1. Level: AP 2

$$\int \frac{3x+1}{x^2-x-6}\,dx = \int\left(\frac{A}{x-3}+\frac{B}{x+2}\right)dx$$

$$\frac{3x+1}{x^2-x-6} = \frac{A}{x-3}+\frac{B}{x+2} \Rightarrow A(x+2)+B(x-3)=3x+1$$

$$\text{Let } x=-2 \Rightarrow A(-2+2)+B(-2-3)=3(-2)+1 \Rightarrow -5B=-5 \Rightarrow B=1$$

$$\text{Let } x=3 \Rightarrow A(3+2)+B(3-3)=3(3)+1 \Rightarrow 5A=10 \Rightarrow A=2$$

$$\boxed{A+B=2+1=3}$$

2. Level: AP 3

$$\int_1^{\infty}\frac{2}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{2}{x^3}\,dx = 2\lim_{b\to\infty}\int_1^b x^{-3}\,dx = 2\lim_{b\to\infty}\left[\frac{x^{-2}}{-2}\right]_1^b = \frac{2}{-2}\lim_{b\to\infty}\left[\frac{1}{x^2}\right]_1^b = -\lim_{b\to\infty}\left[\frac{1}{b^2}-\frac{1}{1^2}\right] = -\left[0-1\right] = \boxed{1}$$

3. Level: AP 2

$$\int 2x\sin(x)\,dx \Rightarrow \boxed{\begin{array}{l} u=2x \\ dv=\sin(x)\,dx \end{array}}$$

4. Level: AP 3

$$\int \frac{1}{x^2+4x-21}\,dx =$$

(A) $\frac{1}{10}\ln\left|\frac{x-3}{x+7}\right|+C$ (boxed)

(B) $\frac{1}{5}\arctan\left(\frac{x+2}{5}\right)+C$

(C) $\frac{1}{10}\ln\left|\frac{x+7}{x-3}\right|+C$

(D) $\frac{1}{10}\left(\frac{1}{x+7}-\frac{1}{x-3}\right)+C$

$$\frac{1}{x^2+4x-21} = \frac{A}{x+7}+\frac{B}{x-3} \Rightarrow A(x-3)+B(x+7)=1$$

$$\text{Let } x=-7 \Rightarrow A(-7-3)+B(-7+7)=1 \Rightarrow -10A=1 \Rightarrow A=-\frac{1}{10}$$

$$\text{Let } x=3 \Rightarrow A(3-3)+B(3+7)=1 \Rightarrow 10B=1 \Rightarrow B=\frac{1}{10}$$

$$\int \frac{1}{x^2+4x-21}\,dx = -\frac{1}{10}\int\frac{1}{x+7}\,dx+\frac{1}{10}\int\frac{1}{x-3}\,dx$$

$$= -\frac{1}{10}\ln|x+7|+\frac{1}{10}\ln|x-3| = \boxed{\frac{1}{10}\ln\left|\frac{x-3}{x+7}\right|+C}$$

5. Level: AP 3

It is known that $\int f(x)\sec^2 x\,dx = f(x)\cdot\tan x - \int 6x^2 \tan x\ dx.$ Which of the following could be $f(x)$?

(A) $18x\cdot\sec^2 x$ (B) $18x$ (C) $2x^3\cdot\sec x\cdot\tan x$ **(D)** $2x^3$

$$\int f(x)\sec^2 x\,dx \Rightarrow \begin{matrix} u = f(x) \\ dv = \sec^2 x\,dx \end{matrix} \Rightarrow \underbrace{f(x)}_{u}\cdot\underbrace{\tan x}_{v} - \int \underbrace{(\tan x)}_{v}\underbrace{(6x^2 dx)}_{du} \Rightarrow \boxed{f(x) = 2x^3}$$

6. Level: AP 4

$$\int_1^4 \frac{1}{x-3}\,dx =$$

(A) $-\ln(2)$ (B) $\frac{3}{2}$ (C) $\frac{3}{20}$ **(D)** The integral diverges.

