Topics in Analytic Geometry

10

Derke O'Hara/Tony Stone Images

Ice in the nucleus of a comet is heated and vaporized by the sun. The escaping gas collects dust particles, forming a tail which always points away from the sun.

The Big Picture

In this chapter you will learn how to

- write the standard equations of parabolas, ellipses, and hyperbolas.
- analyze and sketch the graphs of parabolas, ellipses, and hyperbolas.
- rotate the coordinate axis to eliminate the xy-term in equations of conics and use the discriminant to classify conics.
- solve systems of quadratic equations.
- rewrite sets of parametric equations as rectangular equations and find sets of parametric equations for graphs.
- write equations in polar form.
- graph polar equations and recognize special polar graphs.
- write equations of conics in polar form.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

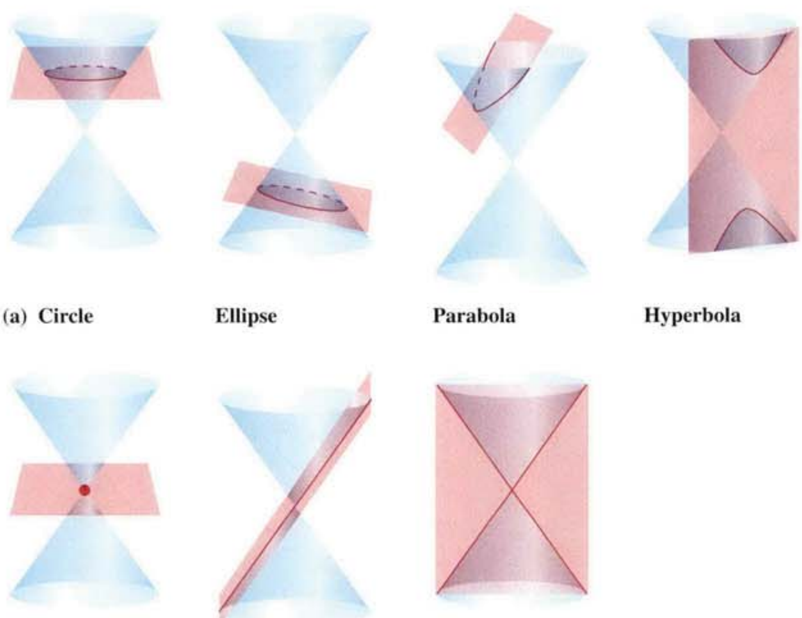
- conic section or conic (pp. 696, 754)
- parabola (pp. 697, 754)
- directrix (p. 697)
- focus or foci (pp. 697, 704, 713)
- tangent (p. 699)
- ellipse (pp. 704, 754)
- vertices (pp. 704, 713)
- major axis (p. 704)
- minor axis (p. 704)
- center (pp. 704, 713)
- eccentricity (pp. 708, 754)
- hyperbola (pp. 713, 754)
- transverse axis (p. 713)
- asymptotes (p. 715)
- conjugate axis (p. 715)
- discriminant (p. 726)
- parameter (p. 731)
- parametric equations (p. 731)
- plane curve (p. 731)
- orientation (p. 732)
- polar coordinate system (p. 739)
- pole or origin (p. 739)
- polar axis (p. 739)
- polar coordinates (p. 739)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

10.1 Introduction to Conics: Parabolas

Conic sections were discovered during the classical Greek period, 600 to 300 B.C. The early Greeks were concerned largely with the geometric properties of conics. It was not until the early 17th century that the broad applicability of conics became apparent, and they then played a prominent role in the early development of calculus.

Each **conic section** (or simply **conic**) is the intersection of a plane and a double-napped cone. Notice in Figure 10.1(a) that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane does pass through the vertex, the resulting figure is a *degenerate conic*, as shown in Figure 10.1(b).



(a) Circle **Ellipse** **Parabola** **Hyperbola**

(b) Point **Line** **Two intersecting lines**

Figure 10.1

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach, in which each of the conics is defined as a **locus** (collection) of points satisfying a geometric property. For example, the definition of a circle as the collection of all points (x, y) that are equidistant from a fixed point (h, k) leads to the standard equation of a circle

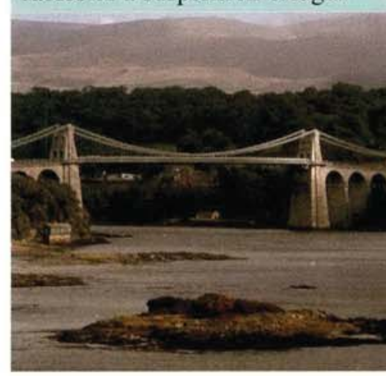
$$(x - h)^2 + (y - k)^2 = r^2. \quad \text{Equation of circle}$$

What You Should Learn:

- How to recognize a conic as the intersection of a plane and a double-napped cone
- How to write equations of parabolas in standard form
- How to use the reflective property of parabolas to solve real-life problems

Why You Should Learn It:

Parabolas can be used to model and solve many types of real-life problems. For instance, in Exercise 58 on page 702, a parabola is used to model the cables of a suspension bridge.



Adam Woolfitt/CORBIS

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Parabolas

The first type of conic is called a **parabola,** and it is defined as follows.

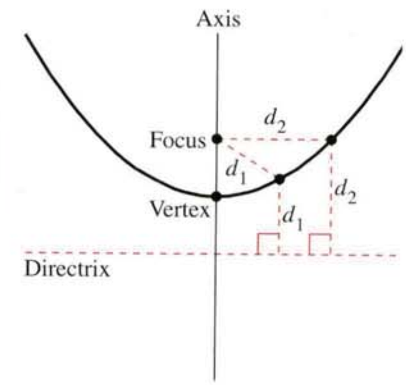


Figure 10.2

Definition of a Parabola

A **parabola** is the set of all points (x, y) that are equidistant from a fixed line (**directrix**) and a fixed point (**focus**) not on the line.

The midpoint between the focus and the directrix is called the **vertex,** and the line passing through the focus and the vertex is called the **axis** of the parabola. Note in Figure 10.2 that a parabola is symmetric with respect to its axis. Using the definition of a parabola, you can derive the following **standard form of the equation of a parabola** whose directrix is parallel to the x-axis or to the y-axis. See Appendix A for a proof.

Standard Equation of a Parabola

The **standard form of the equation of a parabola** with vertex at (h, k) is as follows.

$$(x - h)^2 = 4p(y - k),\ p \neq 0 \qquad \text{Vertical axis; directrix: } y = k - p$$

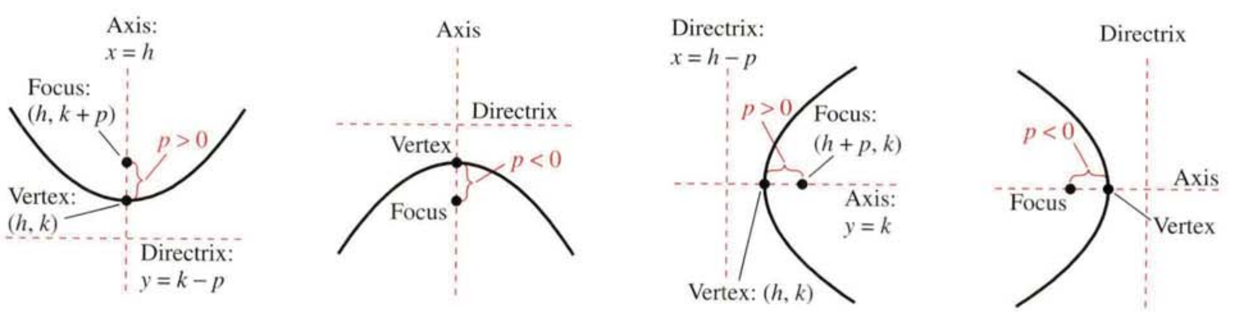
$$(y - k)^2 = 4p(x - h),\ p \neq 0 \qquad \text{Horizontal axis; directrix: } x = h - p$$

The focus lies on the axis p units (*directed distance*) from the vertex. If the vertex is at the origin $(0, 0)$, the equation takes one of the following forms.

$$x^2 = 4py \qquad \text{Vertical axis}$$

$$y^2 = 4px \qquad \text{Horizontal axis}$$

See Figure 10.3.



(a) Vertical axis: $p > 0$ **(b) Vertical axis: $p < 0$**
$(x - h)^2 = 4p(y - k)$

(c) Horizontal axis: $p > 0$ **(d) Horizontal axis: $p < 0$**
$(y - k)^2 = 4p(x - h)$

Figure 10.3

EXAMPLE 1 Finding the Standard Equation of a Parabola

Find the standard form of the equation of the parabola with vertex $(2, 1)$ and focus $(2, 4)$.

Solution

Because the axis of the parabola is vertical, consider the equation

$$(x - h)^2 = 4p(y - k)$$

where $h = 2$, $k = 1$, and $p = 4 - 1 = 3$. So, the standard form is

$$(x - 2)^2 = 4(3)(y - 1) = 12(y - 1).$$

You can obtain the more common quadratic form as follows.

$$(x - 2)^2 = 12(y - 1) \quad \text{Write original equation.}$$

$$x^2 - 4x + 4 = 12y - 12 \quad \text{Multiply.}$$

$$x^2 - 4x + 16 = 12y \quad \text{Add 12 to each side.}$$

$$y = \tfrac{1}{12}(x^2 - 4x + 16) \quad \text{Divide each side by 12.}$$

Try using a graphing utility to confirm the graph of this parabola, as shown in Figure 10.4.

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

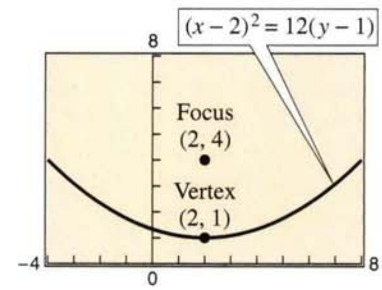


Figure 10.4

EXAMPLE 2 Finding the Focus of a Parabola

Find the focus of the parabola

$$y = -\frac{1}{2}x^2 - x + \frac{1}{2}.$$

Solution

To find the focus, convert to standard form by completing the square.

$$y = -\frac{1}{2}x^2 - x + \frac{1}{2} \quad \text{Write original equation.}$$

$$-2y = x^2 + 2x - 1 \quad \text{Multiply each side by } -2.$$

$$1 - 2y = x^2 + 2x \quad \text{Group terms.}$$

$$1 + 1 - 2y = x^2 + 2x + 1 \quad \text{Complete the square.}$$

$$2 - 2y = x^2 + 2x + 1 \quad \text{Combine like terms.}$$

$$-2(y - 1) = (x + 1)^2 \quad \text{Write in standard form.}$$

Comparing this equation with

$$(x - h)^2 = 4p(y - k)$$

you can conclude that $h = -1$, $k = 1$, and $p = -\frac{1}{2}$. Because p is negative, the parabola opens downward, as shown in Figure 10.5. Therefore, the focus of the parabola is

$$(h, k + p) = \left(-1, \frac{1}{2}\right). \quad \text{Focus}$$

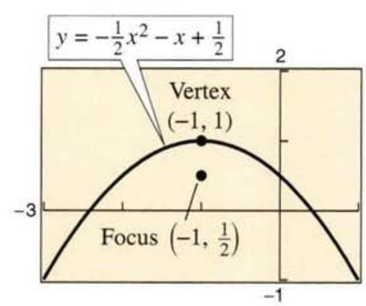


Figure 10.5

EXAMPLE 3 Vertex at the Origin

Find the standard equation of the parabola with vertex at the origin and focus (2, 0).

Solution

The axis of the parabola is horizontal, passing through (0, 0) and (2, 0), as shown in Figure 10.6.

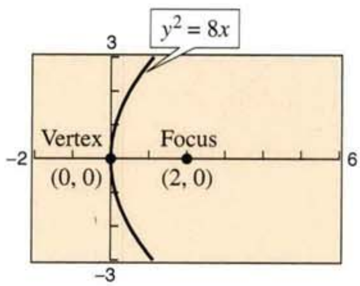


Figure 10.6

So, the standard form is

$$y^2 = 4px$$

where $h = k = 0$ and $p = 2$. Therefore, the equation is

$$y^2 = 8x.$$

STUDY TIP

You can use a graphing utility to confirm the equation found in Example 3. To do this, it helps to split the equation into two parts: $y_1 = \sqrt{8x}$ (upper part) and $y_2 = -\sqrt{8x}$ (lower part).

Applications

A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord.** The specific focal chord perpendicular to the axis of the parabola is called the **latus rectum.**

Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola around its axis. The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola; this is the principle behind the construction of the parabolic mirrors used in reflecting telescopes. Conversely, the light rays emanating from the focus of a parabolic reflector used in a flashlight are all parallel to one another, as shown in Figure 10.7.

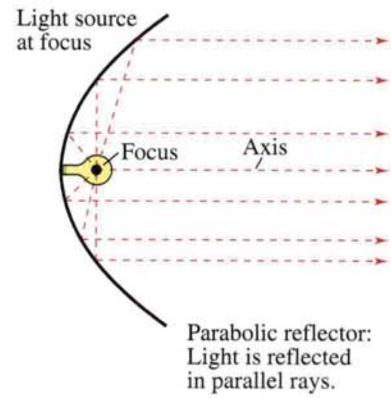


Figure 10.7

A line is **tangent** to a parabola at a point on the parabola if the line intersects, but does not cross, the parabola at the point. Tangent lines to parabolas have special properties that are related to the use of parabolas in constructing reflective surfaces.

Reflective Property of a Parabola

The tangent line to a parabola at a point P makes equal angles with the following two lines (see Figure 10.8).

1. The line passing through P and the focus
2. The axis of the parabola

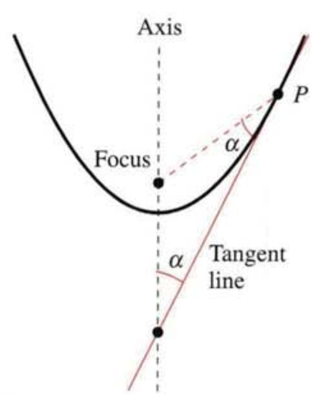


Figure 10.8

EXAMPLE 4 Finding the Tangent Line at a Point on a Parabola

Find the equation of the tangent line to the parabola given by $y = x^2$ at the point $(1, 1)$.

Solution

For this parabola, $p = \frac{1}{4}$ and the focus is $\left(0, \frac{1}{4}\right)$, as shown in Figure 10.9. You can find the y-intercept $(0, b)$ of the tangent line by equating the lengths of the two sides of the isosceles triangle

$$d_1 = \frac{1}{4} - b$$

and

$$d_2 = \sqrt{(1 - 0)^2 + \left[1 - \left(\frac{1}{4}\right)\right]^2} = \frac{5}{4}$$

shown in Figure 10.9. Setting $d_1 = d_2$ produces

$$\frac{1}{4} - b = \frac{5}{4}$$

$$b = -1.$$

So, the slope of the tangent line is

$$m = \frac{1 - (-1)}{1 - 0} = 2$$

and its slope-intercept equation is

$$y = 2x - 1.$$

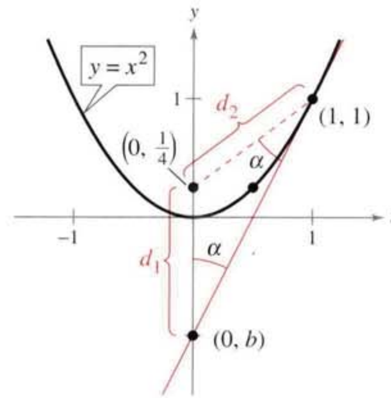


Figure 10.9

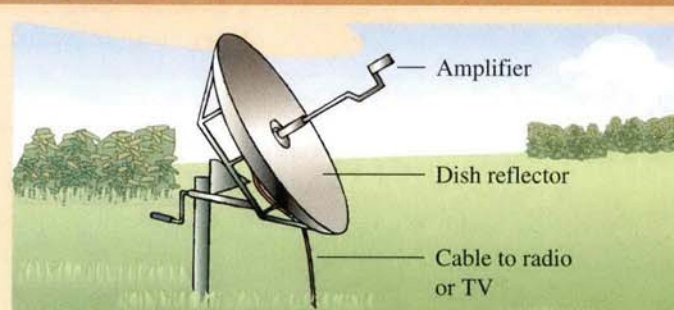
Try using a graphing utility to confirm the result of Example 4. By graphing

$$y = x^2 \qquad \text{and} \qquad y = 2x - 1$$

in the same viewing window, you should be able to see that the line touches the parabola at the point $(1, 1)$.

