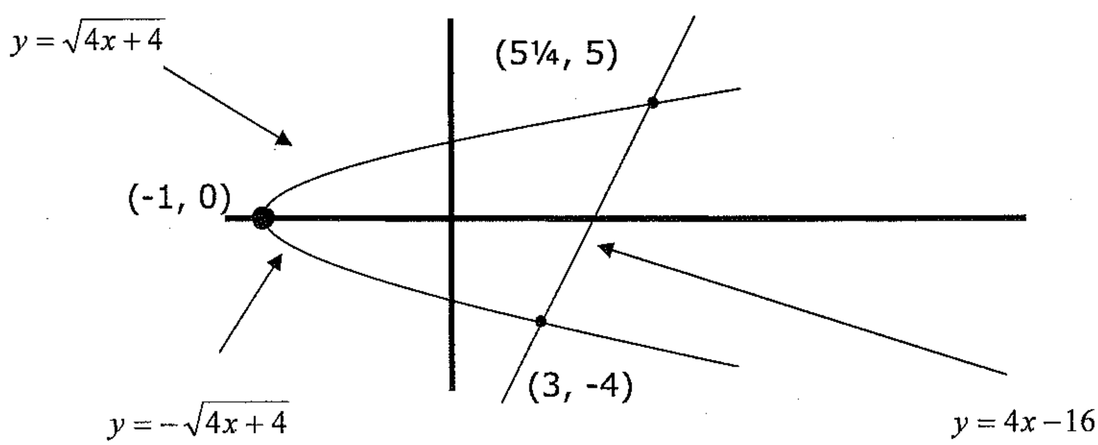
Problem: Let the curves $y^2 - 4x = 4$ and $4x - y = 16$ be the bounds for an enclosed region. SUBDNI the definite integrals for the volume of the solid by rotating this region about the following axes using both the disk/washer method and the shell method (i.e., generate 2 definite integrals that rotate the region about each axis – one using the disk/washer method, the other using the shell method; 8 definite integrals total):

$x = -3$ $\quad x = 6$ $\quad y = -7$ $\quad y = 8$

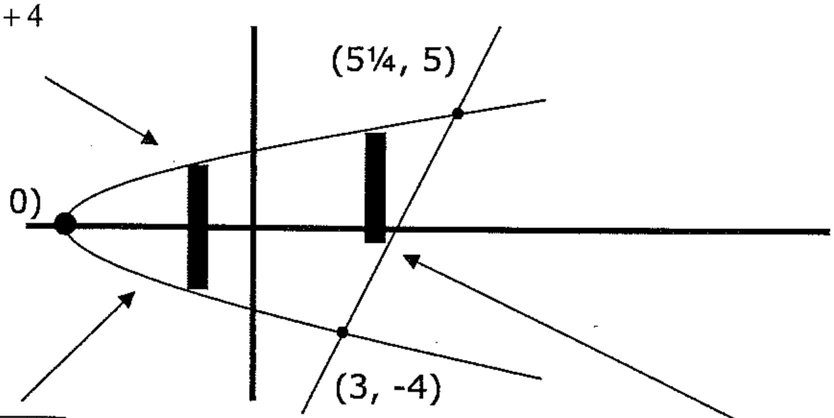
First, graph your region on your calculator; however, in order to do this, you must first convert the two equations into functions. The curve $y^2 - 4x = 4$ needs to be handled like a piece-wise function: $y = \sqrt{4x+4}$ and $y = -\sqrt{4x+4}$; the line is easily rewritten as $y = 4x - 16$. Enter these equations into your calculator and graph them:



Next, you need to find the points of intersection between the three different functions by either setting them equal to each other and solving (ugh . . . not pretty), or use the INTERSECT feature on your graphing calculator (let's use this one). Do it now . . . I can wait . . .



Ok, now we begin . . . lets draw vertical chunks . . . Stink! I notice that the bounds change when x = 3 . . . that means another definite integral . . . might as well start now . . .