$\frac{1}{x-3}$ has a interior discontinuity at $x = 3$

$$\int_1^4 \frac{1}{x-3}\,dx = \lim_{c\to 3^-}\int_1^c \frac{1}{x-3}\,dx + \lim_{c\to 3+}\int_c^4 \frac{1}{x-3}\,dx = \lim_{c\to 3^-}\Big[\ln|x-3|\Big]_1^c + \lim_{c\to 3^+}\Big[\ln|x-3|\Big]_c^4$$

$$= \lim_{c\to 3^-}\Big[\ln|x-3|\Big]_1^c + \lim_{c\to 3^+}\Big[\ln|x-3|\Big]_c^4 = \lim_{c\to 3^-}\Big[\ln|c-3| - \ln|1-3|\Big] + \lim_{c\to 3^+}\Big[\ln|4-3| - \ln|c-3|\Big]$$

$$= \underbrace{\ln(0)}_{-\infty} - \ln(2) + \ln(1) - \underbrace{\ln(0)}_{-\infty} \Rightarrow \boxed{\text{Does not exist}}$$

7. Level: AP 4

$$\int x\cos(2x)dx =$$

(A) $\frac{1}{4}x^2\sin(2x) + C$

(B) $2x\sin(2x) + 4\cos(2x) + C$

(C) $\frac{1}{2}x\sin(2x) - \frac{1}{2}\sin(2x) + C$

(D) $\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$

$u = x \Rightarrow du = dx \qquad dv = \cos(2x)dx \Rightarrow v = \frac{1}{2}\sin(2x)$

$$\int x\cos(2x)dx = \underbrace{x\left(\frac{1}{2}\sin(2x)\right)}_{u\cdot v} - \underbrace{\int \frac{1}{2}\sin(2x)dx}_{v\cdot du}$$

$$= \frac{1}{2}x\sin(2x) - \frac{1}{2}\left(-\frac{1}{2}\cos(2x)\right) = \boxed{\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C}$$

8. Level: AP 4

$$\int_2^5 \frac{6x}{(x+2)(x-1)}\,dx =$$

(A) $-\frac{27}{14}$

(B) $\ln\left(\frac{7^4}{4^2}\right)$

(C) $\ln\left(\frac{7^4}{4^6}\right)$

(D) $\ln(20)$

$$\frac{6x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \Rightarrow A(x-1) + B(x+2) = 6x$$

Let $x = 1 \Rightarrow A(1-1) + B(1+2) = 6(1) \Rightarrow 3B = 6 \Rightarrow B = 2$

Let $x = -2 \Rightarrow A(-2-1) + B(-2+2) = 6(-2) \Rightarrow -3A = -12 \Rightarrow A = 4$

$$\int \frac{6x}{(x+2)(x-1)}\,dx = 4\int \frac{1}{x+2}\,dx + 2\int \frac{1}{x-1}\,dx = 4\ln|x+2| + 2\ln|x-1|$$

$$= \ln\left[(x+2)^4(x-1)^2\right]$$

$$\int_2^5 \frac{6x}{(x+2)(x-1)}\,dx = \ln\left[(x+2)^4(x-1)^2\right]_2^5$$

$$= \ln\left[(5+2)^4(5-1)^2\right] - \ln\left[(2+2)^4(2-1)^2\right] = \ln\left[(7)^4(4)^2\right] - \ln\left[(4)^4(1)^2\right]$$

$$= \ln\left[\frac{(7)^4(4)^2}{(4)^4(1)^2}\right] = \boxed{\ln\left[\frac{(7)^4}{(4)^2}\right]}$$

9. Level: AP 5

$$\int_1^\infty xe^{-x}\,dx =$$

(A) $\frac{1}{2}e^2$ (B) $-\frac{1}{2}e^2$ (C) $\frac{2}{e}$ (D) The integral diverges.

$u = x \Rightarrow du = dx$

$dv = e^{-x}dx \Rightarrow v = -e^{-x}$

$$\int xe^{-x}\,dx = -xe^{-x} - \int e^{-x}\,dx = -xe^{-x} - e^{-x} = -e^{-x}(x+1)$$

$$\int_1^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}(x+1)\right]_1^b = -\lim_{b\to\infty}\left[e^{-b}(b+1) - (2e^{-1})\right]$$

$$= -\lim_{b\to\infty}\left[\left(\frac{b+1}{e^b}\right) - \left(\frac{2}{e}\right)\right] = -\left[(0) - \left(\frac{2}{e}\right)\right] = \boxed{\frac{2}{e}}$$

