Lab 2: Introduction to Limits of Functions

Goals

- To develop an intuitive understanding of the nature of limits.
- To lay the foundation for the frequent use of limits in calculus.
- To experience the power and peril of investigating limits by successively closer evaluation.

In the Lab

In this lab we shall study the behavior of a function f near a specified point. While this is sometimes a straightforward process, it can also be quite subtle; in many instances in calculus the process for finding a limit must be applied carefully. By gaining an intuitive feel for the notion of limits, you will be laying a solid foundation for success in calculus.

1. Consider the function f defined by $f(x) = \dfrac{x^4 - 1}{x - 1}$.

   a. By successive evaluation of f at $x = 1.8$, 1.9, 1.99, 1.999, and 1.9999, what do you think happens to the values of f as x increases towards 2?

   b. Do a similar experiment on f for values of x slightly greater than 2. Again, comment on your results.

   As a shorthand, and anticipating a forthcoming definition, we shall describe what you found in parts a and b by writing $\lim_{x \to 2} f(x) = 15$, or more specifically,

   $$\lim_{x \to 2} \frac{x^4 - 1}{x - 1} = 15.$$

   c. In this particular case you could have "cheated" by immediately evaluating f at 2. Get a computer plot of the function between 1.8 and 2.2 to illustrate what happens in this straightforward situation.

2. Use the same function f as above, but this time consider what happens as x approaches 1.

   a. Study this situation experimentally as you did in parts a and b of Problem 1. To gain some additional feel and respect for the situation, compute the numerator and denominator of f separately for several x values before dividing. What are your conclusions and, in particular, what is $\lim_{x \to 1} f(x)$?

   b. What happens when you try to "cheat" as was done in part c of Problem 1? There are situations in which direct evaluation at the specified point is possible and actually gives the limit. These give rise to a concept called *continuity*. There are many important situations in calculus when this technique will not work, however.

3. By computer or calculator experimentation, try to determine the values of the following limits. Use either graphing or function evaluation at nearby points.

   a. $\displaystyle \lim_{x \to 0} \frac{\sin(10x)}{x}$

   b. $\displaystyle \lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} \dfrac{x^4 - 1}{x - 1}, & \text{for } x < 1, \\ 17, & \text{for } x = 1, \\ 14 - 10/x, & \text{for } x > 1 \end{cases}$$

   Important Comment and Hint: As far as the existence and value of the limit are concerned, the value of $g(1)$ has no relevance.

   c. $\displaystyle \lim_{x \to 0} (1 + x)^{1/x}$

   Record your best guess; the limit is a famous mathematical constant.

   d. $\displaystyle \lim_{t \to 1} \frac{t^n - 1}{t - 1}$ for a general positive integer n

   Hint: Recall Problem 2, try other values of n, and generalize.

4. It is important to be aware that limits can sometimes fail to exist. Investigate the following limits and explain why you think each does not exist. You may find it helpful to use the computer to evaluate the functions at different values of x and to plot graphs of the functions.

   a. $\displaystyle \lim_{x \to 2} \frac{x}{x - 2}$

   b. $\displaystyle \lim_{x \to 0} \frac{\sin(10x)}{x^2}$

c. $\lim\limits_{x \to 0} \sin(1/x)$

d. $\lim\limits_{x \to 0} |x|/x$

e. $\lim\limits_{x \to 4} f(x)$, where

$$f(x) = \begin{cases} (x+2)^3, & \text{for } x < 4, \\ e^x, & \text{for } x > 4 \end{cases}$$

Note: For parts d and e think about the idea of a "one-sided limit" and store your thoughts for future reference. You should also convince yourself that computer evaluation is not really needed to make wise conclusions in these particular situations.

Further Exploration

5. Find $\lim\limits_{x \to \infty} f(x)$, if it exists, for $f(x) = \dfrac{3x^4 - x^2 + 10}{2x^4 + 5/x}$, $f(x) = \dfrac{\sin x}{1 + x^2}$, and $f(x) = \dfrac{4 - 3/x}{\sin x}$. Hints: The limit exists for two of these functions. Answer this question by using computer evaluation, common sense, and perhaps some algebra.

6. Try to determine the limit $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ when $f(x) = \ln x$. Hints: After specifying the quotient for the given f, treat x as a constant and give x a specific value of your own choosing before investigating the limit as h goes to 0. Then repeat for several other x values, make a table of results, and try to see the pattern. What function of x emerges as you compute this limit for values of x? Limits of this particular *difference quotient* are very important in calculus.