For rotation about $y=-7$: disk/washer as the chunks are perpendicular to the axis of revolution (AOR)

$$\pi\int_{-1}^{3}[(\sqrt{4x+4}-(-7))^2-(-\sqrt{4x+4}-(-7))^2]\,dx\ +$$

$$\pi\int_{3}^{\frac{21}{4}}[(\sqrt{4x+4}-(-7))^2-((4x-16)-(-7))^2]\,dx$$

For rotation about $y=8$: disk/washer as the chunks are perpendicular to AOR

$$\pi\int_{-1}^{3}[(8-(-\sqrt{4x+4}))^2-(8-\sqrt{4x+4})^2]\,dx+$$

$$\pi\int_{3}^{\frac{21}{4}}[(8-(4x-16))^2-(8-(\sqrt{4x+4}))^2]\,dx$$

For rotation about $x = -3$: shell method as the chunks are parallel to AOR

$$2\pi \int_{-1}^{3} [(x-(-3)) \cdot (\sqrt{4x+4} - (-\sqrt{4x+4}))]\, dx + 2\pi \int_{3}^{\frac{21}{4}} [(x-(-3)) \cdot (\sqrt{4x+4} - (4x-16))]\, dx$$

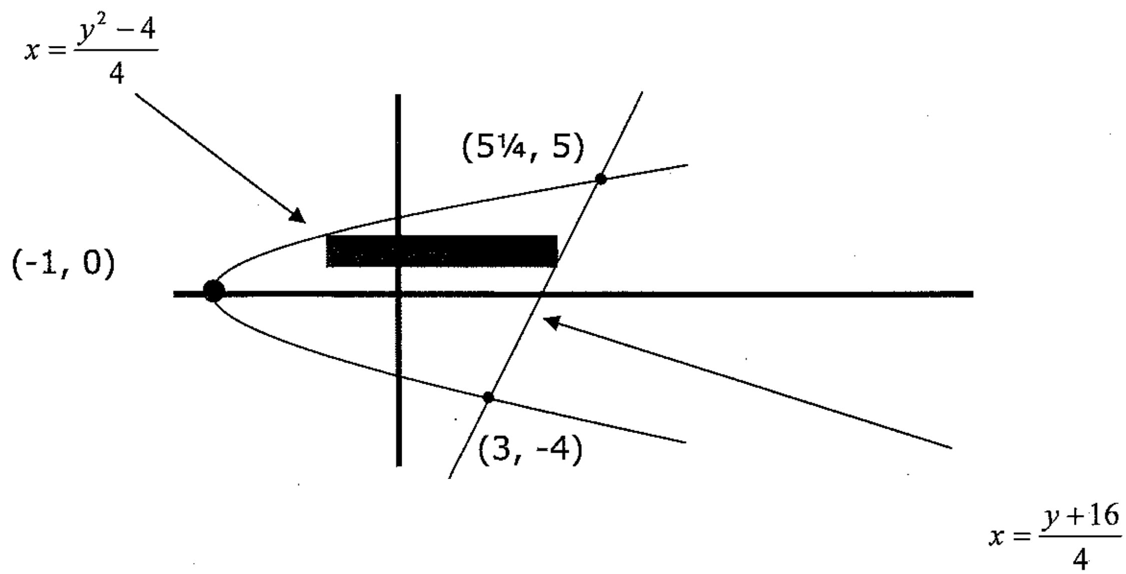
For rotation about $x = 6$: shell method as the chunks are parallel to AOR

$$2\pi \int_{-1}^{3} [(6-x) \cdot (\sqrt{4x+4} - (-\sqrt{4x+4}))]\, dx + 2\pi \int_{3}^{\frac{21}{4}} [(6-x) \cdot (\sqrt{4x+4} - (4x-16))]\, dx$$

GOTTA LOVE THE SHELL METHOD!!! The only thing that really changed was the radius. The height of the chunk, the limits . . . all the same!