Writing About Math *Television Antenna Dishes*

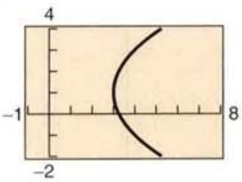
Cross sections of television antenna dishes are parabolic in shape. Write a paragraph describing why these dishes are parabolic. Include a graphical representation of your description.



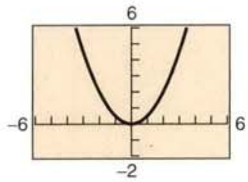
10.1 Exercises

In Exercises 1–6, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

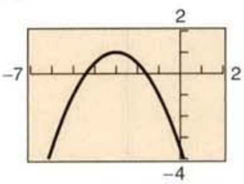
(a)



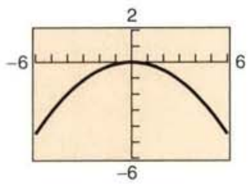
(b)



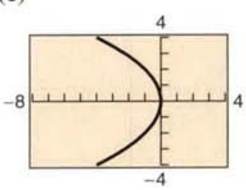
(c)



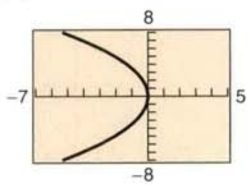
(d)



(e)



(f)



1. $y^2 = -4x$

2. $x^2 = 2y$

3. $x^2 = -8y$

4. $y^2 = -12x$

5. $(y - 1)^2 = 4(x - 3)$

6. $(x + 3)^2 = -2(y - 1)$

In Exercises 7–20, find the vertex, focus, and directrix of the parabola and sketch its graph.

7. $y = \frac{1}{2}x^2$

8. $y = -2x^2$

9. $y^2 = -6x$

10. $y^2 = 3x$

11. $x^2 + 6y = 0$

12. $x + y^2 = 0$

13. $(x - 1)^2 + 8(y + 2) = 0$

14. $(x + 5) + (y - 1)^2 = 0$

15. $\left(x + \frac{3}{2}\right)^2 = 4(y - 2)$

16. $\left(x + \frac{1}{2}\right)^2 = 4(y - 1)$

17. $y = \frac{1}{4}(x^2 - 2x + 5)$

18. $x = \frac{1}{4}(y^2 + 2y + 33)$

19. $y^2 + 6y + 8x + 25 = 0$

20. $y^2 - 4y - 4x = 0$

The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

In Exercises 21–24, find the vertex, focus, and directrix of the parabola and sketch its graph. Use a graphing utility to verify your graph.

21. $x^2 + 4x + 6y - 2 = 0$

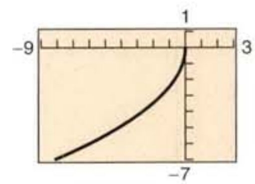
22. $x^2 - 2x + 8y + 9 = 0$

23. $y^2 + x + y = 0$

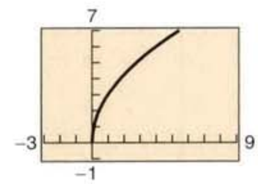
24. $y^2 - 4x - 4 = 0$

In Exercises 25 and 26, change the equation so that its graph matches the given graph.

25. $y^2 = -6x$

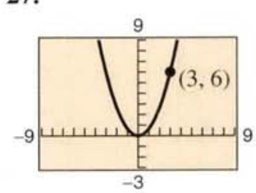


26. $y^2 = 9x$

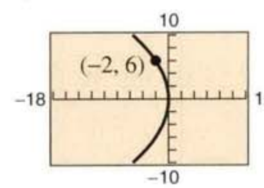


In Exercises 27–38, find the standard form of the equation of the parabola with its vertex at the origin.

27.



28.



29. Focus: $\left(0, -\frac{3}{2}\right)$

30. Focus: $(2, 0)$

31. Focus: $(-2, 0)$

32. Focus: $(0, -2)$

33. Directrix: $y = -1$

34. Directrix: $y = 3$

35. Directrix: $x = 2$

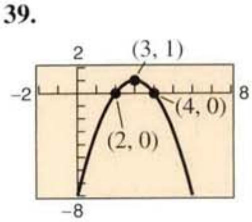
36. Directrix: $x = -3$

37. Horizontal axis and passes through the point $(4, 6)$

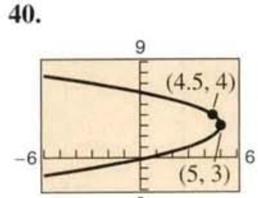
38. Vertical axis and passes through the point $(-3, -3)$

In Exercises 39–48, find the standard form of the equation of the parabola.

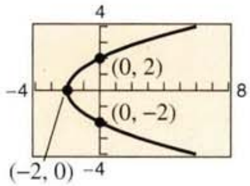
39.



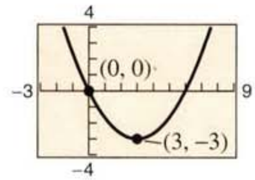
40.



41.



42.



43. Vertex: $(5, 2)$; Focus: $(3, 2)$

44. Vertex: $(-1, 2)$; Focus: $(-1, 0)$

45. Vertex: $(0, 4)$; Directrix: $y = 2$

46. Vertex: $(-2, 1)$; Directrix: $x = 1$

47. Focus: $(2, 2)$; Directrix: $x = -2$

48. Focus: $(0, 0)$; Directrix: $y = 8$

In Exercises 49 and 50, the equations of a parabola and a tangent line to the parabola are given. Use a graphing utility to graph both in the same viewing window. Determine the coordinates of the point of tangency.

	Parabola	*Tangent Line*
49.	$y^2 - 8x = 0$	$x - y + 2 = 0$
50.	$x^2 + 12y = 0$	$x + y - 3 = 0$

In Exercises 51–54, find an equation of the tangent line to the parabola at the given point and find the x-intercept of the line.

51. $x^2 = 2y,\ (4, 8)$

52. $x^2 = 2y,\ \left(-3, \frac{9}{2}\right)$

53. $y = -2x^2,\ (-1, -2)$

54. $y = -2x^2,\ (2, -8)$

55. ***Revenue*** The revenue R generated by the sale of x units of a product is

$$R = 265x - \frac{5}{4}x^2.$$

Use a graphing utility to graph the function and approximate the number of sales that will maximize revenue.

56. ***Revenue*** The revenue R generated by the sale of x units of a product is

$$R = 378x - \frac{7}{5}x^2.$$

Use a graphing utility to graph the function and approximate the number of sales that will maximize revenue.

57. ***Satellite Antenna*** The receiver in a parabolic television dish antenna is 3.5 feet from the vertex and is located at the focus. Find an equation of a cross section of the reflector. (Assume that the dish is directed upward and that the vertex is at the origin.)

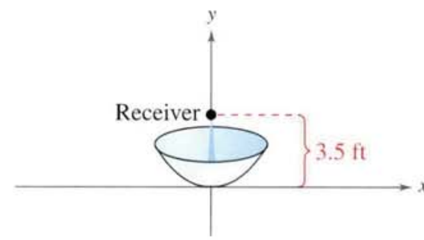


FIGURE FOR 57

58. ***Suspension Bridge*** Each cable of a suspension bridge is suspended (in the shape of a parabola) between two towers that are 120 meters apart, and the top of each tower is 20 meters above the roadway. The cables touch the roadway midway between the towers.

(a) Draw a diagram for the bridge. Draw a rectangular coordinate system on the bridge with the center of the bridge at the origin. Identify the coordinates of the known points.

(b) Find an equation for the parabolic shape of each cable.

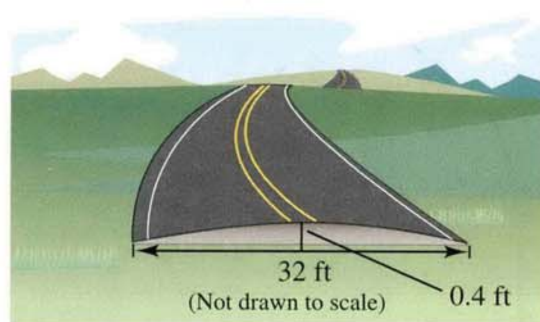
(c) Complete the table by finding the height of the suspension cables above the roadway at a distance of x meters from the center of the bridge.

x	0	20	40	60
y				

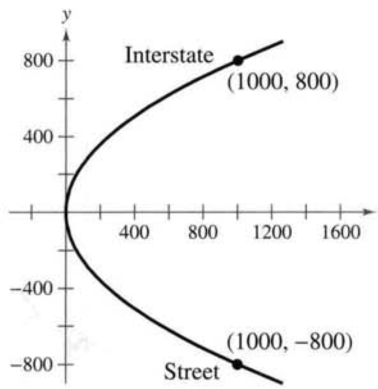
59. ***Road Design*** Roads are often designed with parabolic surfaces to allow rain to drain off. A particular road that is 32 feet wide is 0.4 foot higher in the center than it is on the sides.

(a) Find an equation of the parabola. (Assume that the origin is at the center of the road.)

(b) How far from the center of the road is the road surface 0.1 foot lower than in the middle?



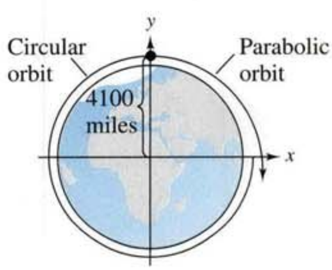
60. ***Highway Design*** Highway engineers design a parabolic curve for an entrance ramp from a straight street to an interstate highway. Find an equation of the parabola.



61. ***Satellite Orbit*** An earth satellite in a 100-mile-high circular orbit around the earth has a velocity of approximately 17,500 miles per hour. If this velocity is multiplied by $\sqrt{2}$, the satellite will have the minimum velocity necessary to escape the earth's gravity and it will follow a parabolic path with the center of the earth as the focus.

(a) Find the escape velocity of the satellite.

(b) Find an equation of its path (assume the radius of the earth is 4000 miles).



62. ***Path of a Projectile*** The path of a softball is given by the equation

$$y = -0.08x^2 + x + 4.$$

The coordinates x and y are measured in feet, with $x = 0$ corresponding to the position where the ball was thrown.

(a) Use a graphing utility to graph the trajectory of the softball.

(b) Move the cursor along the path to approximate the highest point and the range of the trajectory.

Projectile Motion **In Exercises 63 and 64, consider the path of a projectile projected horizontally with a velocity of v feet per second at a height of s feet, where the model for the path is**

$$y = -\frac{16}{v^2}t^2 + s.$$

In this model, air resistance is disregarded and y is the height (in feet) of the projectile t seconds after its release.

63. A ball is thrown from the top of a 75-foot tower with a velocity of 32 feet per second.

(a) Find the equation of the parabolic path.

(b) How far does the ball travel horizontally before striking the ground?

64. A bomber flying due east at 550 miles per hour at an altitude of 42,000 feet releases a bomb. Determine the distance the bomb travels horizontally before striking the ground.

Synthesis

True or False? **In Exercises 65 and 66, determine whether the statement is true or false. Justify your answer.**

65. It is possible for a parabola to intersect its directrix.

66. If the vertex and focus of a parabola are on a horizontal line, then the directrix of the parabola is vertical.

Review

In Exercises 67–70, list the possible rational zeros given by the Rational Zero Test.

67. $f(x) = x^3 - 2x^2 + 2x - 4$

68. $f(x) = 2x^3 + 4x^2 - 3x + 10$

69. $f(x) = 2x^5 + x^2 + 16$

70. $f(x) = 3x^3 - 12x + 22$

10.2 Ellipses

Introduction

The second type of conic is called an **ellipse.** It is defined as follows.

Definition of an Ellipse

An **ellipse** is the set of all points (x, y) the sum of whose distances from two distinct fixed points (**foci**) is constant. (See Figure 10.10.)

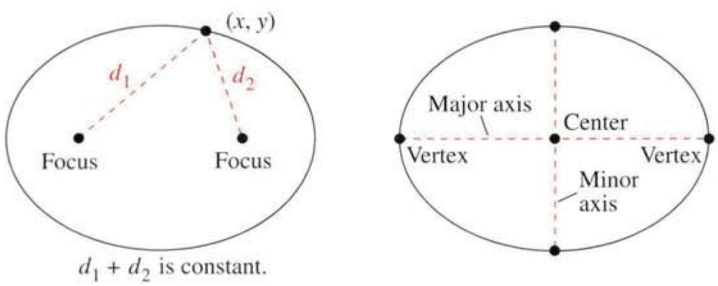


Figure 10.10

The line through the foci intersects the ellipse at two points called **vertices.** The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis** of the ellipse.

To derive the standard form of the equation of an ellipse, consider the ellipse in Figure 10.11 with the following points: center, (h, k); vertices, $(h \pm a, k)$; foci, $(h \pm c, k)$.

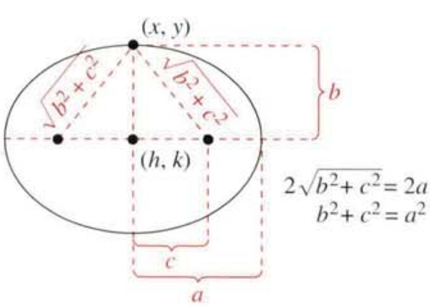


Figure 10.11

The sum of the distances from any point on the ellipse to the two foci is constant. Using a vertex point, this constant sum is

$$(a + c) + (a - c) = 2a \qquad \text{Length of major axis}$$

or simply the length of the major axis.

What You Should Learn:

- How to write equations of ellipses in standard form
- How to use properties of ellipses to model and solve real-life problems
- How to find eccentricities of ellipses

Why You Should Learn It:

Ellipses can be used to model and solve many types of real-life problems. For instance, in Exercise 52 on page 711, an ellipse is used to model the orbit of Halley's comet.

Now, if you let (x, y) be *any* point on the ellipse, the sum of the distances between (x, y) and the two foci must also be $2a$. That is,

$$\sqrt{[x-(h-c)]^2+(y-k)^2}+\sqrt{[x-(h+c)]^2+(y-k)^2}=2a.$$

Finally, using Figure 10.11 and $b^2 = a^2 - c^2$, you obtain the following equation of the ellipse.

$$b^2(x-h)^2+a^2(y-k)^2=a^2b^2$$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

In the development above, you would obtain a similar equation by starting with a vertical major axis. Both results are summarized as follows.

Standard Equation of an Ellipse

The **standard form of the equation of an ellipse,** with center (h, k) and major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad \text{Major axis is horizontal.}$$

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$. If the center is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{Major axis is horizontal.}$$

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad \text{Major axis is vertical.}$$

STUDY T!P

Don't confuse the equation

$$c^2 = a^2 - b^2$$

with the Pythagorean Theorem—there is a sign difference.

Figure 10.12 shows both the vertical and horizontal orientations for an ellipse.

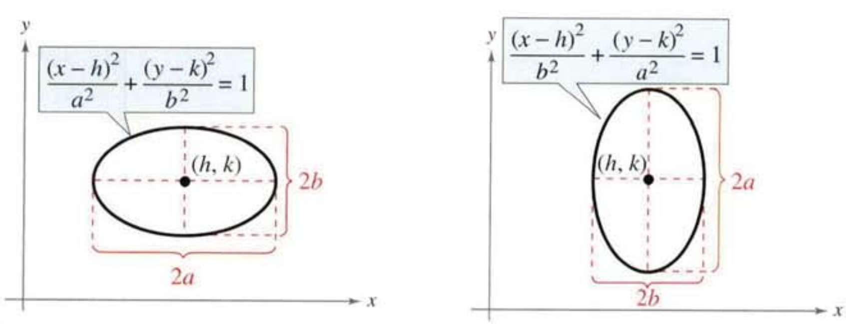


Figure 10.12

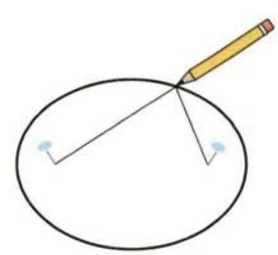


Figure 10.13

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 10.13. If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 1 Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse having foci at $(0, 1)$ and $(4, 1)$, and a major axis of length 6, as shown in Figure 10.14.