Free Response Practice

1. Let $f(x)=\int_3^x \frac{1}{x^2+k}\,dx$, where k is a constant.

(a) Let $k=9$. Find $f\left(\sqrt{3}\right)$.

$$f(x)=\int_3^{\sqrt{3}} \frac{1}{x^2+9}\,dx=\int_3^{\sqrt{3}} \frac{1/9}{\frac{x^2}{9}+1}\,dx=\frac{1}{3}\int_3^{\sqrt{3}} \frac{1}{\underbrace{\left(\frac{x}{3}\right)}_{u}^2+1}\underbrace{\left(\frac{1}{3}dx\right)}_{du}=\left[\frac{1}{3}\arctan(u)\right]_1^{\sqrt{3}/3}$$

$$=\underbrace{\frac{1}{3}\left[\arctan\left(\frac{\sqrt{3}}{3}\right)-\arctan(1)\right]}_{\text{stop here}}=\frac{1}{3}\left(\frac{\pi}{6}-\frac{\pi}{4}\right)=\frac{1}{3}\left(\frac{2\pi-3\pi}{12}\right)=\boxed{-\frac{\pi}{36}}$$

(b) Let $k=-1$. Find $f(4)$.

$$\frac{1}{x^2-1}=\frac{A}{x+1}+\frac{B}{x-1}\Rightarrow A(x-1)+B(x+1)=1$$

Let $x=1\Rightarrow A(1-1)+B(1+1)=1\Rightarrow 2B=1\Rightarrow B=\frac{1}{2}$

Let $x=-11\Rightarrow A(-1-1)+B(-1+1)=1\Rightarrow -2A=1\Rightarrow A=-\frac{1}{2}$

$$\int \frac{1}{x^2-1}\,dx=-\frac{1}{2}\int\frac{1}{x+1}\,dx+\frac{1}{2}\int\frac{1}{x-1}\,dx=-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|$$

$$f(x)=\int_3^4 \frac{1}{x^2-1}\,dx=\left[\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right]_3^4=\underbrace{\frac{1}{2}\left[\ln\left|\frac{4-1}{4+1}\right|-\ln\left|\frac{3-1}{3+1}\right|\right]}_{\text{stop here}}=\frac{1}{2}\left[\ln\left(\frac{3}{5}\right)-\ln\left(\frac{2}{4}\right)\right]=\boxed{\frac{1}{2}\ln\left(\frac{6}{5}\right)}$$

(c) Let $k=0$. Find $\lim_{x\to\infty} f(x)$ or show that the limit does not exist.

$$\lim_{x\to\infty} f(x)=\lim_{b\to\infty}\int_3^b \frac{1}{x^2}\,dx=\lim_{b\to\infty}\left[-x^{-1}\right]_3^b=\lim_{b\to\infty}\left[-\frac{1}{b}+\frac{1}{3}\right]=\frac{1}{3}-\lim_{b\to\infty}\frac{1}{b}=\frac{1}{3}-0=\boxed{\frac{1}{3}}$$

(d) Let $g(x)=f'(x)$. Given that $k \neq 0$, determine if g has a local minimum, a local maximum, or neither at $x=0$. Give a reason for your answer.

$$g(x)=f'(x) \Rightarrow g(x)=\frac{1}{x^2+k}=\left(x^2+k\right)^{-1} \Rightarrow g'(x)=-1\left(x^2+k\right)^{-2}(2x)=-\frac{2x}{\left(x^2+k\right)^2}$$

$$x=0 \Rightarrow g'(0)=0 \qquad g'(x)\ \overset{+}{\text{- - -}}\underset{0}{+}\overset{-}{\text{- - -}}$$

g has a local maximum at $x=0$ because $g'(x)$ changes from positive to negative at $x=0$.