That sucked . . . two chunks can really complicate things . . . if you want to make your life easier, set up your chuck in a fashion where you would only need one chunk to satisfy the conditions. Even if it means using horizontal chunks (the horror of it!) and writing your equations in terms of "y", it is much easier to manipulate.

Let's do that now . . . set up your region in terms of "y" and draw a horizontal chunk:



This will be much easier to manipulate , , , the bounds are fractions, but manageable . . . the chunk will slide between – 4 and 5 . . .

For rotation about $y=-7$: shell method as the chunks are parallel to the AOR

$$2\pi\int_{-4}^{5}(y-(-7))\cdot\left(\left(\frac{y+16}{4}\right)-\left(\frac{y^2-4}{4}\right)\right)dy$$

For rotation about $y=8$: shell method as the chunks are parallel to the AOR

$$2\pi\int_{-4}^{5}(8-y)\cdot\left(\left(\frac{y+16}{4}\right)-\left(\frac{y^2-4}{4}\right)\right)dy$$

For rotation about $x=-3$: disk/washer method as the chunks are perpendicular to the AOR

$$\pi\int_{-4}^{5}\left(\left(\frac{y+16}{4}\right)-(-3)\right)^2-\left(\left(\frac{y^2-4}{4}\right)-(-3)\right)^2 dy$$