Lab 4: Discovering the Derivative

Goals

- To define the slope of a function at a point by zooming in on that point.
- To develop the definition of the derivative of a function at a point by examining slopes of secant lines.
- To understand situations in which the derivative will fail to exist.

In the Lab

You learned in analytic geometry that the slope of a non-vertical straight line is $\dfrac{\Delta y}{\Delta x}$ or $\dfrac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are any two points on the line. Most functions we see in calculus have the property that if we pick a point on the graph of the function and zoom in, we will see a straight line.

1. Graph the function $f(x) = x^3 - 6x + 3$ for $-3 \le x \le 4$. Zoom in on the point $(1, -2)$ until the graph looks like a straight line. Pick another point on the curve other than $(1, -2)$ and estimate the coordinates of this point. Calculate the slope of the straight line through these two points.

The number computed above is an approximation to the *slope of the function* $f(x) = x^3 - 6x + 3$ *at the point* $(1, -2)$. This slope is also called the *derivative of* f *at* $x = 1$, and is denoted by $f'(1)$.

In Problem 1 you learned how to use the graph of a function to estimate the value of its derivative at a specified point. There is an analytic definition of the derivative that you will now develop using a primarily geometric approach.

2. In Figure 1, the straight line intersects the graph of the function f at two points with x-coordinates a and $a + h$. Write expressions for the coordinates of these two points and then write a formula for the slope of the straight line.

3. Again, consider the function f defined by $f(x) = x^3 - 6x + 3$, and let $a = 1$. Its graph over the domain $[0, 2]$ is given in Figure 2.

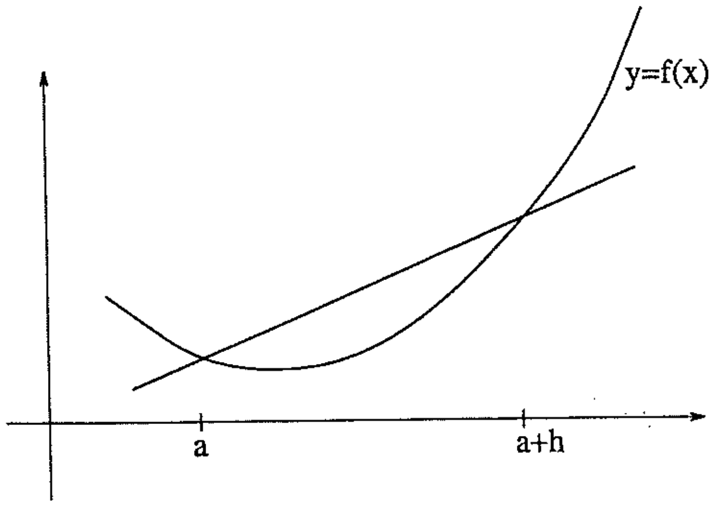

Figure 1

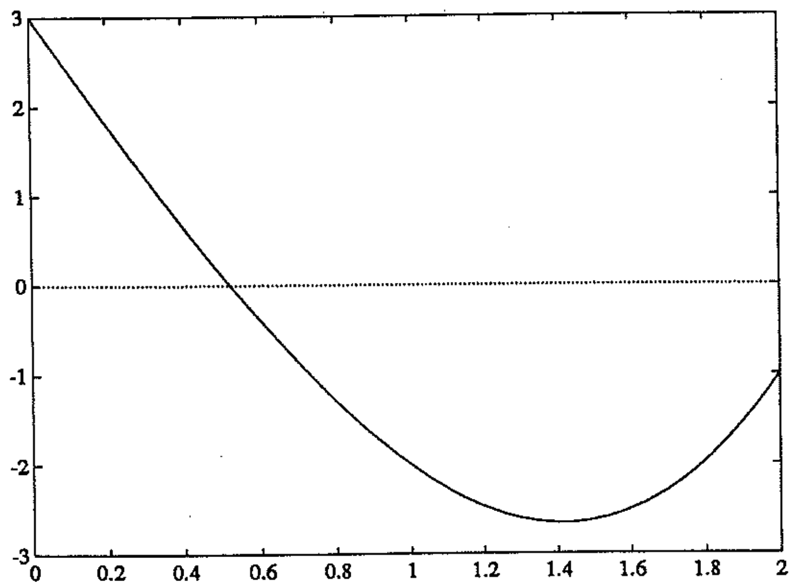

Figure 2

a. For each value of h, sketch by hand, on Figure 2, the straight line through $(1, f(1))$ and $(1+h, f(1+h))$. Make a table containing the values of h and the slopes of the straight lines. Use $h = 0.6$, 0.4, 0.2, -0.6, -0.4, and -0.2.