Additional Multiple-Choice Practice

10. Level: AP 4

$$\int \frac{1}{x^2-4x+20}\,dx=$$

(A) $\dfrac{1}{\frac{x^3}{3}-2x^2+20x}+C$

(B) $\ln\left|x^2+4x-21\right|+C$

(C) $\arcsin\left(\dfrac{x+2}{5}\right)+C$

(D) $\dfrac{1}{4}\arctan\left(\dfrac{x-2}{4}\right)+C$ [boxed]

$$\int \frac{1}{x^2-4x+20}\,dx=\int \frac{1}{\left(x^2-4x+4\right)+16}\,dx=\int \frac{1}{(x-2)^2+4^2}\,dx$$

$$=\int \frac{1/16}{\frac{(x-2)^2}{16}+1}\,dx=\frac{1}{16}\int \frac{1}{\left(\frac{x-2}{4}\right)^2+1}\,dx=\frac{1}{4}\int \frac{1}{\left(\underbrace{\frac{x-2}{4}}_{u}\right)^2+1}\underbrace{\left(\frac{1}{4}dx\right)}_{du}$$

$$=\boxed{\frac{1}{4}\arctan\left(\frac{x-2}{4}\right)+C}$$

11. Level: AP 4

Which of the following three integrals converge?

$$\text{I. } \int_0^1 \frac{1}{\sqrt{x}}\,dx \qquad \text{II. } \int_1^\infty \frac{1}{x^2}\,dx \qquad \text{III. } \int_0^1 \frac{1}{x^4}\,dx$$

(A) I only

(B) I and II only [boxed]

(C) II and III only

(D) I, II, and III

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx=\lim_{a\to 0}\int_a^1 x^{-1/2}\,dx=\lim_{a\to 0}\left[2x^{1/2}\right]_a^1=2\lim_{a\to 0}\left[1^{1/2}-a^{1/2}\right]=2(1)=2$$

$$\int_1^\infty \frac{1}{x^2}\,dx=\lim_{b\to\infty}\int_1^b x^{-2}\,dx=\lim_{b\to\infty}\left[-x^{-1}\right]_1^b=\lim_{b\to\infty}\left[-b^{-1}-\left(-1^{-1}\right)\right]=0+1=1$$

$$\int_0^1 \frac{1}{x^4}\,dx=\lim_{a\to 0}\int_a^1 x^{-4}\,dx=\lim_{a\to 0}\left[\frac{x^{-3}}{-3}\right]_a^1=\lim_{a\to 0}\left[\frac{1^{-3}}{-3}-\frac{a^{-3}}{-3}\right]=-\frac{1}{3}+\lim_{a\to 0}\frac{1}{3a^3}\to -\frac{1}{3}+\frac{1}{0}\to\infty$$

12. Level: AP 3

$$\int \frac{3}{(x-1)(x-4)}\,dx =$$

(A) $\ln\left(\frac{x-4}{x-1}\right)+C$ (boxed)

(B) $3\ln\left(\frac{x-1}{x-4}\right)+C$

(C) $\ln(x-1)-\ln(x-4)+C$

(D) $\ln(x-4)+\ln(x-1)+C$

$$\frac{3}{(x-1)(x-4)} = \frac{A}{x-1}+\frac{B}{x-4} \Rightarrow A(x-4)+B(x-1)=3$$

Let $x=4 \Rightarrow A(4-4)+B(4-1)=3 \Rightarrow 3B=3 \Rightarrow B=1$

Let $x=1 \Rightarrow A(1-4)+B(1-1)=3 \Rightarrow -3A=3 \Rightarrow A=-1$

$$\int \frac{3}{(x-1)(x-4)}\,dx = \int \frac{-1}{x-1}\,dx + \int \frac{1}{x-4}\,dx = -\ln|x-1| + \ln|x-4|$$

$$= \boxed{\ln\left(\frac{x-4}{x-1}\right)+C}$$

13. Level: AP 4

$$\int \frac{x^2}{x^2+1}\,dx =$$

(A) $\ln(x^2+1)+C$

(B) $x+\frac{x^3}{3}+C$

(C) $x-\tan^{-1}(x)+C$ (boxed)

(D) $\frac{x^3}{3}\tan^{-1}(x)+C$

$$\frac{x^2}{x^2+1} = 1-\frac{1}{x^2+1}$$

$$\begin{array}{r} 1 \\ x^2+1\overline{)\,x^2\quad} \\ \underline{x^2+1} \\ -1 \end{array}$$

$$\int \frac{x^2}{x^2+1}\,dx = \int\left(1-\frac{1}{x^2+1}\right)dx = \boxed{x-\arctan(x)+C}$$

14. Level: AP 3

$$\int \frac{x}{x^2+1}\,dx =$$

(A) $\ln(x^2+1)+C$

(B) $\frac{1}{2}\ln(x^2+1)+C$ (boxed)