Solution

Because the foci occur at $(0, 1)$ and $(4, 1)$, the center of the ellipse is $(2, 1)$ and the distance from the center to one of the foci is $c = 2$. Because $2a = 6$ you know that $a = 3$. Now, from $c^2 = a^2 - b^2$, you have

$$\begin{aligned} b &= \sqrt{a^2 - c^2} \\ &= \sqrt{9 - 4} \\ &= \sqrt{5}. \end{aligned}$$

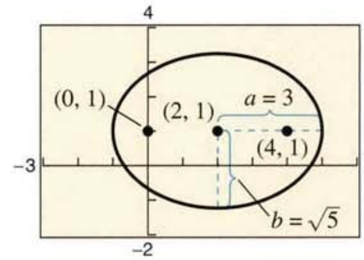


Figure 10.14

Because the major axis is horizontal, the standard equation is

$$\frac{(x-2)^2}{3^2} + \frac{(y-1)^2}{(\sqrt{5})^2} = 1.$$

EXAMPLE 2 Sketching an Ellipse

Sketch the ellipse given by $4x^2 + y^2 = 36$ and identify the vertices.

Algebraic Solution

$$4x^2 + y^2 = 36 \qquad \text{Write original equation.}$$

$$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36} \qquad \text{Divide each side by 36.}$$

$$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1 \qquad \text{Write in standard form.}$$

Because the denominator of the y^2-term is larger than the denominator to the x^2-term, you can conclude that the major axis is vertical. Moreover, because $a = 6$, the vertices are $(0, -6)$ and $(0, 6)$. Finally, because $b = 3$, the endpoints of the minor axis are $(-3, 0)$ and $(3, 0)$, as shown in Figure 10.15.

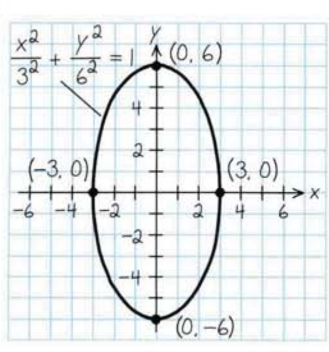


Figure 10.15

Graphical Solution

Solve the equation of the ellipse for y as follows.

$$\begin{aligned} 4x^2 + y^2 &= 36 \\ y^2 &= 36 - 4x^2 \\ y &= \pm\sqrt{36 - 4x^2} \end{aligned}$$

Then use a graphing utility to graph both $y_1 = \sqrt{36 - 4x^2}$ and $y_2 = -\sqrt{36 - 4x^2}$ in the same viewing window. Be sure to use a square setting. From the graph in Figure 10.16, you can see that the major axis is vertical. You can use the *zoom* and *trace* features to approximate the vertices to be $(0, 6)$ and $(0, -6)$.

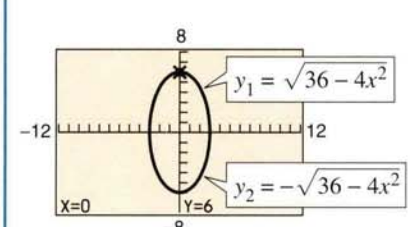


Figure 10.16

EXAMPLE 3 Writing an Equation in Standard Form

Write the equation of the ellipse in standard form and sketch the graph of the ellipse.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0.$$

Solution

To write the given equation in standard form you must complete the square twice. In the fourth step, note that 9 and 4 are added to *both* sides of the equation.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0 \quad \text{Write original equation.}$$

$$(x^2 + 6x + \quad) + (4y^2 - 8y + \quad) = -9 \quad \text{Group terms.}$$

$$(x^2 + 6x + \quad) + 4(y^2 - 2y + \quad) = -9 \quad \text{Factor 4 out of } y\text{-terms.}$$

$$(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -9 + 9 + 4(1)$$

$$(x + 3)^2 + 4(y - 1)^2 = 4 \quad \text{Write in completed square form.}$$

$$\frac{(x + 3)^2}{2^2} + \frac{(y - 1)^2}{1^2} = 1 \quad \text{Write in standard form.}$$

Now you see that the center is at $(h, k) = (-3, 1)$. Because the denominator of the x-term is $a^2 = 2^2$, you can locate the endpoints of the major axis two units to the right and left of the center. Similarly, because the denominator of the y-term is $b^2 = 1^2$, you can locate the endpoints of the minor axis one unit up and down from the center. The graph of this ellipse is shown in Figure 10.17.

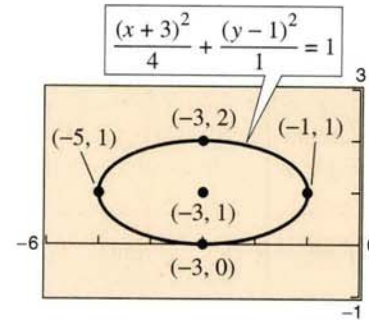


Figure 10.17

EXAMPLE 4 Analyzing an Ellipse

Find the center, vertices, and foci of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$.

Solution

By completing the square, you can write the given equation in standard form.

$$4x^2 + y^2 - 8x + 4y - 8 = 0$$

$$(4x^2 - 8x + \quad) + (y^2 + 4y + \quad) = 8$$

$$4(x^2 - 2x + \quad) + (y^2 + 4y + \quad) = 8$$

$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4(1) + 4$$

$$4(x - 1)^2 + (y + 2)^2 = 16$$

$$\frac{(x - 1)^2}{2^2} + \frac{(y + 2)^2}{4^2} = 1$$

So, the major axis is vertical, where $h = 1$, $k = -2$, $a = 4$, $b = 2$, and

$$c = \sqrt{16 - 4} = 2\sqrt{3}.$$

Therefore, you have the following.

Center: $(1, -2)$ Vertices: $(1, -6)$ Foci: $(1, -2 - 2\sqrt{3})$

$(1, 2)$ $(1, -2 + 2\sqrt{3})$

The graph of the ellipse is shown in Figure 10.18.

STUDY TIP

You can use a graphing utility to graph an ellipse by graphing the upper and lower portions in the same viewing window. For instance, to graph the ellipse in Example 3, first solve for y to get

$$y_1 = 1 + \sqrt{1 - \frac{(x + 3)^2}{4}}$$

and

$$y_2 = 1 - \sqrt{1 - \frac{(x + 3)^2}{4}}.$$

Use a viewing window in which $-6 \le x \le 0$ and $-1 \le y \le 3$. You should obtain the graph shown in Figure 10.17.

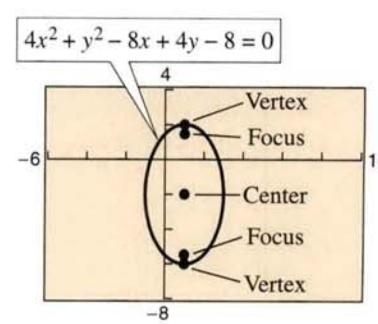


Figure 10.18

Application

Ellipses have many practical and aesthetic uses. For instance, machine gears, supporting arches, and acoustical designs often involve elliptical shapes. The orbits of satellites and planets are also ellipses. Example 5 investigates the elliptical orbit of the moon about the earth.

EXAMPLE 5 An Application Involving an Elliptical Orbit

The moon travels about the earth in an elliptical orbit with the earth at one focus, as shown in Figure 10.19. The major and minor axes of the orbit have lengths of 768,806 kilometers and 767,746 kilometers, respectively. Find the greatest and least distances (the apogee and perigee) from the earth's center to the moon's center.

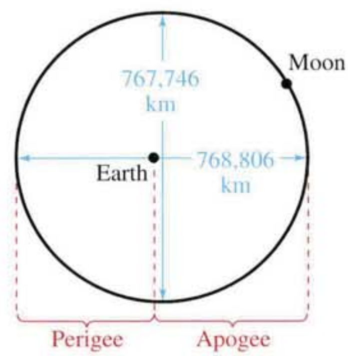


Figure 10.19

Solution

Because $2a = 768{,}806$ and $2b = 767{,}746$, you have $a = 384{,}403$ and $b = 383{,}873$, which implies that

$$\begin{aligned} c &= \sqrt{a^2 - b^2} \\ &= \sqrt{384{,}403^2 - 383{,}873^2} \\ &\approx 20{,}179. \end{aligned}$$

Therefore, the greatest distance between the center of the earth and the center of the moon is

$$\begin{aligned} a + c &\approx 384{,}403 + 20{,}179 \\ &= 404{,}582 \text{ km} \end{aligned}$$

and the least distance is

$$\begin{aligned} a - c &\approx 384{,}403 - 20{,}179 \\ &= 364{,}224 \text{ km}. \end{aligned}$$

Eccentricity

One of the reasons it was difficult for early astronomers to detect that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to their centers, and so the orbits are nearly circular. To measure the ovalness of an ellipse, you can use the concept of **eccentricity.**

Exploration

Use a graphing utility to graph the following ellipses in the same viewing window.

$$\frac{x^2}{9} + \frac{y^2}{8} = 1$$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$$\frac{x^2}{49} + \frac{y^2}{9} = 1$$

Calculate the eccentricity $e = c/a$ for each ellipse. How does the value of e relate to the shape of the ellipse?

Definition of Eccentricity

The **eccentricity** e of an ellipse is given by the ratio

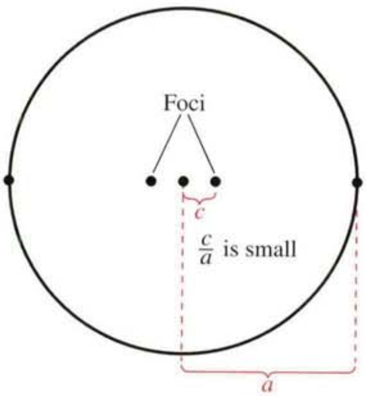
$$e = \frac{c}{a}.$$

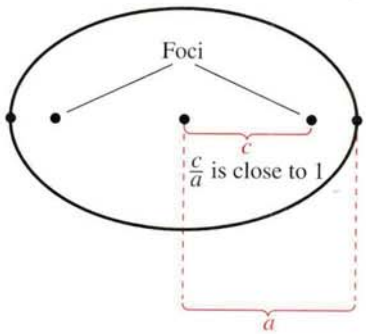
Note that $0 < e < 1$ for *every* ellipse.

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is small [see Figure 10.20(a)]. On the other hand, for an elongated ellipse, the foci are close to the vertices and the ratio c/a is close to 1 [see Figure 10.20(b)].





(a) (b)

Figure 10.20

The orbit of the moon has an eccentricity of $e = 0.0549$, and the eccentricities of the nine planetary orbits are as follows.

Mercury:	$e = 0.2056$	Saturn:	$e = 0.0543$
Venus:	$e = 0.0068$	Uranus:	$e = 0.0460$
Earth:	$e = 0.0167$	Neptune:	$e = 0.0082$
Mars:	$e = 0.0934$	Pluto:	$e = 0.2481$
Jupiter:	$e = 0.0484$		

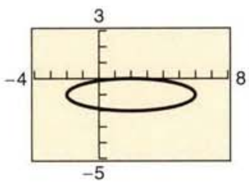
Writing About Math *Graphing Ellipses*

Write an equation of an ellipse in standard form and graph it on graph paper. Do not write the equation on your graph. Exchange graphs with another student. Use the graph you receive to reconstruct the equation of the ellipse it represents. Find the eccentricity of the ellipse. Compare your results and write a short paragraph that discusses your findings.

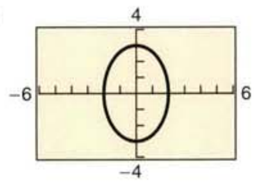
10.2 Exercises

In Exercises 1–6, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

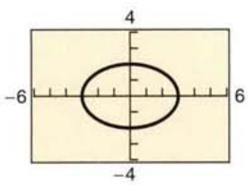
(a)



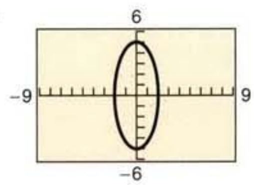
(b)



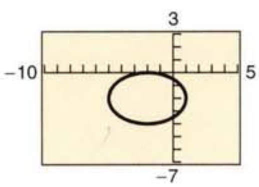
(c)



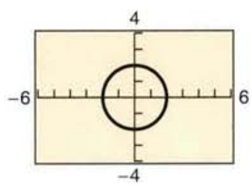
(d)



(e)



(f)



1. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

2. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

3. $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$

4. $\dfrac{y^2}{4} + \dfrac{x^2}{4} = 1$

5. $\dfrac{(x-2)^2}{16} + (y+1)^2 = 1$

6. $\dfrac{(x+2)^2}{9} + \dfrac{(y+2)^2}{4} = 1$

In Exercises 7–22, find the center, vertices, foci, and eccentricity of the ellipse, and sketch its graph. Use a graphing utility to verify your graph.

7. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

8. $\dfrac{x^2}{81} + \dfrac{y^2}{144} = 1$

9. $\dfrac{x^2}{5} + \dfrac{y^2}{9} = 1$

10. $\dfrac{x^2}{64} + \dfrac{y^2}{28} = 1$

11. $\dfrac{(x+3)^2}{16} + \dfrac{(y-5)^2}{25} = 1$

12. $\dfrac{(x-4)^2}{12} + \dfrac{(y+3)^2}{16} = 1$

13. $\dfrac{(x+5)^2}{9/4} + (y-1)^2 = 1$

14. $(x+2)^2 + \dfrac{(y+4)^2}{1/4} = 1$

15. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

16. $9x^2 + 4y^2 - 54x + 40y + 37 = 0$

17. $x^2 + 5y^2 - 8x - 30y - 39 = 0$

18. $3x^2 + y^2 + 18x - 2y - 8 = 0$

19. $6x^2 + 2y^2 + 18x - 10y + 2 = 0$

20. $x^2 + 4y^2 - 6x + 20y - 2 = 0$

21. $16x^2 + 25y^2 - 32x + 50y + 16 = 0$

22. $9x^2 + 25y^2 - 36x - 50y + 61 = 0$

In Exercises 23–26, use a graphing utility to graph the ellipse. Find the center, foci, and vertices. (*Hint:* Use two equations.)

23. $5x^2 + 3y^2 = 15$

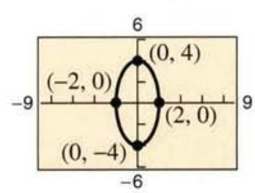
24. $3x^2 + 4y^2 = 12$

25. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

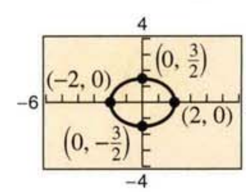
26. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

In Exercises 27–34, find the standard form of the equation of the ellipse with its center at the origin.

27.



28.



29. Vertices: $(\pm 6, 0)$; Foci: $(\pm 2, 0)$

30. Vertices: $(0, \pm 8)$; Foci: $(0, \pm 4)$

31. Foci: $(\pm 5, 0)$; Major axis of length 12

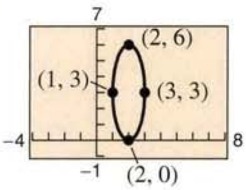
32. Foci: $(\pm 2, 0)$; Major axis of length 8

33. Vertices: $(0, \pm 5)$; Passes through the point $(4, 2)$

34. Major axis is vertical; Passes through the points $(0, 4)$ and $(2, 0)$

In Exercises 35–46, find the standard form of the equation of the specified ellipse.

35.