b. As the h values get smaller and smaller (positive or negative), what geometrical property do the lines we have drawn and the line to which they tend seem to have? By inspecting your graph and the slopes you have computed, try to guess a limiting value for the slopes of the straight lines.

c. Based on your answer to Problem 2 and your thoughts on parts a and b, write an expression (in terms of a general f) for the slope of the limiting (that's a hint) straight line referred to in b. We call this expression the *derivative of f at a*, and we denote it by $f'(a)$.

d. Use the computer to estimate or evaluate this limit for the specific function $f(x) = x^3 - 6x + 3$ and $a = 1$. Is the answer consistent with the slope estimate you made in part b?

We now have an analytic definition corresponding to the geometric picture of the derivative of a function at a point. It is an important aspect of calculus that there is an "algorithm" for computing the derivatives of familiar functions. Not surprisingly, your computer algebra system has a command to give values for the derivative $f'(a)$.

4. Use the computer to investigate (1) geometrically via zooming, (2) analytically via use of limits, and (3) directly via your computer algebra system's derivative command, the value of $f'(a)$ for each of the situations given below. Do the various methods give answers that are consistent and reasonable?

   a. $f(x) = x^4 - x^2 + 3$ at $(1, 3)$

   b. $f(x) = \sin x$ at $(\pi, 0)$

5. So far in this lab the functions and points considered have always had derivatives that were defined. Sometimes, however, a function does not have a well-defined slope at a point. Discuss what is happening with $f'(a)$, and hence with the slope of the appropriate tangent line, in each of the situations below. Use zooming and then the derivative command. Support your answers with appropriate sketches of the graphs of both functions.

   a. $f(x) = (x - 1)^{1/3}$ at $a = 1$

   b. $f(x) = |x^2 - 4|$ at a general a

   For what values of a does $f'(a)$ exist and for what values does it fail to exist?

Further Exploration

6. Explain in your own words how calculating the slope of a function at the point $(a, f(a))$ by repeated zooming is related to the definition of the derivative $f'(a)$ that you discovered in Problem 3*c*.

Lab 6: Relationship between a Function and Its Derivative

Goal

- Given the graph of a function, to be able to visualize the graph of its derivative.

Before the Lab

In this laboratory, you will be asked to compare the graph of a function like the one in Figure 1 to that of its derivative. This exercise will develop your understanding of the geometric information that f' carries.

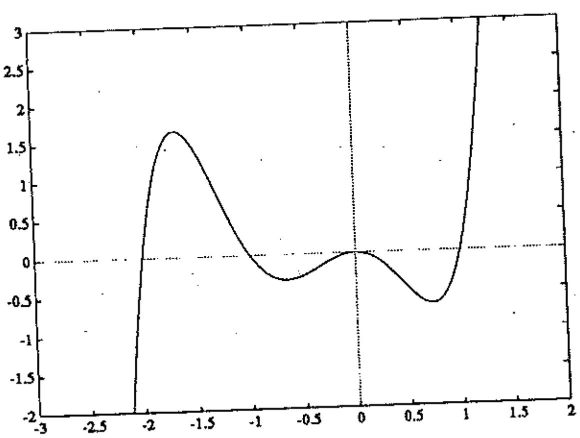

Figure 1: $f(x) = x^2(x-1)(x+1)(x+2)$

You will need to bring an example of such a function into the lab with you, one whose graph meets the x-axis at four or five places over the interval $[-3, 2]$. During the lab, your partner will be asked to look at the graph of your function and describe the shape of its derivative (and you will be asked to do the same for your partner's function). One way to make such a function is to write a polynomial in its factored form. For example, $f(x) = x^2(x+1)(x-1)(x+2)$ is the factored form of the function in Figure 1. Its zeros are at 0, 1, -1, and -2.

1. Give another example of a polynomial g of degree at least 5 with four or five real zeros between -3 and 2. You will use this polynomial in Problem 4.

   a. Your polynomial: $g(x) =$ ____________________________.

   b. Its zeros: ____________________________.

   c. Its derivative: $g'(x) =$ ____________________________. If your function is complicated, you may want to use the computer to calculate its derivative.

In the Lab

2. Let $f(x) = x^2(x^2 - 1)(x + 2) = x^5 + 2x^4 - x^3 - 2x^2$.

   a. Find the derivative of f. Plot the graphs of both f and f' in the same viewing rectangle over the interval $[-2.5, 1.5]$.