(C) $2\ln(x^2+1)+C$

(D) $\frac{1}{2}\tan^{-1}(x)+C$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int\left(\frac{1}{\underbrace{x^2+1}_{u}}\right)\underbrace{(2x\,dx)}_{du} = \boxed{\frac{1}{2}\ln(x^2+1)+C}$$

15. Level: AP 3

$$\int \frac{1}{x^2+1}\,dx =$$

(A) $\ln\left(x^2+1\right)+C$

(B) $\dfrac{\ln\left(x^2+1\right)}{2x}+C$

$\boxed{\text{(C)}}$ $\tan^{-1}(x)+C$

(D) $\sin^{-1}(x)+C$

16. Level: AP 4

$$\int \frac{1}{x^2-1}\,dx =$$

(A) $\ln\left(\dfrac{x-1}{x+1}\right)+C$

$\boxed{\text{(B)}}$ $\dfrac{1}{2}\ln\left(\dfrac{x-1}{x+1}\right)+C$

(C) $\dfrac{1}{2}\ln\left(x^2-1\right)+C$

(D) $\tan^{-1}(x)+C$

$$\frac{1}{x^2-1}=\frac{A}{x+1}+\frac{B}{x-1} \Rightarrow A(x-1)+B(x+1)=1$$

$$\text{Let } x=1 \Rightarrow A(1-1)+B(1+1)=1 \Rightarrow 2B=1 \Rightarrow B=\frac{1}{2}$$

$$\text{Let } x=-11 \Rightarrow A(-1-1)+B(-1+1)=1 \Rightarrow -2A=1 \Rightarrow A=-\frac{1}{2}$$

$$\int \frac{1}{x^2-1}\,dx=-\frac{1}{2}\int \frac{1}{x+1}\,dx+\frac{1}{2}\int \frac{1}{x-1}\,dx=-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|$$

$$=\boxed{\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)+C}$$

17. Level: AP 4

$$\int \frac{x}{x^2-1}\,dx =$$

(A) $\ln\left(x^2-1\right)+C$

(B) $\dfrac{1}{2}\ln\left(\dfrac{x-1}{x+1}\right)+C$

$\boxed{\text{(C)}}$ $\dfrac{1}{2}\ln\left(x^2-1\right)+C$

(D) $\ln(x)-\dfrac{x^2}{2}+C$

$$\int \frac{x}{x^2-1}\,dx=\frac{1}{2}\int\left(\frac{1}{\underbrace{x^2-1}_{u}}\right)\underbrace{(2x\,dx)}_{du}=\boxed{\frac{1}{2}\ln\left(x^2-1\right)+C}$$

Additional Open Response Practice

18. Level: AP 5

Given $\int_{2A}^{3A} \frac{x+A}{x(x-A)}\,dx = \ln\left(\frac{A}{3}\right)$, find A.

$$\frac{x+A}{x(x-A)} = \frac{B}{x} + \frac{C}{x-A} \Rightarrow B(x-A) + Cx = x + A$$

Let $x = 0 \Rightarrow B(0-A) + C(0) = 0 + A \Rightarrow (-A)B = A \Rightarrow B = -1$

Let $x = A \Rightarrow B(A-A) + C(A) = A + A \Rightarrow (A)C = 2A \Rightarrow C = 2$

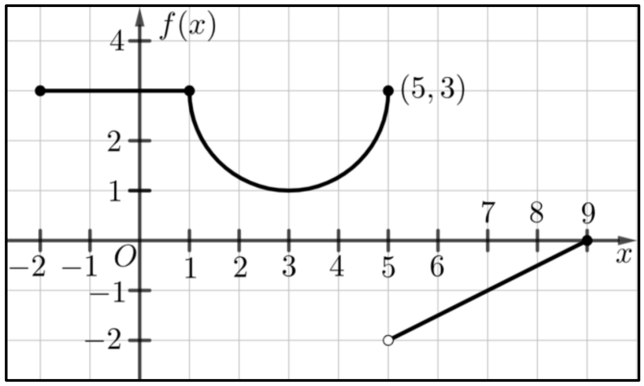
$$\int \frac{x+A}{x(x-A)}\,dx = \int \left(\frac{-1}{x} + \frac{2}{x-A}\right) dx = -\ln|x| + 2\ln|x-A|$$