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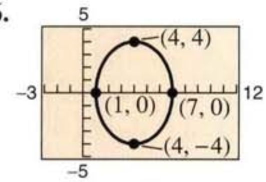
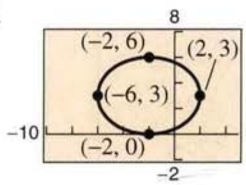
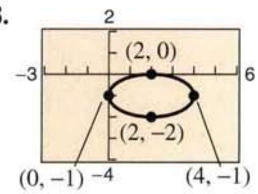


37.



38.



39. Vertices: $(0, 4), (4, 4)$; Minor axis of length 2

40. Foci: $(0, 0), (4, 0)$; Major axis of length 8

41. Foci: $(0, 0), (0, 8)$; Major axis of length 16

42. Center: $(2, -1)$; Vertex: $\left(2, \frac{1}{2}\right)$; Minor axis of length 2

43. Vertices: $(3, 1), (3, 9)$; Minor axis of length 6

44. Center: $(3, 2)$; $a = 3c$; Foci: $(1, 2), (5, 2)$

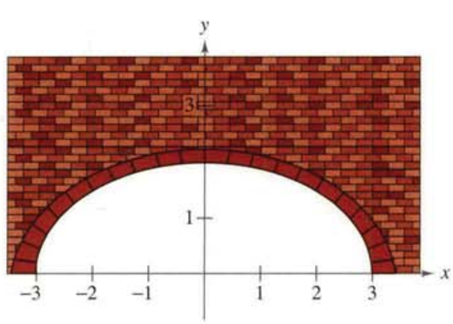
45. Center: $(0, 4)$; $a = 2c$; Vertices: $(-4, 4), (4, 4)$

46. Vertices: $(5, 0), (5, 12)$; Endpoints of the minor axis: $(0, 6), (10, 6)$

47. Find an equation of the ellipse with vertices $(\pm 5, 0)$ and eccentricity $e = \frac{3}{5}$.

48. Find an equation of the ellipse with vertices $(0, \pm 8)$ and eccentricity $e = \frac{1}{2}$.

49. ***Fireplace Arch*** A fireplace arch is to be built in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 6 feet along the base. The contractor draws the outline of the ellipse using the method described on page 705. Give the required positions of the tacks and the length of the string.



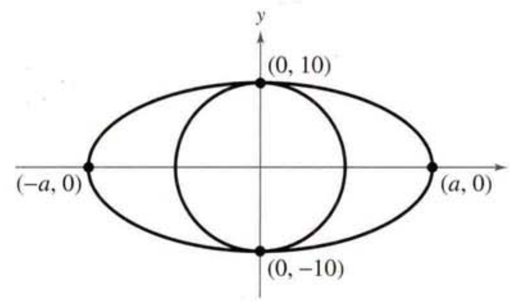
50. ***Mountain Travel*** A semielliptical arch over a tunnel for a road through a mountain has a major axis of 80 feet and a height at the center of 30 feet.

(a) Draw a rectangular coordinate system on a sketch of the tunnel with the center of the road entering the tunnel at the origin. Identify the coordinates of the known points.

(b) Find an equation of the elliptical tunnel.

(c) Determine the height of the arch 5 feet from the edge of the tunnel.

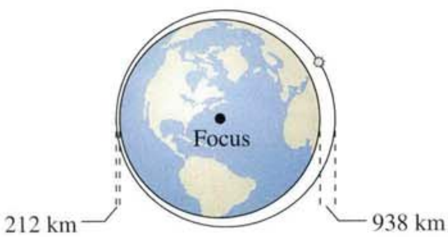
51. ***Geometry*** The area of the ellipse in the figure is twice the area of the circle. What is the length of the major axis? (*Hint:* $A = \pi ab$ for an ellipse.)



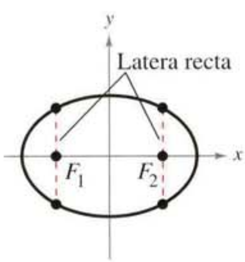
52. ***Comet Orbit*** Halley's comet has an elliptical orbit with the sun at one focus. The eccentricity of the orbit is approximately 0.97. The length of the major axis of the orbit is about 36.23 astronomical units. (An astronomical unit is about 93 million miles.) Find an equation for the orbit. Place the center of the orbit at the origin and place the major axis on the x-axis.

53. ***Comet Orbit*** The comet Encke has an elliptical orbit with the sun at one focus. Encke ranges from 0.34 to 4.08 astronomical units from the sun. Find an equation of the orbit. Place the center of the orbit at the origin and place the major axis on the x-axis.

54. ***Satellite Orbit*** The first artificial satellite to orbit earth was *Sputnik I* (launched by Russia in 1957). Its highest point above earth's surface was 938 kilometers, and its lowest point was 212 kilometers. The radius of earth is 6378 kilometers. Find the eccentricity of the orbit.



55. ***Geometry*** A line segment through a focus with endpoints on the ellipse and perpendicular to the major axis is called a **latus rectum** of the ellipse. Therefore, an ellipse has two latera recta. Knowing the length of the latera recta is helpful in sketching an ellipse because it yields other points on the curve. Show that the length of each latus rectum is $2b^2/a$.



In Exercises 56–59, sketch the graph of the ellipse, making use of the latera recta (see Exercise 55).

56. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$ **57.** $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

58. $9x^2 + 4y^2 = 36$ **59.** $5x^2 + 3y^2 = 15$

Synthesis

True or False? **In Exercises 60–63, determine whether the statement is true or false. Justify your answer.**

60. The graph of $(x^2/4) + y^4 = 1$ is an ellipse.

61. It is easier to distinguish the graph of an ellipse from the graph of a circle if the eccentricity of the ellipse is large (close to 1).

62. The area of a circle with diameter $d = 2r = 8$ is greater than the area of an ellipse with major axis $2a = 8$.

63. It is possible for the foci of an ellipse to occur outside the ellipse.

64. ***Think About It*** At the beginning of this section it was noted that an ellipse can be drawn using two thumbtacks, a string of fixed length (greater than the distance between the two tacks), and a pencil (see Figure 10.13). If the ends of the string are fastened at the tacks and the string is drawn taut with a pencil, the path traced by the pencil is an ellipse.

(a) What is the length of the string in terms of a?

(b) Explain why the path is an ellipse.

65. ***Exploration*** The area A of the ellipse

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $A = \pi ab$.

For a particular application, $a + b = 20$.

(a) Write the area of the ellipse as a function of a.

(b) Find the equation of an ellipse with an area of 264 square centimeters.

(c) Complete the table and make a conjecture about the shape of the ellipse with a maximum area.

a	8	9	10	11	12	13
A						

(d) Use a graphing utility to graph the area function, and use the graph to make a conjecture about the shape of the ellipse that yields a maximum area.

Review

In Exercises 66–69, determine whether the sequence is arithmetic, geometric, or neither.

66. 66, 55, 44, 33, 22, . . . **67.** 80, 40, 20, 10, 5, . . .

68. $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, \ldots$ **69.** $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$

In Exercises 70–73, find a formula for a_n for the arithmetic sequence.

70. $a_1 = 13,\ d = 3$ **71.** $a_1 = 0,\ d = -\frac{1}{4}$

72. $a_1 = 5,\ a_4 = 9.5$ **73.** $a_3 = 27,\ a_8 = 72$

In Exercises 74–77, find the sum.

74. $\displaystyle\sum_{n=0}^{6} 3^n$ **75.** $\displaystyle\sum_{n=0}^{6} (-3)^n$

76. $\displaystyle\sum_{n=1}^{10} 4\left(\frac{3}{4}\right)^{n-1}$ **77.** $\displaystyle\sum_{n=0}^{10} 5\left(\frac{4}{3}\right)^n$

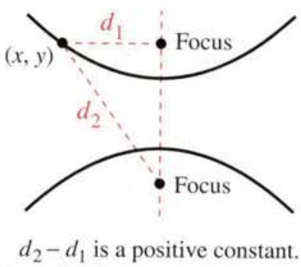
10.3 Hyperbolas

Introduction

The definition of a hyperbola parallels that of an ellipse. The difference is that for an ellipse, the *sum* of the distances between the foci and a point on the ellipse is fixed; whereas for a hyperbola, the *difference* of these distances is fixed.

Definition of a Hyperbola

A **hyperbola** is the set of all points (x, y) the difference of whose distances from two distinct fixed points (**foci**) is a positive constant. (See Figure 10.21.)



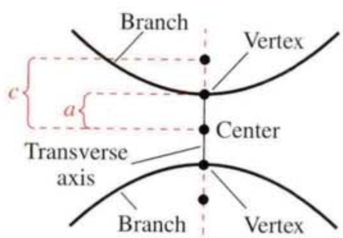


Figure 10.21

Every hyperbola has two disconnected **branches.** The line through the two foci intersects a hyperbola at its two **vertices.** The line segment connecting the vertices is called the **transverse axis,** and the midpoint of the transverse axis is called the **center** of the hyperbola. The development of the **standard form of the equation of a hyperbola** is similar to that of an ellipse. Note that a, b, and c are related differently for hyperbolas than for ellipses.

Standard Equation of a Hyperbola

The **standard form of the equation of a hyperbola** with center at (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.}$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \quad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$. If the center of the hyperbola is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.} \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{Transverse axis is vertical.}$$

What You Should Learn:

- How to write equations of hyperbolas in standard form
- How to find asymptotes of hyperbolas
- How to use properties of hyperbolas to solve real-life problems
- How to classify conics from their general equations

Why You Should Learn It:

Hyperbolas can be used to model and solve many types of real-life problems. For instance, in Exercise 41 on page 721, hyperbolas are used to locate the position of an explosion that was recorded by three listening stations.

Figure 10.22 shows both the horizontal and vertical orientations for a hyperbola.

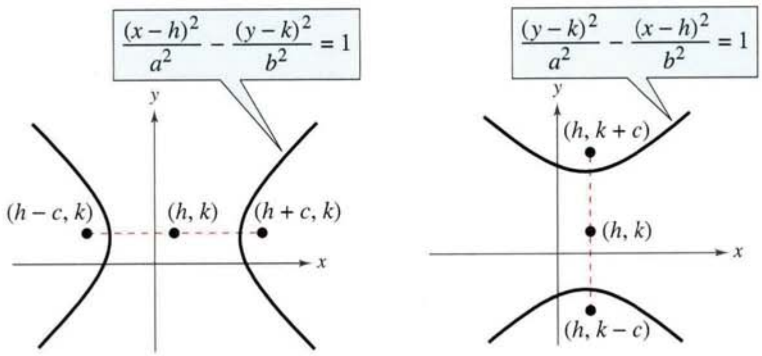


Figure 10.22

EXAMPLE 1 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with foci at $(-1, 2)$ and $(5, 2)$, and vertices at $(0, 2)$ and $(4, 2)$.

Solution

By the Midpoint Formula, the center of the hyperbola occurs at the point $(2, 2)$. Furthermore, $c = 3$ and $a = 2$, and it follows that

$$\begin{aligned} b &= \sqrt{c^2 - a^2} \\ &= \sqrt{3^2 - 2^2} \\ &= \sqrt{9 - 4} \\ &= \sqrt{5}. \end{aligned}$$

So, the equation of the hyperbola is

$$\frac{(x-2)^2}{2^2} - \frac{(y-2)^2}{(\sqrt{5})^2} = 1.$$

Figure 10.23 shows the hyperbola.

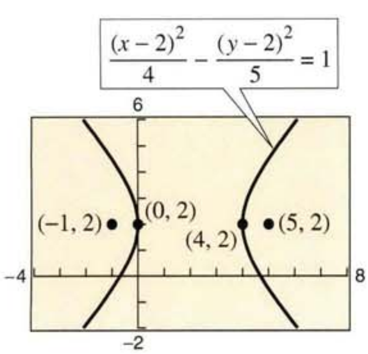


Figure 10.23

Exploration

Most graphing utilities have a *parametric* mode. Try using *parametric* mode to graph the hyperbola $x = 2 + 2 \sec t$ and $y = 2 + \sqrt{5} \tan t$. How does the result compare with the graph given in Figure 10.23? (Let the parameter vary from $0 \le t \le 6.28$, with an increment of t of 0.13.)

Asymptotes of a Hyperbola

Each hyperbola has two **asymptotes** that intersect at the center of the hyperbola, as shown in Figure 10.24. The asymptotes pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k).

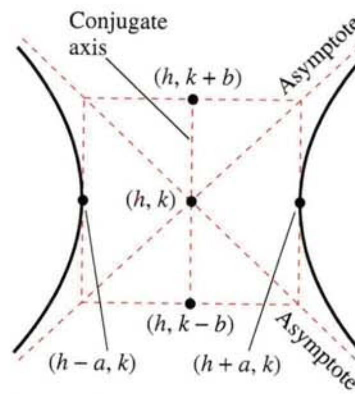


Figure 10.24

Asymptotes of a Hyperbola

$y = k \pm \dfrac{b}{a}(x - h)$	Asymptotes for horizontal transverse axis	$y = k \pm \dfrac{a}{b}(x - h)$	Asymptotes for vertical transverse axis

The **conjugate axis** of a hyperbola is the line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ if the transverse axis is horizontal, and the line segment of length $2b$ joining $(h + b, k)$ and $(h - b, k)$ if the transverse axis is vertical.

EXAMPLE 2 Sketching the Graph of a Hyperbola

Sketch the hyperbola whose equation is $4x^2 - y^2 = 16$.

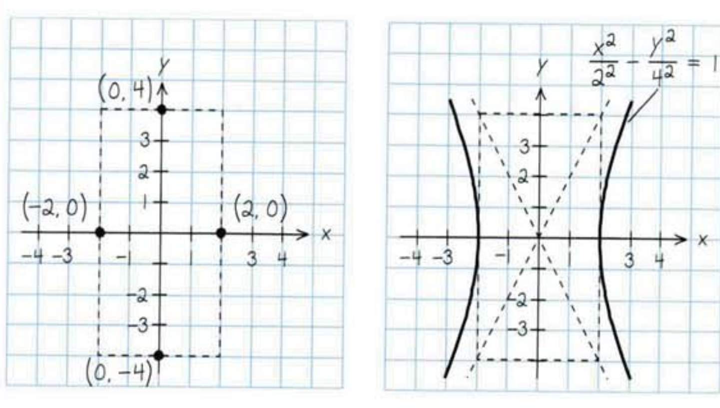
Algebraic Solution

$$4x^2 - y^2 = 16 \qquad \text{Write original equation.}$$

$$\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16} \qquad \text{Divide each side by 16.}$$

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1 \qquad \text{Write in standard form.}$$

Because the x^2-term is positive, you can conclude that the transverse axis is horizontal. So, the vertices occur at $(-2, 0)$ and $(2, 0)$, and the endpoints of the conjugate axis occur at $(0, -4)$ and $(0, 4)$. Using these four points, you can sketch the rectangle shown in Figure 10.25(a). Finally, by drawing the asymptotes through the corners of this rectangle, you can complete the sketch, as shown in Figure 10.25(b).



(a) (b)

Figure 10.25

Graphical Solution

Solve the equation of the hyperbola for y as follows.

$$4x^2 - y^2 = 16$$

$$4x^2 - 16 = y^2$$

$$\pm\sqrt{4x^2 - 16} = y$$

Then use a graphing utility to graph $y_1 = \sqrt{4x^2 - 16}$ and $y_2 = -\sqrt{4x^2 - 16}$ in the same viewing window. Be sure to use a square setting. From the graph in Figure 10.26, you can see that the transverse axis is horizontal. You can use the *zoom* and *trace* features to approximate the vertices to be $(-2, 0)$ and $(2, 0)$.

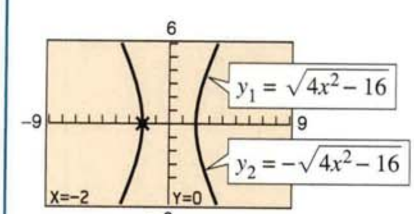


Figure 10.26

EXAMPLE 3 Finding the Asymptotes of a Hyperbola

Sketch the hyperbola given by $4x^2 - 3y^2 + 8x + 16 = 0$ and find the equations of its asymptotes.

Solution

$4x^2 - 3y^2 + 8x + 16 = 0$ Write original equation.