   Answer the following questions by inspection of this graph:

   b. Over what intervals does the graph of f appear to be rising as you move from left to right?

   c. Over what intervals does the graph of f' appear to be above the x-axis?

   d. Over what intervals does the graph of f appear to be falling as you move from left to right?

   e. Over what intervals does the graph of f' appear to be below the x-axis?

   f. What are the x-coordinates of all of the high points and low points of the graph of f?

   g. For what values of x does the graph of f' appear to meet the x-axis?

3. Let $f(x) = \dfrac{x}{1 + x^2}$.

   a. Find the derivative of f. Plot the graphs of both f and f' in the same viewing rectangle over the interval $[-3, 3]$.

   b. Answer the same set of questions as in parts *b*-*g* above.

4. On the basis of your experience so far, write a statement that relates where a function is rising, is falling, and has a high point or low point to properties you have observed about the graph of its derivative.

5. Now let g be the function that your lab partner brought into the lab. (If you have no partner, just use your own function.) In this problem you will use your statement from Problem 4 to predict the shape of the graph of g', given only the shape of the graph of g.

   a. Have your lab partner produce a plot of the graph of g over the interval $[-3, 2]$. Your partner may need to adjust the height of the window to capture all of the action. On the basis of this plot, use your conjecture to imagine the shape of the graph of g'. In particular, find where g' is above, where g' is below, and where g' meets the x-axis. Carefully sketch a graph of both g and your version of g' on your data sheet, labeling each graph.

   b. Now have your lab partner plot the graph of g'. Add a sketch of the actual graph of g' onto your drawing. Compare your graph with the computer drawn graph. How did you do?

   c. Reverse roles with your lab partner and do parts a and b again.

6. Consider the function $f(x) = |x^2 - 4|$. A graph of the function will help you answer these questions.

   a. There are two values of x for which the derivative does not exist. What are these values, and why does the derivative not exist there?

   b. Find the derivative of f at those values of x where it exists. To do this, recall that f can be defined by $f(x) = \begin{cases} x^2 - 4, & |x| \geq 2, \\ 4 - x^2, & |x| < 2. \end{cases}$ You can compute the derivative for each part of the definition separately.

   c. Give a careful sketch of f and f' (disregarding the places where f' is not defined) over the interval $[-4, 4]$. Does your conjecture from Problem 4 still hold? Do you need to make any modifications?

Further Exploration

7. Consider the function $f(x) = 2^x$. Some people think that $f'(x) = x2^{x-1}$. On the basis of your conjecture explain why this cannot be true.

8. This laboratory has given you experience in using what you know about the shape of the graph of a function f to visualize the shape of its derivative function f'. What about going backwards? Suppose that your partner had given you the graph of f', would you be able to reconstruct the shape of the graph of f? If f' is positive, for example, does your conjecture enable you to rule out certain possibilities for the shape of f? The graph in Figure 2 is a sketch of the derivative of f. Use your conjecture to construct a possible graph for the function f itself. The important part of this problem is neither the actual shape that you come up with, nor its position in the xy-plane, but your reasons for choosing it. Why isn't there a unique function that has f' for its derivative?

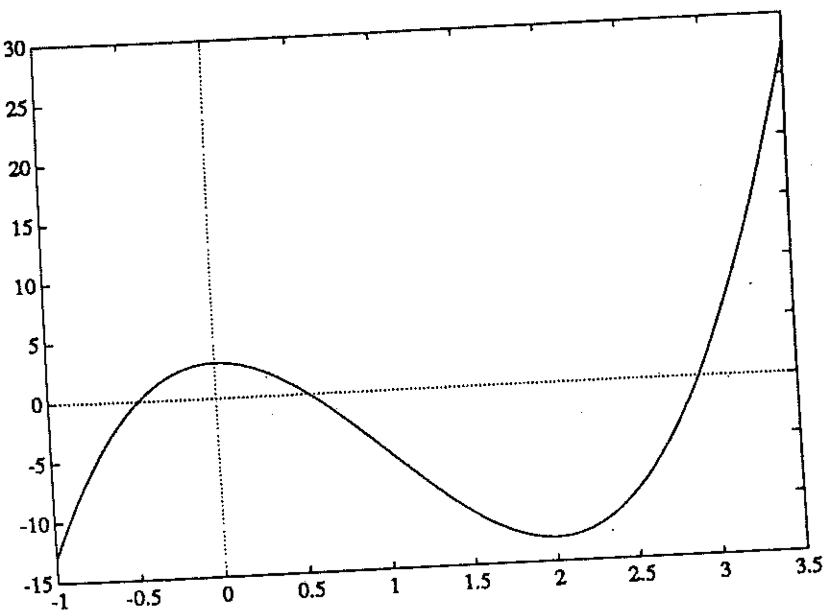

Figure 2: The graph of $y = f'(x)$

9. Figure 3 shows the graphs of three functions. One is the position of a car at time t minutes, one is the velocity of that car, and one is its acceleration. Identify which graph represents which function and explain your reasoning.