$$\int_{2A}^{3A} \frac{x+A}{x(x-A)}\,dx = \Big[2\ln|x-A| - \ln|x|\Big]_{2A}^{3A} = \Big[2\ln|3A-A| - \ln|3A|\Big] - \Big[2\ln|2A-A| - \ln|2A|\Big]$$

$$= \Big[2\ln|2A| - \ln|3A|\Big] - \Big[2\ln|A| - \ln|2A|\Big] = \ln\left(\frac{(2A)^2}{3A}\right) - \ln\left(\frac{A^2}{2A}\right) = \ln\left(\frac{4A}{3}\cdot\frac{2}{A}\right) = \ln\left(\frac{8}{3}\right)$$

$$\int_{2A}^{3A} \frac{x+A}{x(x-A)}\,dx = \ln\left(\frac{A}{3}\right) = \ln\left(\frac{8}{3}\right) \Rightarrow \boxed{A = 8}$$

Additional Free Response Practice



2. The function $f(x)$ is defined for $-2 \le x \le 9$ and consists of two line segments and a semi-circle as shown in the figure above.

(a) $\int_9^1 \left(f(x)+3\right)dx =$

$$\int_9^1 \left(f(x)+3\right)dx = -\int_1^9 f(x)\,dx - \int_1^9 3\,dx = -\left(\underbrace{\int_1^5 f(x)\,dx}_{\text{rectangle} - \text{semi-circle}} + \underbrace{\int_5^9 f(x)\,dx}_{\text{triangle}}\right) - 3(9-1)$$

$$= \underbrace{-\left[(3)(4) - \frac{1}{2}\pi(2)^2 + \frac{1}{2}(4)(-2)\right]}_{\text{stop here}} - 24 = -\left[12 - 2\pi - 4\right] - 24 = -\left[8 - 2\pi\right] - 24 = \boxed{2\pi - 32}$$

(b) $\int_{-2}^{0} x f'\left(x^2-1\right)dx =$

$$\int_{-2}^{0} x f'\left(x^2-1\right)dx = \frac{1}{2}\int_{-2}^{0} f'\big(\underbrace{x^2-1}_{u}\big)\big(\underbrace{2x\,dx}_{du}\big) = \frac{1}{2}\int_{(-2)^2-1}^{0^2-1} f'(u)\,du = \frac{1}{2}\left[f(u)\right]_3^{-1} = \frac{1}{2}\left[f(-1) - f(3)\right]$$

$$= \frac{1}{2}\left[(3) - (1)\right] = \boxed{1}$$

(c) $\int_0^3 x f'(x)\,dx =$

$$\int \underbrace{x}_{u} \underbrace{f'(x)\,dx}_{dv} \qquad\qquad u = x \Rightarrow du = dx$$
$$dv = f'(x)\,dx \Rightarrow v = f(x)$$

$$= \underbrace{x\,f(x)}_{u \cdot v} - \int \underbrace{f(x)}_{v} \underbrace{dx}_{du}$$

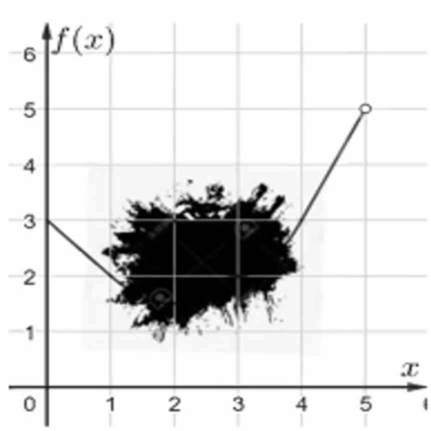
$$\int_0^3 x\,f'(x)\,dx = \left[x\,f(x)\right]_0^3 - \underbrace{\int_0^3 f(x)\,dx}_{\text{rectangle} - \frac{1}{4}\text{circle}} = \left[(3)\,f(3)\right] - \left[(0)\,f(0)\right] - \left[(3)(3) - \frac{1}{4}\pi 2^2\right]$$

$$= \left[(3)(1)\right] - \left[(0)(3)\right] - \left[9 - \pi\right] = 3 - 9 + \pi = \boxed{\pi - 6}$$