$4(x^2 + 2x) - 3y^2 = -16$ Subtract 16 from each side; factor.

$4(x^2 + 2x + 1) - 3y^2 = -16 + 4$ Complete the square.

$4(x + 1)^2 - 3y^2 = -12$ Write in completed square form.

$\dfrac{y^2}{2^2} - \dfrac{(x + 1)^2}{(\sqrt{3})^2} = 1$ Write in standard form.

From this equation you can conclude that the hyperbola is centered at $(-1, 0)$ and has vertices at $(-1, 2)$ and $(-1, -2)$, and that the ends of the conjugate axis occur at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$. To sketch the hyperbola, draw a rectangle through these four points. The asymptotes are the lines passing through the corners of the rectangle, as shown in Figure 10.27. Finally, using $a = 2$ and $b = \sqrt{3}$, you can conclude that the equations of the asymptotes are

$$y = \frac{2}{\sqrt{3}}(x + 1) \qquad \text{and} \qquad y = -\frac{2}{\sqrt{3}}(x + 1).$$

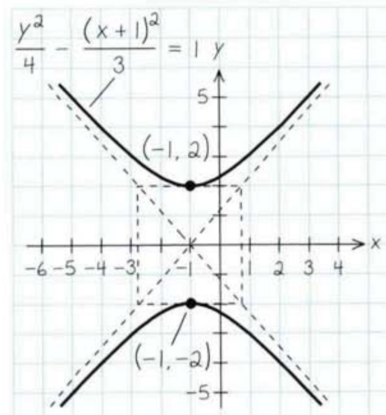


Figure 10.27

If the constant term F in the equation in Example 3 had been $F = 4$ instead of 16, you would have obtained the following degenerate case.

Two Intersecting Lines: $\dfrac{y^2}{4} - \dfrac{(x + 1)^2}{3} = 0$

STUDY TIP

You can use a graphing utility to graph a hyperbola by graphing the upper and lower portions in the same viewing window. For instance, to graph the hyperbola in Example 3, first solve for y to get

$$y_1 = 2\sqrt{1 + \frac{(x + 1)^2}{3}}$$

and

$$y_2 = -2\sqrt{1 + \frac{(x + 1)^2}{3}}.$$

Use a viewing window in which $-8 \le x \le 6$ and $-6 \le y \le 6$. You should obtain the graph shown in Figure 10.28.

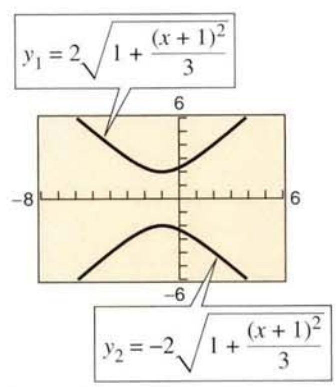


Figure 10.28

EXAMPLE 4 Using Asymptotes to Find the Standard Equation

Find the standard form of the equation of the hyperbola having vertices at $(3, -5)$ and $(3, 1)$ and with asymptotes

$$y = 2x - 8 \qquad \text{and} \qquad y = -2x + 4$$

as shown in Figure 10.29.

Solution

By the Midpoint Formula, the center of the hyperbola is at $(3, -2)$. Furthermore, the hyperbola has a vertical transverse axis with $a = 3$. From the given equations, you can determine the slopes of the asymptotes to be

$$m_1 = 2 = \frac{a}{b}$$

and

$$m_2 = -2 = -\frac{a}{b}$$

and because $a = 3$, you can conclude that $b = \frac{3}{2}$. So, the standard equation is

$$\frac{(y+2)^2}{3^2} - \frac{(x-3)^2}{(3/2)^2} = 1.$$

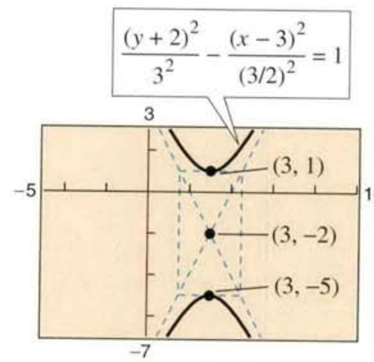


Figure 10.29

As with ellipses, the *eccentricity* of a hyperbola is

$$e = \frac{c}{a} \qquad \text{Eccentricity}$$

and because $c > a$ it follows that $e > 1$. If the eccentricity is large, the branches of the hyperbola are nearly flat. If the eccentricity is close to 1, the branches of the hyperbola are more pointed. See Figure 10.30.

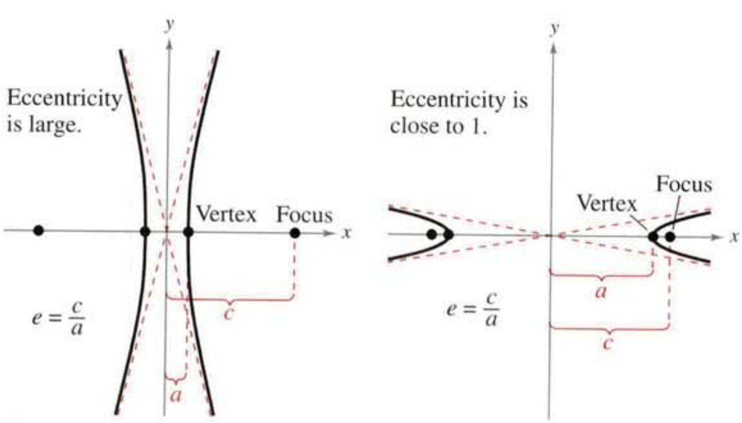


Figure 10.30

Applications

The following application was developed during World War II. It shows how the properties of hyperbolas can be used in radar and other detection systems.

EXAMPLE 5 An Application Involving Hyperbolas

Two microphones, 1 mile apart, record an explosion. Microphone A received the sound 2 seconds before microphone B. Where was the explosion?

Solution

Assuming sound travels at 1100 feet per second, you know that the explosion took place 2200 feet further from B than from A, as shown in Figure 10.31. The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

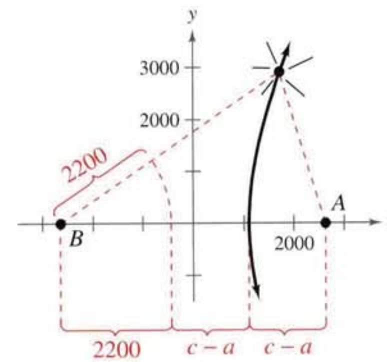
$$c = \frac{5280}{2} = 2640$$

and

$$a = \frac{2200}{2} = 1100.$$

So, $b^2 = c^2 - a^2 = 5{,}759{,}600$, and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola

$$\frac{x^2}{1{,}210{,}000} - \frac{y^2}{5{,}759{,}600} = 1.$$



$$2c = 5280$$

$$2200 + 2(c - a) = 5280$$

Figure 10.31

Another interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure 10.32. Undoubtedly, there have been many comets with parabolic or hyperbolic orbits that have not been identified. You get to see such comets only *once*. Comets with elliptical orbits, such as Halley's comet, are the only ones that remain in our solar system.

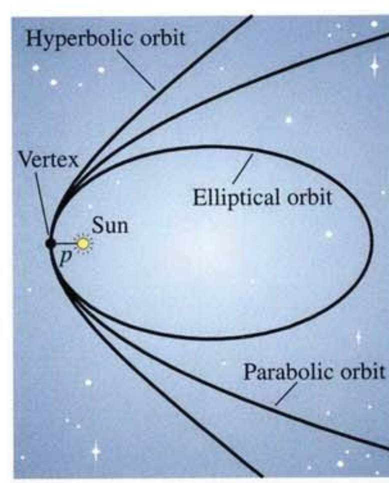


Figure 10.32

If p is the distance between the vertex and the focus in meters, and v is the velocity of the comet at the vertex in meters per second, the type of orbit is determined as follows.

1. Ellipse: $v < \sqrt{2GM/p}$
2. Parabola: $v = \sqrt{2GM/p}$
3. Hyperbola: $v > \sqrt{2GM/p}$

In each of these equations, $M \approx 1.991 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 6.67 \times 10^{-11}$ cubic meters per kilogram-second squared (the universal gravitational constant).

Classifying a Conic from Its General Equation

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is one of the following.

1. Circle:	$A = C$	$A \neq 0$
2. Parabola:	$AC = 0$	$A = 0$ or $C = 0$, but not both.
3. Ellipse:	$AC > 0$	A and C have like signs.
4. Hyperbola:	$AC < 0$	A and C have unlike signs.

The test above is valid *if* the graph is a conic. The test does not apply to equations such as $x^2 + y^2 = -1$, which is not a conic.

EXAMPLE 6 Classifying Conics from General Equations

Classify each graph.

a. $4x^2 - 9x + y - 5 = 0$

b. $4x^2 - y^2 + 8x - 6y + 4 = 0$

c. $2x^2 + 4y^2 - 4x + 12y = 0$

d. $2x^2 + 2y^2 - 8x + 12y + 2 = 0$

Solution

a. For the equation $4x^2 - 9x + y - 5 = 0$, you have

$AC = 4(0) = 0.$ Parabola

So, the graph is a parabola.

b. For the equation $4x^2 - y^2 + 8x - 6y + 4 = 0$, you have

$AC = 4(-1) < 0.$ Hyperbola

So, the graph is a hyperbola.

c. For the equation $2x^2 + 4y^2 - 4x + 12y = 0$, you have

$AC = 2(4) > 0.$ Ellipse

So, the graph is an ellipse.

d. For the equation $2x^2 + 2y^2 - 8x + 12y + 2 = 0$, you have

$A = C = 2.$ Circle

So, the graph is a circle.

The Granger Collection

The first woman to be credited with detecting a new comet was the English astronomer **Caroline Herschel (1750–1848).** During her long life, Caroline Herschel discovered a total of eight new comets.

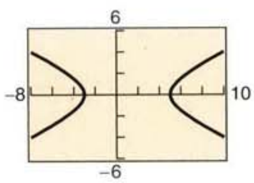
Writing About Math *Identifying Equations of Conics*

Use the Internet to research information about the orbits of comets in our solar system. What can you find about the orbits of comets that have been identified since 1970? Write a summary of your results. Identify your source. Does it seem reliable?

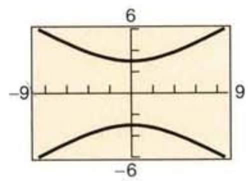
10.3 Exercises

In Exercises 1–4, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

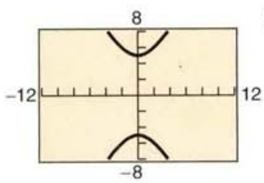
(a)



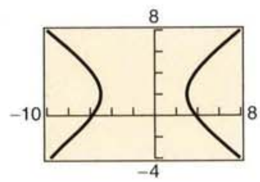
(b)



(c)



(d)



1. $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$

2. $\dfrac{y^2}{25} - \dfrac{x^2}{9} = 1$

3. $\dfrac{(x-1)^2}{16} - \dfrac{y^2}{4} = 1$

4. $\dfrac{(x+1)^2}{16} - \dfrac{(y-2)^2}{9} = 1$

In Exercises 5–18, find the center, vertices, foci, and asymptotes of the hyperbola, and sketch its graph, using the asymptotes as an aid. Use a graphing utility to verify your graph.

5. $x^2 - y^2 = 1$

6. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$

7. $\dfrac{y^2}{1} - \dfrac{x^2}{4} = 1$

8. $\dfrac{y^2}{9} - \dfrac{x^2}{1} = 1$

9. $\dfrac{y^2}{25} - \dfrac{x^2}{81} = 1$

10. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$

11. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$

12. $\dfrac{(x+3)^2}{144} - \dfrac{(y-2)^2}{25} = 1$

13. $(y+6)^2 - (x-2)^2 = 1$

14. $\dfrac{(y-1)^2}{1/4} - \dfrac{(x+3)^2}{1/16} = 1$

15. $9x^2 - y^2 - 36x - 6y + 18 = 0$

16. $x^2 - 9y^2 + 36y - 72 = 0$

17. $x^2 - 9y^2 + 2x - 54y - 80 = 0$

18. $16y^2 - x^2 + 2x + 64y + 63 = 0$

In Exercises 19–22, find the center, vertices, foci, and the equations of the asymptotes of the hyperbola. Use a graphing utility to graph the hyperbola and its asymptotes.

19. $2x^2 - 3y^2 = 6$

20. $6y^2 - 3x^2 = 18$

21. $9y^2 - x^2 + 2x + 54y + 62 = 0$

22. $9x^2 - y^2 + 54x + 10y + 55 = 0$

In Exercises 23–40, find the standard form of the equation of the specified hyperbola.

23. Vertices: $(0, \pm 2)$; Foci: $(0, \pm 4)$

24. Vertices: $(\pm 2, 0)$; Foci: $(\pm 5, 0)$

25. Vertices: $(\pm 1, 0)$; Asymptotes: $y = \pm 5x$

26. Vertices: $(0, \pm 3)$; Asymptotes: $y = \pm 3x$

27. Foci: $(0, \pm 8)$; Asymptotes: $y = \pm 4x$

28. Foci: $(\pm 10, 0)$; Asymptotes: $y = \pm \frac{3}{4}x$

29. Vertices: $(2, 0), (6, 0)$; Foci: $(0, 0), (8, 0)$

30. Vertices: $(2, 3), (2, -3)$; Foci: $(2, 5), (2, -5)$

31. Vertices: $(4, 1), (4, 9)$; Foci: $(4, 0), (4, 10)$

32. Vertices: $(-2, 1), (2, 1)$; Foci: $(-3, 1), (3, 1)$

33. Vertices: $(2, 3), (2, -3)$;
Passes through the point $(0, 5)$

34. Vertices: $(-2, 1), (2, 1)$;
Passes through the point $(5, 4)$

35. Vertices: $(0, 4), (0, 0)$;
Passes through the point $(\sqrt{5}, 5)$

36. Vertices: $(1, 2), (1, -2)$;
Passes through the point $(0, \sqrt{5})$

37. Vertices: $(1, 2), (3, 2)$;
Asymptotes: $y = x$, $y = 4 - x$

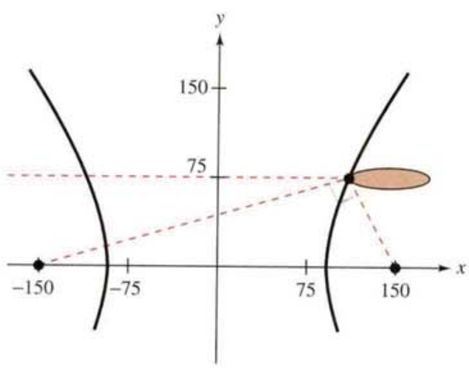
38. Vertices: $(3, 0), (3, -6)$;
Asymptotes: $y = x - 6$, $y = -x$

39. Vertices: $(0, 2), (6, 2)$;
Asymptotes: $y = \frac{2}{3}x$, $y = 4 - \frac{2}{3}x$

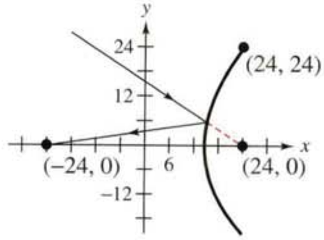
40. Vertices: $(3, 0), (3, 4)$;
Asymptotes: $y = \frac{2}{3}x$, $y = 4 - \frac{2}{3}x$

41. ***Sound Location*** Three listening stations located at (3300, 0), (3300, 1100), and (−3300, 0) monitor an explosion. If the last two stations detect the explosion 1 second and 4 seconds after the first, respectively, determine the coordinates of the explosion. (Assume that the coordinate system is measured in feet and that sound travels at 1100 feet per second.)

42. ***Navigation*** Long distance radio navigation for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations, 300 miles apart, are positioned on the rectangular coordinate system at points with coordinates (−150, 0) and (150, 0), and that a ship is traveling on a path with coordinates $(x, 75)$. Find the x-coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).



43. ***Hyperbolic Mirror*** A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at a focus will be reflected to the other focus. The focus of a hyperbolic mirror has coordinates (24, 0). Find the vertex of the mirror if its mount has coordinates (24, 24).