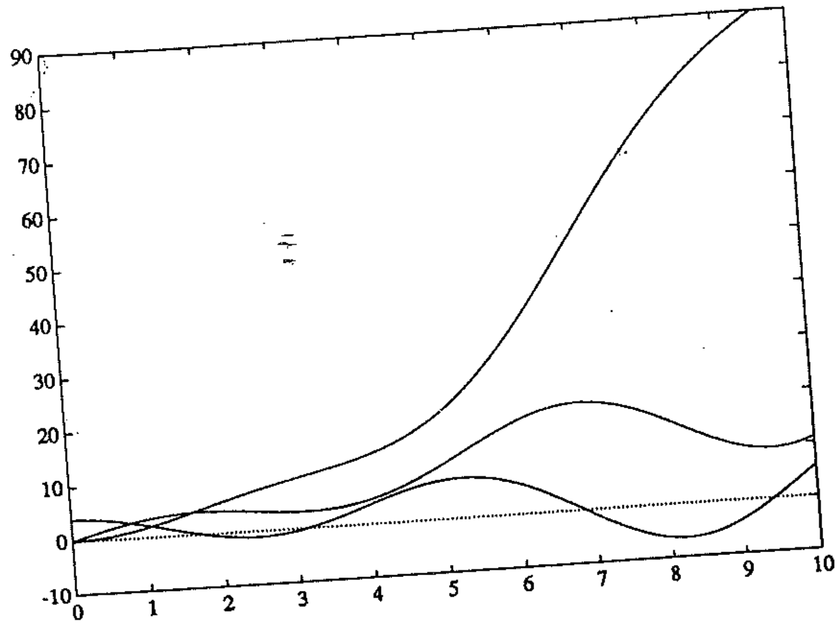

Figure 3: Position, velocity and acceleration graphs

Lab 7: Linking Up with the Chain Rule

Goal

- To understand the Chain Rule for computing derivatives.

In the Lab

1. The Power Rule for computing derivatives says that $\frac{d}{dx}x^2 = 2x$. Thus we should not be surprised to learn that one factor of $\frac{d}{dx}\big(f(x)\big)^2$ is $2f(x)$. The purpose of this problem is to determine the other factor needed to yield the correct derivative.

   a. Functions of the form $g(x) = (ax + b)^2$ are simple enough that we can algebraically determine the other factor in the derivative. Expand $(ax + b)^2$ and differentiate the resulting expression term by term. Rewrite your answer to $g'(x)$ in the form $2(ax + b)k$. What is k? What part of the original expression can be differentiated to give k?

   b. For functions of the form $g(x) = (ax^2 + bx + c)^2$ we can use a computer or symbolic calculator to follow the same procedure of expanding and differentiating term by term to write the derivative as $g'(x) = 2(ax^2 + bx + c)k(x)$. What is $k(x)$? What part of the original expression can be differentiated to give $k(x)$?

The next two problems use the method of zooming to approximate the derivative of a function f at a point a. To do this, use the computer to graph the function f on an interval containing $x = a$. Then zoom in around the point $\big(a, f(a)\big)$ until your graph resembles a straight line. Pick a point on the curve other than the point $\big(a, f(a)\big)$ and estimate its coordinates. Calculate the slope of the line through these two points. This will be a numerical approximation to $f'(a)$.