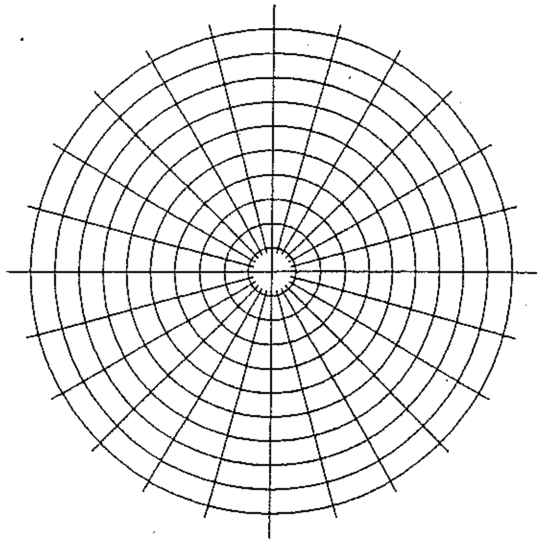
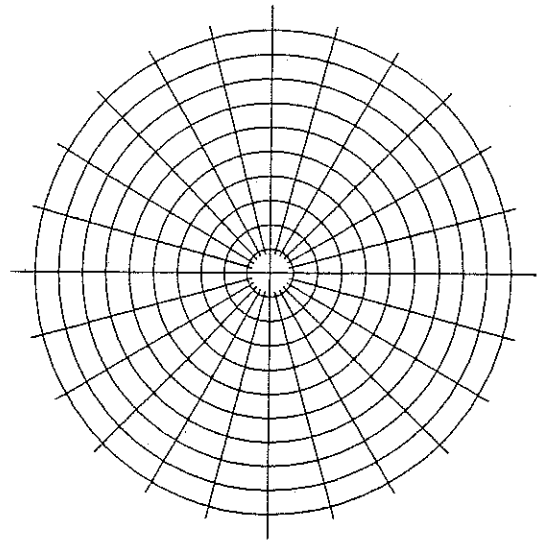
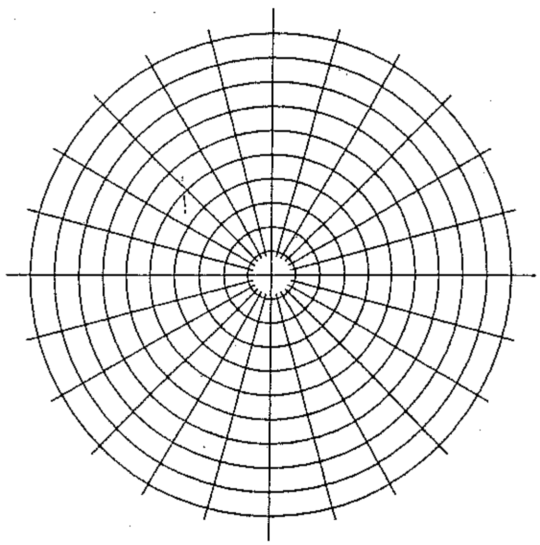
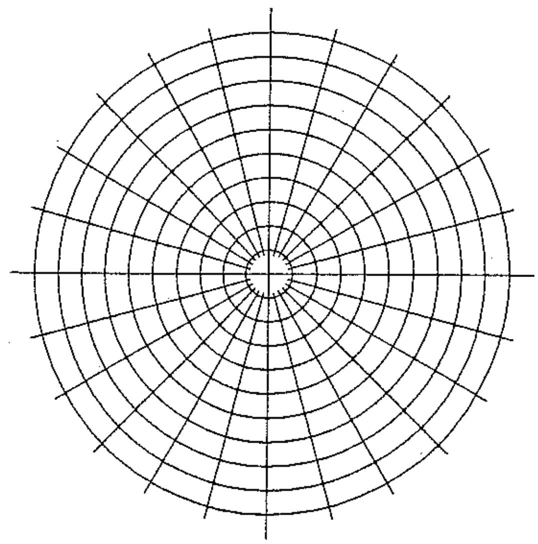
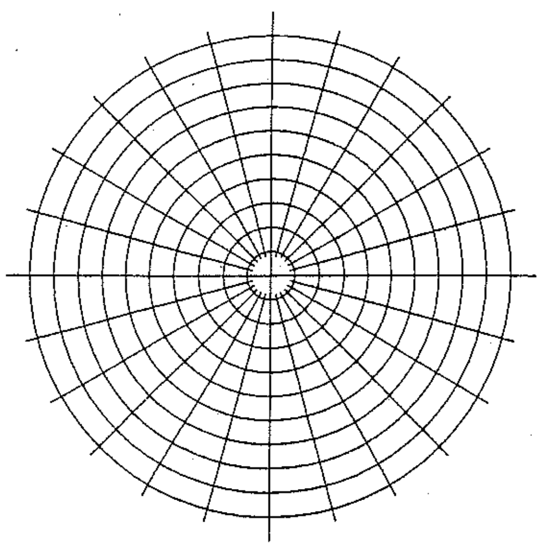
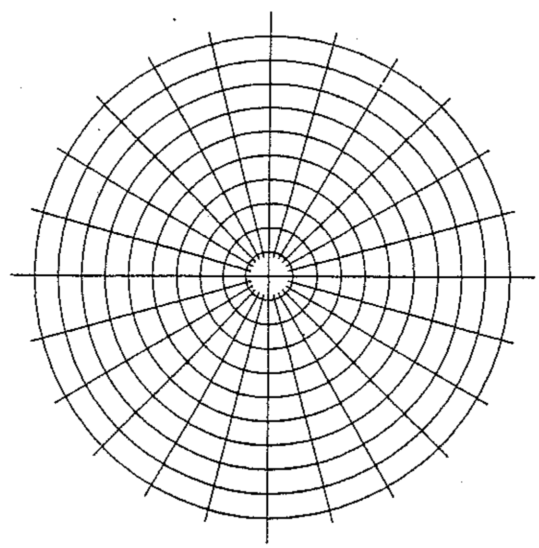
For rotation about $x=6$: disk/washer method as the chunks are perpendicular to the AOR

$$\pi\int_{-4}^{5}\left(6-\left(\frac{y^2-4}{4}\right)\right)^2-\left(6-\left(\frac{y+16}{4}\right)\right)^2 dy$$

If you really understand this, you understand rotational volume for the AP Exam! Well done!+