In Exercises 44–51, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

44. $x^2 + y^2 - 6x + 4y + 9 = 0$

45. $x^2 + 4y^2 - 6x + 16y + 21 = 0$

46. $4x^2 - y^2 - 4x - 3 = 0$

47. $y^2 - 4y - 4x = 0$

48. $4x^2 + 3y^2 + 8x - 24y + 51 = 0$

49. $4y^2 - 2x^2 - 4y - 8x - 15 = 0$

50. $25x^2 - 10x - 200y - 119 = 0$

51. $4x^2 + 4y^2 - 16y + 15 = 0$

Synthesis

True or False? **In Exercises 52 and 53, determine whether the statement is true or false. Justify your answer.**

52. In the standard form of the equation of a hyperbola, the larger the ratio of b to a, the larger the eccentricity of the hyperbola.

53. In the standard form of the equation of a hyperbola, the trivial solution of two intersecting lines occurs when $b = 0$.

54. ***Think About It*** Consider a hyperbola centered at the origin with a horizontal transverse axis. Use the definition of a hyperbola to derive its standard form.

55. ***Think About It*** Explain how the central rectangle of a hyperbola can be used to sketch its asymptotes.

Review

In Exercises 56–59, perform the indicated polynomial operation.

56. Subtract: $(x^3 - 3x^2) - (6 - 2x - 4x^2)$

57. Multiply: $\left(3x - \frac{1}{2}\right)(x + 4)$

58. Divide: $\dfrac{x^3 - 3x + 4}{x + 2}$

59. Expand: $[(x + y) + 3]^2$

In Exercises 60–65, factor the polynomial.

60. $x^3 - 16x$

61. $x^2 + 14x + 49$

62. $2x^3 - 24x^2 + 72x$

63. $6x^3 - 11x^2 - 10x$

64. $16x^3 + 54$

65. $4 - x + 4x^2 - x^3$

10.4 Rotation and Systems of Quadratic Equations

What You Should Learn:

- How to rotate the coordinate axes to eliminate the xy-term in equations of conics
- How to use the discriminant to classify conics
- How to solve systems of quadratic equations

Why You Should Learn It:

As illustrated in Exercises 5–16 on page 729, rotation of the coordinate axes can help you identify the graph of a general second-degree equation.

Rotation

In the previous section you learned that the equation of a conic with axes parallel to the coordinate axes has a standard form that can be written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \quad \text{Horizontal or vertical axes}$$

In this section you will study the equations of conics whose axes are rotated so that they are not parallel to either the x-axis or the y-axis. The general equation for such conics contains an xy-term.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad \text{Equation in } xy\text{-plane}$$

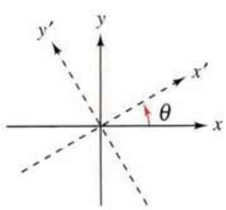


Figure 10.33

To eliminate this xy-term, you can use a procedure called rotation of axes. The objective is to rotate the x- and y-axes until they are parallel to the axes of the conic. The rotated axes are denoted as the x'-axis and the y'-axis, as shown in Figure 10.33. After the rotation, the equation of the conic in the new $x'y'$-plane will have the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0. \quad \text{Equation in } x'y'\text{-plane}$$

Because this equation has no xy-term, you can obtain a standard form by completing the square. The following theorem identifies how much to rotate the axes to eliminate the xy-term and also the equations for determining the new coefficients A', C', D', E', and F'. For a proof of this theorem, see Appendix A.

Rotation of Axes to Eliminate an xy-Term

The general second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where $\cot 2\theta = \dfrac{A - C}{B}$.

The coefficients of the new equation are obtained by making the substitutions

$$x = x'\cos\theta - y'\sin\theta \qquad \text{and} \qquad y = x'\sin\theta + y'\cos\theta.$$

EXAMPLE 1 Rotation of Axes for a Hyperbola

Rotate the axes to eliminate the xy-term in the equations $xy - 1 = 0$. Then write the equation in standard form.

Solution

Because $A = 0$, $B = 1$, and $C = 0$, you have

$$\cot 2\theta = \frac{A - C}{B} = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

which implies that

$$\begin{aligned} x &= x' \cos \frac{\pi}{4} - y' \sin \frac{\pi}{4} \\ &= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{x' - y'}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} y &= x' \sin \frac{\pi}{4} + y' \cos \frac{\pi}{4} \\ &= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{x' + y'}{\sqrt{2}}. \end{aligned}$$

The equation in the $x'y'$-system is obtained by substituting these expressions into the equation $xy - 1 = 0$.

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 1 = 0$$

$$\frac{(x')^2 - (y')^2}{2} - 1 = 0$$

$$\frac{(x')^2}{(\sqrt{2})^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1 \qquad \text{Write in standard form.}$$

In the $x'y'$-system, this is a hyperbola centered at the origin with vertices at $(\pm\sqrt{2}, 0)$, as shown in Figure 10.34. To find the coordinates of the vertices in the xy-system, substitute the coordinates $(\pm\sqrt{2}, 0)$ into the equations

$$x = \frac{x' - y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x' + y'}{\sqrt{2}}.$$

This substitution yields the vertices $(1, 1)$ and $(-1, -1)$ in the xy-system. Note also that the asymptotes of the hyperbola have equations $y' = \pm x'$, which correspond to the original x- and y-axes.

STUDY TIP

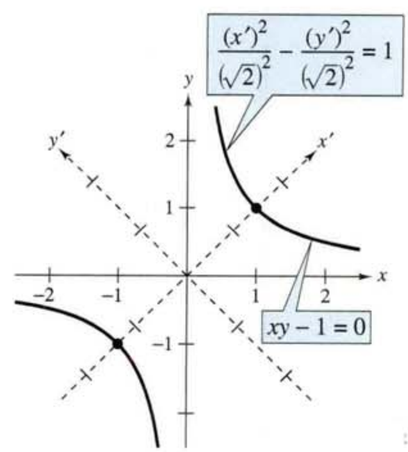
Remember that the substitutions

$$x = x' \cos \theta - y' \sin \theta$$

and

$$y = x' \sin \theta + y' \cos \theta$$

were developed to eliminate the $x'y'$-term in the rotated system. You can use this as a check on your work. In other words, if your final equation contains an $x'y'$-term, you know that you made a mistake.



Vertices:
In x'y'-system: $(\sqrt{2}, 0), (-\sqrt{2}, 0)$
In xy-system: $(1, 1), (-1, -1)$
Figure 10.34

EXAMPLE 2 Rotation of Axes for an Ellipse

Rotate the axes to eliminate the xy-term in the equation

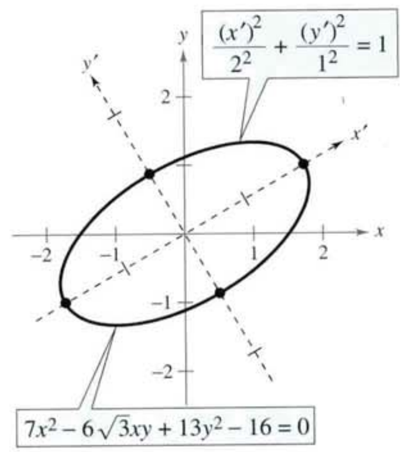
$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0.$$

Then write the equation in standard form and sketch its graph.

Solution

Because $A = 7$, $B = -6\sqrt{3}$, and $C = 13$, you have

$$\cot 2\theta = \frac{A - C}{B}$$
$$= \frac{7 - 13}{-6\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}$$



Vertices:
In x'y'-system: $(\pm 2, 0)$, $(0, \pm 1)$
In xy-system: $(\sqrt{3}, 1)$, $(-3, -1)$, $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Figure 10.35

which implies that $\theta = \pi/6$. The equation in the $x'y'$-system is obtained by making the substitutions

$$x = x' \cos\frac{\pi}{6} - y' \sin\frac{\pi}{6}$$
$$= x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{3}x' - y'}{2}$$

and

$$y = x' \sin\frac{\pi}{6} + y' \cos\frac{\pi}{6}$$
$$= x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{x' + \sqrt{3}y'}{2}$$

into the original equation. So, you have

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$$
$$7\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) + 13\left(\frac{x' + \sqrt{3}y'}{2}\right)^2 - 16 = 0$$

which simplifies to

$$4(x')^2 + 16(y')^2 - 16 = 0$$
$$4(x')^2 + 16(y')^2 = 16$$
$$\frac{(x')^2}{2^2} + \frac{(y')^2}{1^2} = 1. \qquad \text{Write in standard form.}$$

This is the equation of an ellipse centered at the origin with vertices $(\pm 2, 0)$ in the $x'y'$-system, as shown in Figure 10.35.

EXAMPLE 3 Rotation of Axes for a Parabola

Rotate the axes to eliminate the xy-term in the equation

$$x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0.$$

Then write the equation in standard form and sketch its graph.

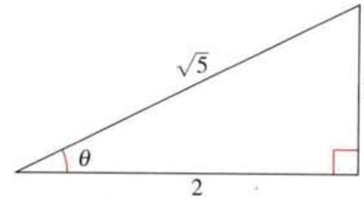


Figure 10.36

Solution

Because $A = 1$, $B = -4$, and $C = 4$, you have

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 4}{-4} = \frac{3}{4}.$$

Using the identity $\cot 2\theta = (\cot^2 \theta - 1)/(2 \cot \theta)$ produces

$$\cot 2\theta = \frac{3}{4} = \frac{\cot^2 \theta - 1}{2 \cot \theta}$$

from which you obtain the equation

$$4 \cot^2 \theta - 4 = 6 \cot \theta$$

$$4 \cot^2 \theta - 6 \cot \theta - 4 = 0$$

$$(2 \cot \theta - 4)(2 \cot \theta + 1) = 0.$$

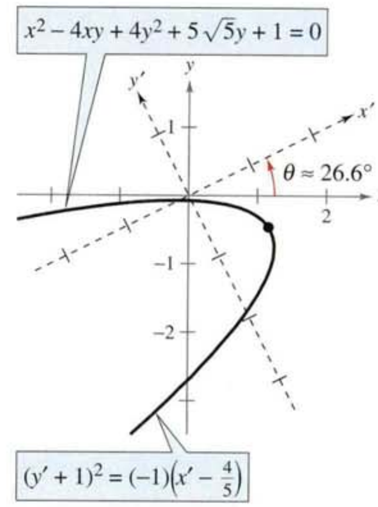
Considering $0 < \theta < \pi/2$, you have $2 \cot \theta = 4$. So,

$$\cot \theta = 2 \quad \Longrightarrow \quad \theta \approx 26.6^\circ.$$

From the triangle in Figure 10.36, you obtain $\sin \theta = 1/\sqrt{5}$ and $\cos \theta = 2/\sqrt{5}$. So, you use the substitutions

$$x = x' \cos \theta - y' \sin \theta = x'\left(\frac{2}{\sqrt{5}}\right) - y'\left(\frac{1}{\sqrt{5}}\right) = \frac{2x' - y'}{\sqrt{5}}$$

$$y = x' \sin \theta + y' \cos \theta = x'\left(\frac{1}{\sqrt{5}}\right) + y'\left(\frac{2}{\sqrt{5}}\right) = \frac{x' + 2y'}{\sqrt{5}}.$$



Vertex:
In $x'y'$-system: $\left(\frac{4}{5}, -1\right)$
In xy-system: $\left(\frac{13}{5\sqrt{5}}, -\frac{6}{5\sqrt{5}}\right)$

Figure 10.37

Substituting these expressions into the original equation, you have

$$x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0$$

$$\left(\frac{2x' - y'}{\sqrt{5}}\right)^2 - 4\left(\frac{2x' - y'}{\sqrt{5}}\right)\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 4\left(\frac{x' + 2y'}{\sqrt{5}}\right)^2 + 5\sqrt{5}\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 1 = 0$$

which simplifies as follows.

$$5(y')^2 + 5x' + 10y' + 1 = 0$$

$$5[(y')^2 + 2y'] = -5x' - 1 \qquad \text{Group terms.}$$

$$5[(y')^2 + 2y' + 1] = -5x' - 1 + 5 \qquad \text{Complete the square.}$$

$$5(y' + 1)^2 = -5x' + 4 \qquad \text{Write in completed square form.}$$

$$(y' + 1)^2 = (-1)\left(x' - \frac{4}{5}\right) \qquad \text{Write in standard form.}$$

The graph of this equation is a parabola with vertex at $\left(\frac{4}{5}, -1\right)$. Its axis is parallel to the x'-axis in the $x'y'$-system, as shown in Figure 10.37.

Invariants Under Rotation

In the rotation of axes theorem listed at the beginning of this section, note that the constant term is the same in both equations—that is, $F' = F$. Such quantities are **invariant under rotation.** The next theorem lists some other rotation invariants.

Rotation Invariants

The rotation of the coordinate axes through an angle θ that transforms the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ into the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

has the following rotation invariants.

1. $F = F'$
2. $A + C = A' + C'$
3. $B^2 - 4AC = (B')^2 - 4A'C'$

You can use the results of this theorem to classify the graph of a second-degree equation *with* an xy-term in much the same way you do for a second-degree equation *without* an xy-term. Note that because $B' = 0$, the invariant $B^2 - 4AC$ reduces to

$$B^2 - 4AC = -4A'C'. \qquad \text{Discriminant}$$

This quantity is called the **discriminant** of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Now, from the classification procedure given in Section 10.3, you know that the sign of $A'C'$ determines the type of graph for the equation

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0.$$

Consequently, the sign of $B^2 - 4AC$ will determine the type of graph for the original equation, as given in the following classification.

Classification of Conics by the Discriminant

The graph of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is, except in degenerate cases, determined by its discriminant as follows.

1. Ellipse or circle: $B^2 - 4AC < 0$
2. Parabola: $B^2 - 4AC = 0$
3. Hyperbola: $B^2 - 4AC > 0$

For example, in the general equation

$$3x^2 + 7xy + 5y^2 - 6x - 7y + 15 = 0$$

you have $A = 3$, $B = 7$, and $C = 5$. So, the discriminant is

$$B^2 - 4AC = 7^2 - 4(3)(5) = 49 - 60 = -11.$$

Because $-11 < 0$, the graph of the equation is an ellipse or a circle.

EXAMPLE 4 Rotations and Graphing Utilities

For each of the following, classify the graph, use the quadratic formula to solve for y, and then use a graphing utility to graph the equation.

a. $2x^2 - 3xy + 2y^2 - 2x = 0$

b. $x^2 - 6xy + 9y^2 - 2y + 1 = 0$

c. $3x^2 + 8xy + 4y^2 - 7 = 0$

Solution

a. Because $B^2 - 4AC = 9 - 16 < 0$, the graph is a circle or an ellipse. Solve for y as follows.

$$2x^2 - 3xy + 2y^2 - 2x = 0 \qquad \text{Write original equation.}$$

$$2y^2 - 3xy + (2x^2 - 2x) = 0 \qquad \text{Quadratic form } ay^2 + by + c = 0$$

$$y = \frac{-(-3x) \pm \sqrt{(-3x)^2 - 4(2)(2x^2 - 2x)}}{2(2)}$$

$$y = \frac{3x \pm \sqrt{x(16 - 7x)}}{4}$$

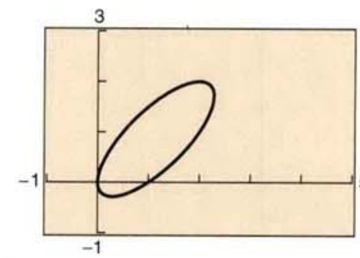


Figure 10.38

Graph both of the equations to obtain the ellipse in Figure 10.38.

$$y = \frac{3x + \sqrt{x(16 - 7x)}}{4} \qquad \text{Top half of ellipse}$$

$$y = \frac{3x - \sqrt{x(16 - 7x)}}{4} \qquad \text{Bottom half of ellipse}$$

b. Because $B^2 - 4AC = 36 - 36 = 0$, the graph is a parabola.

$$x^2 - 6xy + 9y^2 - 2y + 1 = 0 \qquad \text{Write original equation.}$$

$$9y^2 - (6x + 2)y + (x^2 + 1) = 0 \qquad \text{Quadratic form } ay^2 + by + c = 0$$

$$y = \frac{(6x + 2) \pm \sqrt{(6x + 2)^2 - 4(9)(x^2 + 1)}}{18}$$

$$y = \frac{3x + 1 \pm \sqrt{2(3x - 4)}}{9}$$

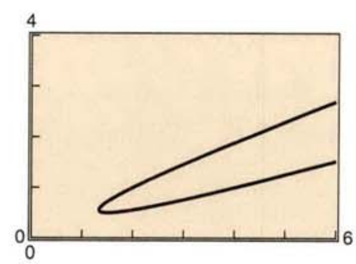


Figure 10.39

Graphing the resulting two equations gives the parabola in Figure 10.39.

c. Because $B^2 - 4AC = 64 - 48 > 0$, the graph is a hyperbola.

$$3x^2 + 8xy + 4y^2 - 7 = 0 \qquad \text{Write original equation.}$$

$$4y^2 + 8xy + (3x^2 - 7) = 0 \qquad \text{Quadratic form } ay^2 + by + c = 0$$

$$y = \frac{-8x \pm \sqrt{(8x)^2 - 4(4)(3x^2 - 7)}}{8}$$

$$y = \frac{-2x \pm \sqrt{x^2 + 7}}{2}$$

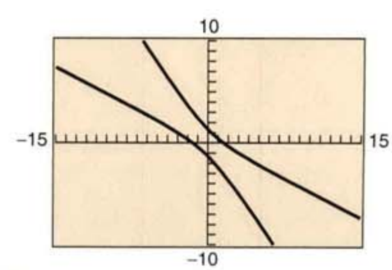


Figure 10.40

The graph of the resulting two equations yields the hyperbola in Figure 10.40.