2. The derivative of $f(x) = (g(x))^n$ contains the factor $n(g(x))^{n-1}$. In this problem we will determine what the other factor might be.

   a. Complete the following table. In the first column you are given functions of the form $f(x) = (g(x))^n$ where $g(x) = x^2 - 3x$. The evaluation of the second column can be done by hand. Use zooming to approximate the slope of f at $x = -1$ for the third column. In the last column enter your best guess of an integer or simple fraction you can multiply $n(g(-1))^{n-1}$ by to get $f'(-1)$.

function $f(x) = (g(x))^n$	*value of* $n(g(x))^{n-1}$ at $x = -1$	*approximation to* $f'(-1)$	*correction factor*
$(x^2 - 3x)^2$			
$(x^2 - 3x)^3$			
$\sqrt{x^2 - 3x}$			

   b. In this problem the correction factor is the same for all three functions in part a. Where does this number come from?

   c. Consider the function $f(x) = (x^3 - 2x^2 - 2x + 6)^3$. Without using the computer, make a conjecture as to the value of $f'(1)$. Explain the reasoning behind your answer.

   d. In general $\dfrac{d}{dx}(g(x))^n = n(g(x))^{n-1}k(x)$. What is $k(x)$? What part of the original expression can be differentiated to give $k(x)$?

3. This problem examines the derivative of $f(x) = \sin(g(x))$ for several choices of $g(x)$. Since $\dfrac{d}{dx}\sin x = \cos x$, we expect, and rightly so, that $\cos(g(x))$ will be one factor of $\dfrac{d}{dx}\sin(g(x))$. The question is: What else is needed?

   a. Complete the following table. In the first column you are given functions of the form $f(x) = \sin(g(x))$. The evaluation in the second column can be done by calculator or computer. Use zooming to approximate the slope of f at $x = 3$ for the third column. In the fourth column enter your best guess of an integer or simple fraction you can multiply $\cos(g(3))$ by to get $f'(3)$. The last column asks for the value of $g'(3)$. This can be computed exactly by hand.

function $f(x) = \sin(g(x))$	*value of* $\cos(g(x))$ *at* $x = 3$	*approximation to* $f'(3)$	*correction factor*	*value of* $g'(3)$
$\sin(2x)$				
$\sin(\frac{1}{2}x + 3)$				
$\sin(x^2)$				

b. Without using the computer, make a conjecture as to the form of the derivative of $f(x) = \sin(x^3)$ at $x = 3$? Explain your reasoning.

c. In general, $\frac{d}{dx}\sin(g(x)) = k(x)\cos(g(x))$ for some function $k(x)$. What do you think $k(x)$ equals? What part of the original expression can be differentiated to give $k(x)$?

Further Exploration

4. a. Write $(g(x))^2$ as $g(x)g(x)$ and apply the product rule to find the derivative of $(g(x))^2$.

   b. Write $(g(x))^3$ as $(g(x))^2 g(x)$ and apply the product rule and your result in part a to find the derivative of $(g(x))^3$.

   c. Let n be a positive integer. Suppose the pattern you observed in parts a and b continues to hold for the derivative of $(g(x))^{n-1}$. Write an expression for $\frac{d}{dx}(g(x))^{n-1}$. It should be in the form given in Problem 2d.

   d. Now write $(g(x))^n$ as $(g(x))^{n-1} g(x)$. Use the product rule and the result of part c to find the derivative of $(g(x))^n$ in the form given in Problem 2d. By the Principle of Mathematical Induction, you are now justified in concluding that this pattern continues from one natural number to the next. Thus it holds for all natural numbers.

Template for Lab #7 – *Linking up with the Chain Rule*

For this lab, you will complete 1a, 2abc, 3abc, 4abd. Use the tables provided below to complete 2a and 3a. You may use the back of this handout to record your answers.

2a.

Function in the form $f(x)=(g(x))^n$ (Think: what is the **Derivative Rule (DR)** for this function?) $f'(x)\rightarrow$	Value of $f'(-1)$ determined with the calculator	Value of the **Derivative Rule (DR)** at $x=-1$ … in this case … $n\cdot(\,g(-1)\,)^{n-1}$	***Correction factor***, or what is needed to be multiplied to the **Derivative Rule (DR)** at $x=-1$ to equal the value of $f'(-1)$
$(x^2-3x)^2\rightarrow$			
$(x^2-3x)^3\rightarrow$			
$\sqrt{x^2-3x}\rightarrow$			

3a.

Function in the form $f(x)=\sin(g(x))$ (Think: what is the **Derivative Rule (DR)** for this function?) $f'(x)\rightarrow$	Value of $f'(3)$ determined with the calculator	Value of the **Derivative Rule (DR)** at $x=3$ … in this case … $\cos(g(3))$	***Correction factor***, or what is needed to be multiplied to the **Derivative Rule (DR)** at $x=3$ to equal the value of $f'(3)$	Value of $g'(3)$ (Think: what do you notice?)
$\sin(2x)\rightarrow$				
$\sin\left(\frac{1}{2}x+3\right)\rightarrow$				
$\sin(x^2)\rightarrow$				