Name ______________________ Group ________________ Date __________

Polar Coordinate Paper

Area Bound by Polar Curves

We know how to find the area of a region bound by "X-Y" functions: find the points of intersection of the functions for the interval (or limits) for the region, determine which function "dominates" the other over the interval to establish the height of a representative "chunk" (rectangle) for the region, set up a definite integral where the integrand is the height of the representative "chunk" and the width is the differential, establish the limits of the integral to the interval of the region. We also need to keep several other points in mind: make sure the representative "chunk" slides from the lower limit to the upper limit and observe if the representative "chunk" changes bounds over the interval, note the representative "chunk" is perpendicular to the axis which provides the differential, make sure the functions are in terms of the proper variable (i.e., if the representative "chunk" is perpendicular to the X-axis, then all functions must be written in terms of "x" and the differential is dx; if the representative "chunk" is perpendicular to the Y-axis, then all functions must be written in terms of "y" and the differential is dy). The definite integral is the "summing" process (i.e., the Riemann Sum) for an infinite number of chunks whose width is approaching zero.

The process for finding area bound by polar curves has many similarities to finding the area bound by "X-Y" functions. We will find "chunk" the region and add up the areas of the individual "chunks" (using a definite integral) to find the area of the region; we will find points of intersection between the curves for the limits of the definite integral; we will notice in some cases which curves "dominate" the other. However, there are some distinct differences which will need development.

The first distinction is how we "chunk" the region. We will use "sectors" instead of rectangles. This drives us to general equation for area bound by polar curves: $\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$. The second distinction is the limits α and β are angles whose rays originate at the center. There is only one kind of orientation for the chunk: a sector.

Types of polar area problems

There are three kinds of polar area problems: "One Function / One Region," "Inside / Outside" and "Overlapping."

One Function / One Region

In this type of problem, you will be given a single function, and may even be given the limits. You will find that simply plugging into the general equation for polar area will easily help you determine the solution.

Example

Find the area of the region bound by $r = Cos\,\theta$ between $\frac{\pi}{6}$ and $\frac{4\pi}{3}$.

We are given r, α and β; we are good to go!

$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{4\pi}{3}} (Cos\,\theta)^2\, d\theta = 0.916$$

This is a pretty simple problem. You are getting some more difficult problems below, but the premise is the same – you will be given r, and maybe even α and β. If you are not given α and β, you must find them! BIG HINT (you can tell because I used caps and larger font . . . lol): try to use symmetry as much as possible! It will make your life dramatically easier!

Practice problems (set up the integral, integrate with your calculator, and record your answer)

Find the area of the region bound by $r = 4$ between $-\frac{2\pi}{3}$ and $\frac{\pi}{2}$.

Find the area of the region bound by $r = [(Sin\,\theta)*(Cos\,\theta) + Sin\,\theta]$ between 0 and $\frac{5\pi}{12}$.

Find the area of the region (*in the 2nd and 3rd quadrants*) bound by $r = 1 - Sin\,\theta$ and the Y-axis. (Careful here with your limits . . . make sure you know exactly what θ gives you the first point of the region you desire (i.e., the lower limit))

Practice problems (con't)
Find the area of one pedal of function $r = 2Cos3\theta$
(Again, be sure to identify your limits! Using symmetry is very helpful on this one!)

Inside / Outside
For this type of problem, you are literally told you want the region inside one curve and outside another. This kind of problem is very intuitive – we will simply take the difference of the areas. Allow the greater function (the one the region is inside) to be labeled R and the smaller function (the one the region is outside) to be labeled r. Hence, the area of the region will be

$$\frac{1}{2}\int_{\alpha}^{\beta}\left(R^2 - r^2\right)d\theta .$$

Example
Find the area of the region inside the function $r = 2 - 2Sin\theta$ and outside $r = 2$.
In this case, $R = 2 - 2Sin\theta$ and $r = 2$. We have our integrand ready to go: $(2-2Sin\theta)^2 - (2)^2$.
Now, we need to find the values of α and β.

Look at the graph. You have a circle partially inside a cardioid. Do you notice symmetry? Where does it start? Where does it end? This is the trick - - - using symmetry to establish the region. I am looking for the far "right" point of intersection for the functions. Where does it occur? That's right, when $\theta = 0$. Now, where is it on the other side? At what θ does that occur? Let's hope it occurs at the same θ, because that would make life easier . . . does it? Ok, it does . . . now figure it out . . .