Systems of Quadratic Equations

To find the points of intersection of two conics, you can use elimination or substitution, as demonstrated in Examples 5 and 6.

EXAMPLE 5 Solving a Quadratic System

Solve the system of quadratic equations.

$$\begin{cases} x^2 + y^2 - 16x + 39 = 0 & \text{Equation 1} \\ x^2 - y^2 - 9 = 0 & \text{Equation 2} \end{cases}$$

Algebraic Solution

You can eliminate the y^2-term by adding the two equations. The resulting equation can then be solved for x.

$$2x^2 - 16x + 30 = 0$$

$$2(x - 3)(x - 5) = 0$$

There are two real solutions: $x = 3$ and $x = 5$. The corresponding y-values are $y = 0$ and $y = \pm 4$. So, the graphs have three points of intersection:

$$(3, 0),\ (5, 4),\ \text{and}\ (5, -4).$$

Graphical Solution

Begin by solving each equation for y as follows.

$$y = \pm\sqrt{-x^2 + 16x - 39} \qquad y = \pm\sqrt{x^2 - 9}$$

Use a graphing utility to graph all four equations $y_1 = \sqrt{-x^2 + 16x - 39}$, $y_2 = -\sqrt{-x^2 + 16x - 39}$, $y_3 = \sqrt{x^2 - 9}$, and $y_4 = -\sqrt{x^2 - 9}$ in the same viewing window. In Figure 10.41, you can see that the graphs appear to intersect at the points (3, 0), (5, 4), and (5, −4). Use the *intersect* feature to confirm this.

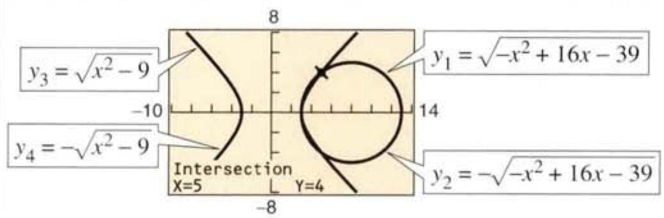


Figure 10.41

EXAMPLE 6 Solving a Quadratic System by Substitution

Solve the system of quadratic equations.

$$\begin{cases} x^2 + 4y^2 - 4x - 8y + 4 = 0 & \text{Equation 1} \\ x^2 + 4y - 4 = 0 & \text{Equation 2} \end{cases}$$

Solution

Because Equation 2 has no y^2-term, solve the equation for y to obtain $y = 1 - (1/4)x^2$. Next, substitute this into Equation 1 and solve for x.

$$x^2 + 4\left(1 - \frac{1}{4}x^2\right)^2 - 4x - 8\left(1 - \frac{1}{4}x^2\right) + 4 = 0$$

$$x^2 + 4 - 2x^2 + \frac{1}{4}x^4 - 4x - 8 + 2x^2 + 4 = 0$$

$$x^4 + 4x^2 - 16x = 0$$

$$x(x - 2)(x^2 + 2x + 8) = 0$$

In factored form, you can see that the equation has two real solutions: $x = 0$ and $x = 2$. The corresponding values of y are $y = 1$ and $y = 0$. This implies that the solutions of the system of equations are (0, 1) and (2, 0), as shown in Figure 10.42.

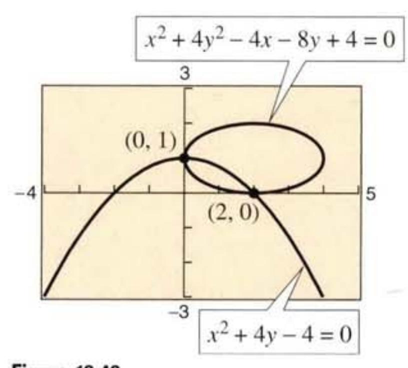


Figure 10.42

10.4 Exercises

In Exercises 1–4, the $x'y'$-coordinate system has been rotated θ degrees from the xy-coordinate system. The coordinates of a point on the xy-coordinate system are given. Find the coordinates of the point on the rotated coordinate system.

1. $\theta = 90°, (0, 4)$
2. $\theta = 45°, (3, 3)$
3. $\theta = 30°, (1, 6)$
4. $\theta = 60°, (5, 1)$

In Exercises 5–16, rotate the axes to eliminate the xy-term in the equation. Then write the equation in standard form. Sketch the graph of the equation, showing both sets of axes.

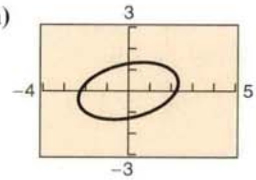
5. $xy + 1 = 0$
6. $xy - 2 = 0$
7. $x^2 - 8xy + y^2 + 1 = 0$
8. $xy + x - 2y + 3 = 0$
9. $xy - 2y - 4x = 0$
10. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$
11. $5x^2 - 6xy + 5y^2 - 12 = 0$
12. $2x^2 - 3xy - 2y^2 + 10 = 0$
13. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$
14. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$
15. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$
16. $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$

In Exercises 17–22, use a graphing utility to graph the conic. Determine the angle θ through which the axes are rotated. Explain how you used the graphing utility to obtain the graph.

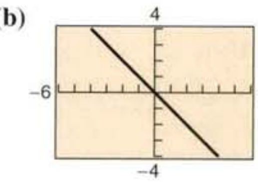
17. $x^2 + xy + y^2 = 12$
18. $x^2 - 4xy + 2y^2 = 10$
19. $17x^2 + 32xy - 7y^2 = 75$
20. $40x^2 + 36xy + 25y^2 = 52$
21. $32x^2 + 50xy + 7y^2 = 52$
22. $4x^2 - 12xy + 9y^2 + (4\sqrt{13} - 12)x - (6\sqrt{13} + 8)y = 91$

In Exercises 23–28, match the graph with its equation. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

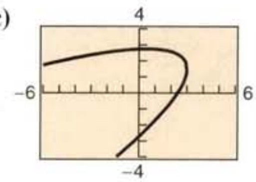
(a)



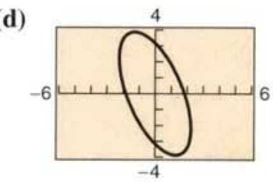
(b)



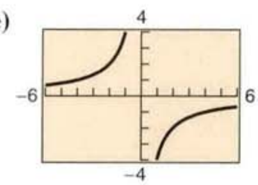
(c)



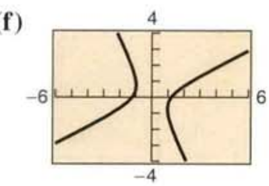
(d)



(e)



(f)



23. $xy + 4 = 0$
24. $x^2 + 2xy + y^2 = 0$
25. $-2x^2 + 3xy + 2y^2 + 3 = 0$
26. $x^2 - xy + 3y^2 - 5 = 0$
27. $3x^2 + 2xy + y^2 - 10 = 0$
28. $x^2 - 4xy + 4y^2 + 10x - 30 = 0$

In Exercises 29–36, (a) use the discriminant to classify the graph, (b) use the quadratic formula to solve for y, and (c) use a graphing utility to graph the equation.

29. $16x^2 - 24xy + 9y^2 - 30x - 40y = 0$
30. $x^2 - 8xy - 2y^2 - 6 = 0$
31. $15x^2 - 8xy + 7y^2 - 45 = 0$
32. $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$
33. $x^2 - 6xy - 5y^2 + 4x - 22 = 0$
34. $36x^2 - 60xy + 25y^2 + 9y = 0$
35. $x^2 + 4xy + 4y^2 - 5x - y - 3 = 0$
36. $x^2 + xy + 4y^2 + x + y - 4 = 0$

In Exercises 37–40, sketch (if possible) the graph of the degenerate conic.

37. $y^2 - 16x^2 = 0$
38. $x^2 + y^2 - 2x + 6y + 10 = 0$
39. $x^2 + 2xy + y^2 - 4 = 0$
40. $x^2 - 10xy + y^2 = 0$

In Exercises 41–48, solve the system of quadratic equations algebraically by the method of elimination. Then verify your results by using a graphing utility to graph the equations and find any points of intersection of the graphs.

41. $\begin{cases} -x^2 + y^2 + 4x - 6y + 4 = 0 \\ x^2 + y^2 - 4x - 6y + 12 = 0 \end{cases}$

42. $\begin{cases} -x^2 - y^2 - 8x + 20y - 7 = 0 \\ x^2 + 9y^2 + 8x + 4y + 7 = 0 \end{cases}$

43. $\begin{cases} -4x^2 - y^2 - 16x + 24y - 16 = 0 \\ 4x^2 + y^2 + 40x - 24y + 208 = 0 \end{cases}$

44. $\begin{cases} x^2 - 4y^2 - 20x - 64y - 172 = 0 \\ 16x^2 + 4y^2 - 320x + 64y + 1600 = 0 \end{cases}$

45. $\begin{cases} x^2 - y^2 - 12x + 16y - 64 = 0 \\ x^2 + y^2 - 12x - 16y + 64 = 0 \end{cases}$

46. $\begin{cases} x^2 + 4y^2 - 2x - 8y + 1 = 0 \\ -x^2 + 2x - 4y - 1 = 0 \end{cases}$

47. $\begin{cases} -16x^2 - y^2 + 24y - 80 = 0 \\ 16x^2 + 25y^2 - 400 = 0 \end{cases}$

48. $\begin{cases} 16x^2 - y^2 + 16y - 128 = 0 \\ y^2 - 48x - 16y - 32 = 0 \end{cases}$

In Exercises 49–54, solve the system of quadratic equations algebraically by the method of substitution. Then verify your results by using a graphing utility to graph the equations and find any points of intersection of the graphs.

49. $\begin{cases} x^2 + y^2 - 4 = 0 \\ 3x - y^2 = 0 \end{cases}$

50. $\begin{cases} 4x^2 + 9y^2 - 36y = 0 \\ x^2 + 9y - 27 = 0 \end{cases}$

51. $\begin{cases} x^2 + 2y^2 - 4x + 6y - 5 = 0 \\ -x + y - 4 = 0 \end{cases}$

52. $\begin{cases} x^2 + 2y^2 - 4x + 6y - 5 = 0 \\ x^2 - 4x - y + 4 = 0 \end{cases}$

53. $\begin{cases} xy + x - 2y + 3 = 0 \\ x^2 + 4y^2 - 9 = 0 \end{cases}$

54. $\begin{cases} 5x^2 - 2xy + 5y^2 - 12 = 0 \\ x + y - 1 = 0 \end{cases}$

Synthesis

True or False? **In Exercises 55 and 56, determine whether the statement is true or false. Justify your answer.**

55. The graph of $x^2 + xy + ky^2 + 6x + 10 = 0$, where k is any constant less than $\frac{1}{4}$, is a hyperbola.

56. After using a rotation of axes to eliminate the xy-term from an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

the coefficients of the x^2- and y^2-terms remain A and B, respectively.

57. Show that the equation $x^2 + y^2 = r^2$ is invariant under rotation of axes.

58. Find the lengths of the major and minor axes of the ellipse in Exercise 10.

Review

In Exercises 59–62, sketch the graph of the rational function. Identify all intercepts and asymptotes.

59. $g(x) = \dfrac{2}{2 - x}$

60. $f(x) = \dfrac{2x}{2 - x}$

61. $h(t) = \dfrac{t^2}{2 - t}$

62. $g(s) = \dfrac{2}{4 - s^2}$

In Exercises 63–66, find (a) AB, (b) BA, and, if possible, (c) A^2.

63. $A = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 6 \\ 5 & -1 \end{bmatrix}$

64. $A = \begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ -3 & 8 \end{bmatrix}$

65. $A = \begin{bmatrix} 4 & -2 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$

66. $A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 1 & 5 \\ 3 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -3 \\ -4 & 5 & -1 \\ 6 & 3 & 2 \end{bmatrix}$

In Exercises 67–70, find the coefficient a of the given term in the expansion of the binomial.

	Binomial	*Term*
67.	$(x + 8)^7$	ax^2
68.	$(3x - y)^6$	ax^3y^3
69.	$(x - 4y)^{10}$	ax^6y^4
70.	$(3x + 2y)^8$	ax^2y^6

10.5 Parametric Equations

Plane Curves

Up to this point, you have been representing a graph by a single equation involving *two* variables such as x and y. In this section, you will study situations in which it is useful to introduce a *third* variable to represent a curve in the plane.

To see the usefulness of this procedure, consider the path followed by an object that is propelled into the air at an angle of 45°. If the initial velocity of the object is 48 feet per second, it can be shown that the object follows the parabolic path

$$y = -\frac{x^2}{72} + x \qquad \text{Rectangular equation}$$

as shown in Figure 10.43. However, this equation does not tell the whole story. Although it does tell us *where* the object has been, it doesn't tell us *when* the object was at a given point (x, y) on the path. To determine this time, you can introduce a third variable t, which is called a **parameter.** It is possible to write both x and y as functions of t to obtain the **parametric equations**

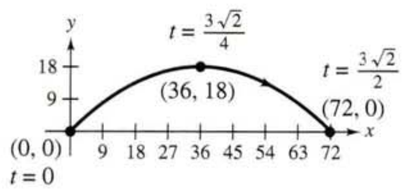
$$x = 24\sqrt{2}t \qquad \text{Parametric equation for } x$$

$$y = -16t^2 + 24\sqrt{2}t. \qquad \text{Parametric equation for } y$$

From this set of equations you can determine that at time $t = 0$, the object is at the point $(0, 0)$. Similarly, at time $t = 1$, the object is at the point $(24\sqrt{2}, 24\sqrt{2} - 16)$, and so on.

Rectangular equation:
$y = -\frac{x^2}{72} + x$

Parametric equations:
$x = 24\sqrt{2}t$
$y = -16t^2 + 24\sqrt{2}t$



Curvilinear motion: two variables for position, one variable for time
Figure 10.43

For this particular motion problem, x and y are continuous functions of t, and the resulting path is a **plane curve.** (Recall that a *continuous function* is one whose graph can be traced without lifting the pencil from the paper.)