(d) Let g be a differentiable function where $g'(x) = f(x)$ and $g(7) = 2$. Find $g(5)$.

$$\int_5^7 g'(x)\,dx = g(7) - g(5) = \int_5^7 f(x)\,dx$$

$$2 - g(5) = \int_5^7 f(x)\,dx \Rightarrow g(5) = 2 - \underbrace{\int_5^7 f(x)\,dx}_{\text{trapezoid}} = 2 - \left[-\frac{1}{2}(2+1)(2)\right] = 2 + 3 = \boxed{5 = g(5)}$$



Graph created by Bryan Passwater

3. The function f is continuous and differentiable on the interval $0 \le x \le 5$. A portion of the graph of $f(x)$ is obscured by a coffee stain. It is known that $\int_0^4 f(x)\,dx = 8$ and $\int_4^3 f(x)\,dx = -3$ and $f(x)$ is linear on the intervals $(0,1)$ and $(4,5)$. For $x \ge 5$, $f(x) = \dfrac{4}{(x-a)^2}$, where a is a positive number.

(a) Find $\int_0^4 2xf'(x)\,dx$.

$$2\int \underbrace{x}_{u}\,\underbrace{f'(x)\,dx}_{dv} \qquad u = x \Rightarrow du = dx$$

$$dv = f'(x)\,dx \Rightarrow v = f(x)$$

$$= 2\left[\underbrace{x\,f(x)}_{u\cdot v} - \int \underbrace{f(x)}_{v}\,\underbrace{dx}_{du}\right]$$

$$\int_0^4 2x\,f'(x)\,dx = 2\left[x\,f(x)\right]_0^4 - 2\int_0^4 f(x)\,dx = 2\left[4\,f(4) - (0)\,f(0)\right] - 2\int_0^4 f(x)\,dx$$

$$= 2\left[4(3) - 0\right] - 2(8) = 24 - 16 = \boxed{8}$$

(b) Find $\int_1^2 x\,f(x^2 - 1)\,dx$.

$$\frac{1}{2}\int_1^2 f\Big(\underbrace{x^2-1}_{u}\Big)\underbrace{(2x\,dx)}_{du} = \frac{1}{2}\int_{1^2-1}^{2^2-1} f(u)\,du = \frac{1}{2}\int_0^3 f(u)\,du = \frac{1}{2}\left[\int_0^4 f(u)\,du - \int_3^4 f(u)\,du\right] = \frac{1}{2}\left[8 - (3)\right] = \boxed{\frac{5}{2}}$$

(c) Find $\int_{0}^{\pi/2} \cos x\, f'(\sin x)\, dx$.

$$\int_{0}^{\pi/2} f'\Big(\underbrace{\sin x}_{u}\Big)\underbrace{\left(\cos x dx\right)}_{du} = \int_{\sin 0}^{\sin \pi/2} f'(u)\,du = \int_{0}^{1} f'(u)\,du = f(1) - f(0) = 2 - 3 = \boxed{-1}$$

(d) It is known that $\int_{3}^{\infty} f(x)\,dx = 9$, find the value of a.

$$\int_{3}^{\infty} f(x)dx = \int_{3}^{5} f(x)dx + \int_{5}^{\infty} f(x)dx = \int_{3}^{4} f(x)dx + \int_{4}^{5} f(x)dx + \int_{5}^{\infty} f(x)dx$$

$$= \left(-(-3)\right) + \left[\frac{1}{2}(5+3)(1)\right] + \lim_{b\to\infty}\int_{5}^{b} f(x)dx = 7 + \lim_{b\to\infty}\int_{5}^{b} \frac{4}{(x-a)^2}dx = 7 + \lim_{b\to\infty}\int_{5}^{b} 4(x-a)^{-2}dx$$

$$= 7 + \lim_{b\to\infty}\left[-4(x-a)^{-1}\right]_5^b = 7 + \lim_{b\to\infty}\left[\frac{-4}{b-a} - \frac{-4}{5-a}\right] = 7 + \left[(0) - \frac{-4}{5-a}\right] = 7 + \frac{4}{5-a}$$

$$7 + \frac{4}{5-a} = 9 \Rightarrow \frac{4}{5-a} = 2 \Rightarrow 5 - a = 2 \Rightarrow a = 3$$