Now, you can tell the points of intersection, set up your integral and solve . . . wait! What is my lower bound? Is it 0 or π? Is the limit 0, or is it *really* 2π? Stink! I will let you do this for practice - - - try to use symmetry (and where would that be?) - - - it is easier that way!

Practice problems (set up the integral, integrate with your calculator, and record your answer)
Find the area of the region inside the function $r = 1 + Cos\,\theta$ and outside $r = Cos\,\theta$. This one is a little tricky - - - why? I'll bet you can do it anyway!

Find the area of the region inside the circle $r = 1$ and outside the cardioid $r = 1 - Cos\,\theta$.
Use symmetry! It is easy that way . . .

Find the area of the region inside the circle $r = 2Cos\,\theta$ and outside the circle $r = Cos\,\theta$
Use symmetry! It is easy that way . . .

Overlapping

The last type of area we will examine is called "Overlapping." You will be told to find the area of a region shared by two curves, or a "common" area. The trick here is to know which function is the bound ***at any given angle***. This is best explained by example.

Example

Find the area of the region bound by the functions $r = Cos\,\theta$ and $r = Sin\,\theta$.

It is easy to see the functions intersect at the center, but what is the other point? It appears to be in the first quadrant. How do we find this point? (count down from five, four, three, two one).

Yes, it is $\frac{\pi}{4}$, but how? Yes again, let $Cos\,\theta = Sin\,\theta$ and solve. Now, over the interval $\left[0, \frac{\pi}{4}\right)$, which of the two bounds is being utilized? Look closely; the $Sin\,\theta$ is the only bound over that interval. Now look at the interval from $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$; only $Cos\,\theta$ applies. So from here, we set up our two integrals and evaluate:

$$\frac{1}{2}\int_0^{\frac{\pi}{4}} (Sin\,\theta)^2\, d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (Cos\,\theta)^2\, d\theta = 0.1426$$

Practice problems (set up the integral, integrate with your calculator, and record your answer)

Find the area of the shared region bound by the functions $r = 1$ and $r = 2\,Sin\,\theta$.

Find the area of the shared region bound by the functions $r = Cos\,\theta$ and $r = 2\,Sin\,\theta$.

Practice problems (con't)

Find the area of the shared region bound by the functions $r = 1 + Cos\,\theta$ and $r = 1 - Cos\,\theta$.

The trick on any polar area problem is finding the limits and knowing what function dominates over what interval. I WILL be asking how you found your limits, and you must have a correct method short of cheating off my "minimalists" . . .

Now, suppose we were to go at this non-calculator . . . ☺ ☹

OMG!

WTH! (not f)

GTG

ttfn

AP

AP® Calculus AB/BC
2014 Free-Response Questions

CollegeBoard

CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

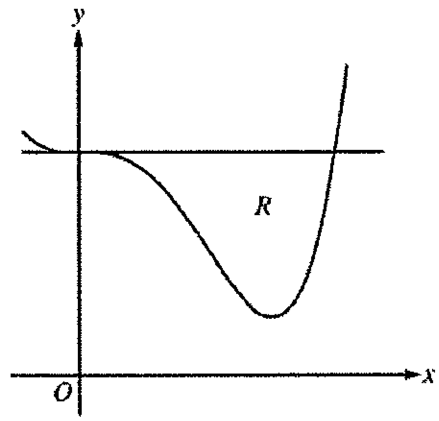
1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30,$ the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t,$ where $A(t)$ is measured in pounds and t is measured in days.

   (a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30.$ Indicate units of measure.

   (b) Find the value of $A'(15).$ Using correct units, interpret the meaning of the value in the context of the problem.

   (c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30.$

   (d) For $t > 30,$ $L(t),$ the linear approximation to A at $t = 30,$ is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.



2. Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

   (a) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

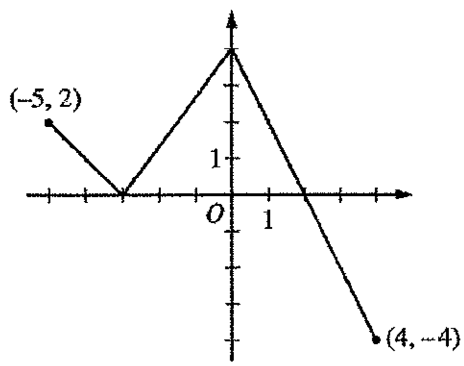
   (b) Region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in R. Find the volume of the solid.

   (c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k.

END OF PART A OF SECTION II

CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\, dt$.

   (a) Find $g(3)$.

   (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

   (c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

   (d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	–120	–150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

   (a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

   (b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

   (c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

   (d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

---

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

5. The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

   (a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

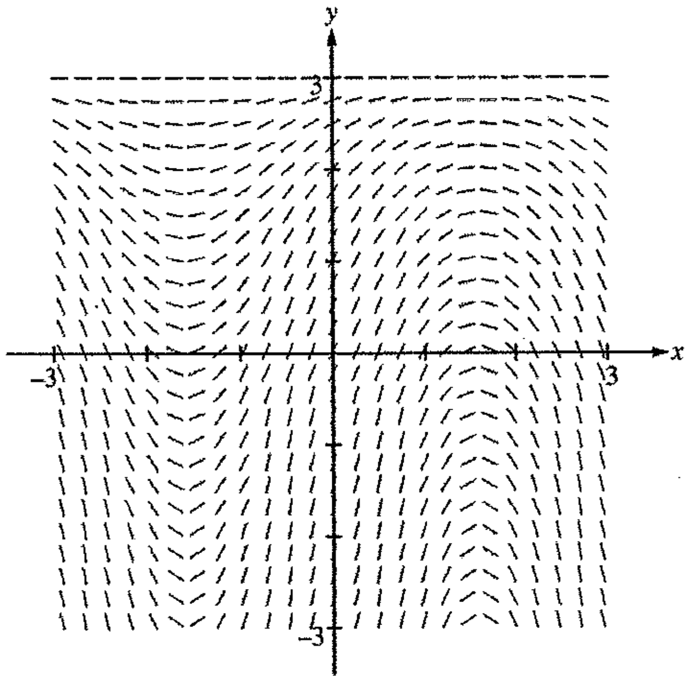
   (b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

   (c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

   (d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

6. Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

   (a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.

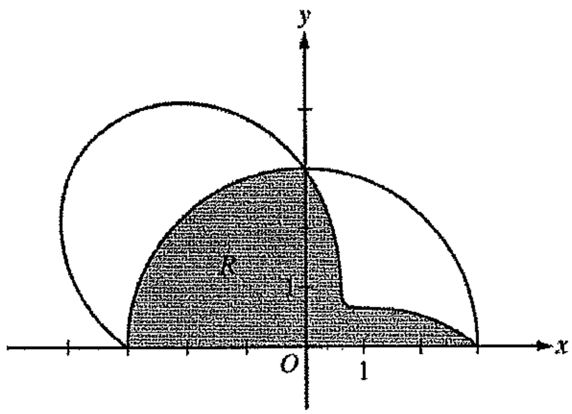


   (b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

   (c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

STOP

END OF EXAM



2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

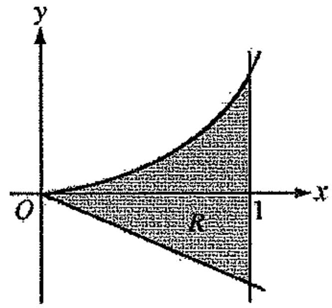
(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

END OF PART A OF SECTION II



5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

   (a) Find the area of R.

   (b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

   (c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

   (a) Find the value of R.

   (b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

   (c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

STOP

END OF EXAM