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, the set of ordered pairs $(f(t), g(t))$ is a **plane curve** C. The equations

$$x = f(t) \qquad \text{and} \qquad y = g(t)$$

are **parametric equations** for C, and t is the **parameter.**

What You Should Learn:

- How to evaluate sets of parametric equations for given values of the parameter
- How to graph curves that are represented by sets of parametric equations
- How to rewrite sets of parametric equations as single rectangular equations by eliminating the parameter
- How to find sets of parametric equations for graphs

Why You Should Learn It:

Parametric equations are useful for modeling the path of an object. For instance, in Exercise 59 on page 738, a set of parametric equations is used to model the path of a baseball.

Jonathan Daniel/Allsport

A computer animation of this concept appears in th *Interactive* CD-ROM and *Internet* versions of this text.

Sketching a Plane Curve

One way to sketch a curve represented by a pair of parametric equations is to plot points in the xy-plane. Each set of coordinates (x, y) is determined from a value chosen for the parameter t. By plotting the resulting points in the order of *increasing* values of t, you trace the curve in a specific direction. This is called the **orientation** of the curve.

Library of Functions

Parametric equations consist of a pair of functions $x = f(t)$ and $y = g(t)$, each of which is a function of the parameter t. These equations define a plane curve, which might not be the graph of a function, as in Example 1. Most graphing utilities have a *parametric* mode.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 1 Sketching a Plane Curve

Sketch the curve given by the parametric equations

$$x = t^2 - 4 \qquad \text{and} \qquad y = \frac{t}{2}, \qquad -2 \le t \le 3.$$

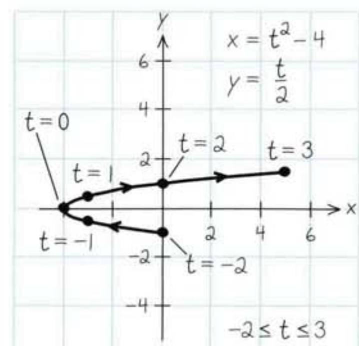
Describe the orientation of the curve.

Solution

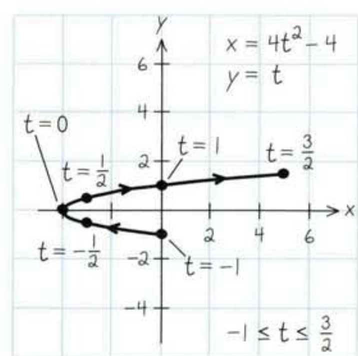
Using values of t in the given interval, the parametric equations yield the points (x, y) shown in the table.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

By plotting these points in the order of increasing t, you obtain the curve shown in Figure 10.44(a). The arrows on the curve indicate its orientation as t increases from -2 to 3. So, if a particle were moving on this curve, it would start at $(0, -1)$ and then move along the curve to the point $\left(5, \frac{3}{2}\right)$.



(a)



(b)

Figure 10.44

The graph shown in Figure 10.44(a) does not define y as a function of x. This points out one benefit of parametric equations—they can be used to represent graphs that are more general than graphs of functions.

Two different sets of parametric equations can have the same graph. For example, the set of parametric equations

$$x = 4t^2 - 4 \qquad \text{and} \qquad y = t, \qquad -1 \le t \le \frac{3}{2}$$

has the same graph as the set given in Example 1. [See Figure 10.44(b).] However, by comparing the values of t in Figures 10.44(a) and (b), you can see that this second graph is traced out more *rapidly* (considering t as time) than the first graph. So, in applications, different parametric representations can be used to represent various *speeds* at which objects travel along a given path.

Another way to display a curve represented by a pair of parametric equations is to use a graphing utility, as shown in Example 2.

EXAMPLE 2 Using a Graphing Utility in Parametric Mode

Use a graphing utility to graph the curves represented by the parametric equations. For which curve is y a function of x? (Use $-4 \le t \le 4$.)

a. $x = t^2$, $y = t^3$ **b.** $x = t$, $y = t^3$ **c.** $x = t^2$, $y = t$

Solution

Begin by setting the graphing utility to *parametric* mode. When choosing a viewing window, you must set not only minimum and maximum values of x and y but also minimum and maximum values of t.

a. Enter the parametric equations for x and y.

$$X_{1T} = T^2, \qquad Y_{1T} = T^3$$

The curve is shown in Figure 10.45(a). From the graph, you can see that y *is not* a function of x.

b. Enter the parametric equations for x and y.

$$X_{1T} = T, \qquad Y_{1T} = T^3$$

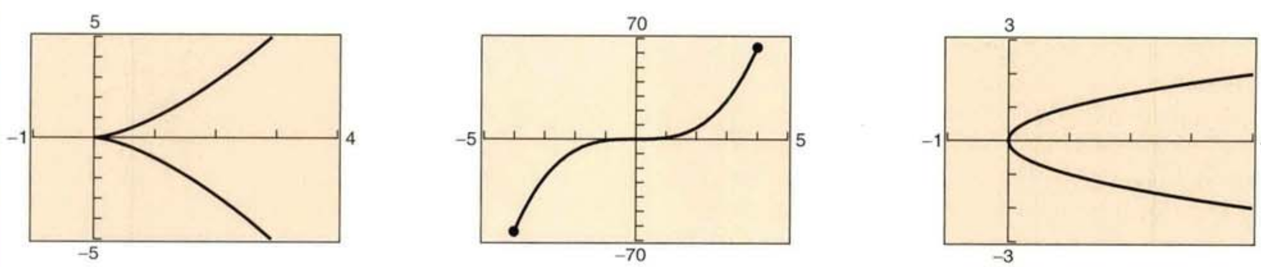
The curve is shown in Figure 10.45(b). From the graph, you can see that y *is* a function of x.

c. Enter the parametric equations for x and y.

$$X_{1T} = T^2, \qquad Y_{1T} = T$$

The curve is shown in Figure 10.45(c). From the graph, you can see that y *is not* a function of x.

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.



(a) (b) (c)

Figure 10.45

Exploration

Use a graphing utility set in *parametric* mode to graph the curve

$X_{1T} = T$ and $Y_{1T} = 1 - T^2$.

Set the viewing window so that $-4 \le x \le 4$ and $-12 \le y \le 2$. Now, graph the curve with various settings for t. Use the following.

a. $0 \le t \le 3$ **b.** $-3 \le t \le 0$ **c.** $-3 \le t \le 3$

Compare the curves given by the different t settings. Repeat this experiment using $X_{1T} = -T$. How does this change the results?

Eliminating the Parameter

Many curves that are represented by sets of parametric equations have graphs that can also be represented by rectangular equations (in x and y). The process of finding the rectangular equation is called **eliminating the parameter.**

Parametric equations	→	Solve for t in one equation.	→	Substitute in second equation.	→	Rectangular equation
$x = t^2 - 4$ $y = \frac{1}{2}t$		$t = 2y$		$x = (2y)^2 - 4$		$x = 4y^2 - 4$

After eliminating the parameter, you can recognize that the curve is a parabola with a horizontal axis and vertex at $(-4, 0)$.

Converting equations from parametric to rectangular form can change the ranges of x and y. In such cases, you should restrict x and y in the rectangular equation so that its graph matches the graph of the parametric equations.

STUDY TIP

It is important to realize that eliminating the parameter is primarily an aid to identifying the curve. If the parametric equations represent the path of a moving object, the graph alone is not sufficient to describe the object's motion. You still need the parametric equations to determine the *position*, *direction*, and *speed* at a given time.

EXAMPLE 3 Eliminating the Parameter

Identify the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad y = \frac{t}{t+1}.$$

Solution

Solving for t in the equation for x produces

$$x^2 = \frac{1}{t+1} \quad \text{or} \quad \frac{1}{x^2} = t + 1$$

which implies that $t = (1/x^2) - 1$. Substituting in the equation for y, you obtain

$$\begin{aligned} y &= \frac{t}{t+1} \\ &= \frac{\left(\frac{1}{x^2}\right) - 1}{\left(\frac{1}{x^2}\right) - 1 + 1} \\ &= \frac{\frac{1 - x^2}{x^2}}{\left(\frac{1}{x^2}\right)} \cdot \frac{x^2}{x^2} \\ &= 1 - x^2. \end{aligned}$$

From the rectangular equation, you can recognize the curve to be a parabola that opens downward and has its vertex at $(0, 1)$, as shown in Figure 10.46(a). The rectangular equation is defined for all values of x. The parametric equation for x, however, is defined only when $t > -1$. From the graph of the parametric equation, you can see that x is always positive, as shown in Figure 10.46(b). So, you should restrict the domain of x to positive values, as shown in Figure 10.46(c).

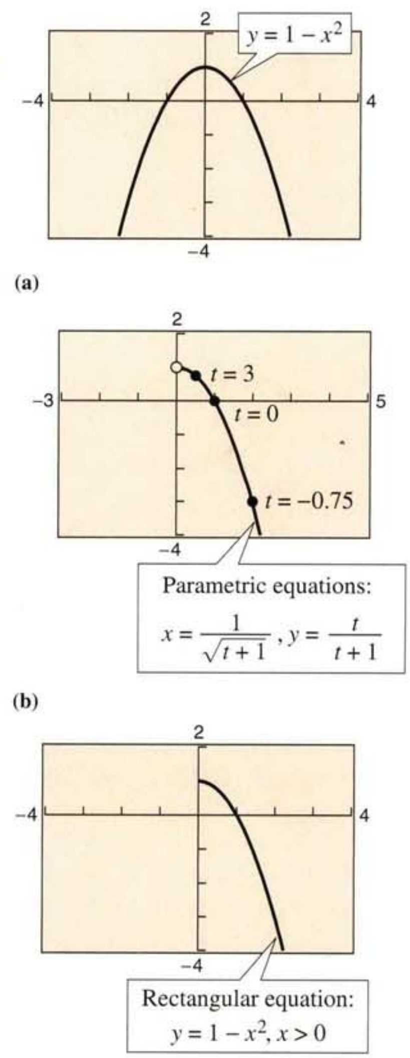


Figure 10.46

It is not necessary for the parameter in a set of parametric equations to represent time. The next example uses an *angle* as the parameter.

EXAMPLE 4 Eliminating the Parameter

Sketch the curve represented by $x = 3\cos\theta$ and $y = 4\sin\theta$, $0 \le \theta \le 2\pi$, by eliminating the parameter.

Solution

Begin by solving for $\cos\theta$ and $\sin\theta$ in the given equations.

$$\cos\theta = \frac{x}{3} \quad \text{and} \quad \sin\theta = \frac{y}{4} \qquad \text{Solve for } \cos\theta \text{ and } \sin\theta.$$

Make use of the identity $\sin^2\theta + \cos^2\theta = 1$ to form an equation involving only x and y.

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{Pythagorean identity}$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \qquad \text{Substitute.}$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 \qquad \text{Rectangular equation}$$

From this rectangular equation, you can see that the graph is an ellipse centered at $(0, 0)$, with vertices at $(0, 4)$ and $(0, -4)$, and minor axis of length $2b = 6$, as shown in Figure 10.47. Note that the elliptic curve is traced out *counterclockwise* as θ varies from 0 to 2π.

Exploration

In Example 4 you made use of the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ to sketch an ellipse. Which trigonometric identity would you use to obtain the graph of a hyperbola? Sketch the curve represented by $x = 3\sec\theta$ and $y = 4\tan\theta$, $0 \le \theta \le 2\pi$, by eliminating the parameter.

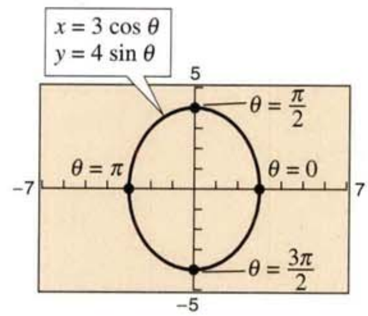


Figure 10.47

Finding Parametric Equations for a Graph

How can you determine a set of parametric equations for a given graph or a given physical description? From the discussion following Example 1, you know that such a representation is not unique. This is further demonstrated in Example 5.

EXAMPLE 5 Finding Parametric Equations for a Given Graph

Find a set of parametric equations to represent the graph of $y = 1 - x^2$ using the following parameters. **a.** $t = x$ **b.** $t = 1 - x$

Solution

a. Letting $t = x$, you obtain the parametric equations $x = t$ and $y = 1 - t^2$. The graph of these equations is shown in Figure 10.48(a).

b. Letting $t = 1 - x$, you obtain the following parametric equations.

$$x = 1 - t \qquad \text{Parametric equation for } x$$

$$y = 1 - (1 - t)^2 \qquad \text{Substitute } 1 - t \text{ for } x.$$

$$= 2t - t^2 \qquad \text{Parametric equation for } y$$

The graph of these equations is shown in Figure 10.48(b). In this figure, note how the resulting curve is oriented by the increasing values of t. In Figure 10.48(a), the curve has the opposite orientation.

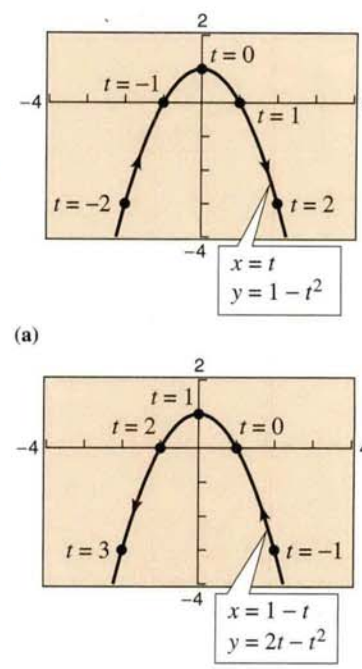
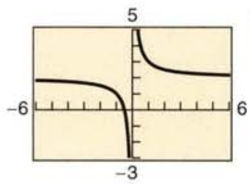


Figure 10.48

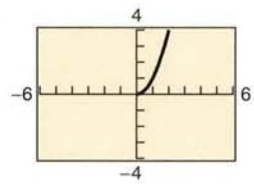
10.5 Exercises

In Exercises 1–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

(a)



(b)



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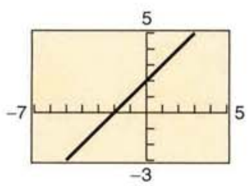

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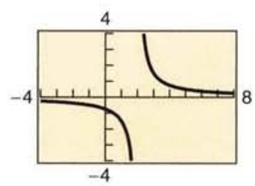
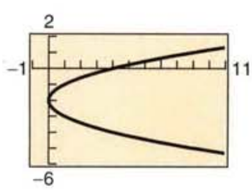
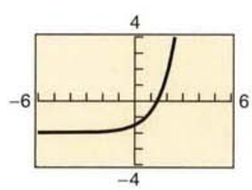


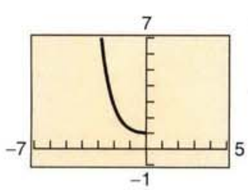
(e)



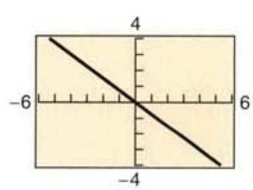
(f)



(g)



(h)



1. $x = t$
$y = t + 2$

2. $x = t$
$y = -\frac{3}{4}t$

3. $x = \sqrt{t}$
$y = t$

4. $x = t^2$
$y = t - 2$

5. $x = \dfrac{1}{t}$
$y = t + 2$

6. $x = \dfrac{1}{2}t$
$y = \dfrac{3}{t - 4}$

7. $x = \ln t$
$y = \frac{1}{2}t - 2$

8. $x = -2\sqrt{t}$
$y = e^t$

9. Consider the parametric equations

$x = \sqrt{t}$ and $y = 2 - t$.

(a) Complete the table.

t	0	1	2	3	4
x					
y					

(b) Plot the points (x, y) generated in part (a) and sketch a graph of the parametric equations.

(c) Use a graphing utility to graph the curve represented by the parametric equations.

(d) Find the rectangular equation by eliminating the parameter. Sketch its graph. How do the graphs differ from those in parts (b) and (c)?

10. Consider the parametric equations

$x = 4\cos^2\theta$ and $y = 2\sin\theta$.

(a) Complete the table.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
x					
y					

(b) Plot the points (x, y) generated in part (a) and sketch a graph of the parametric equations.

(c) Use a graphing utility to graph the curve represented by the parametric equations.

(d) Find the rectangular equation by eliminating the parameter. Sketch its graph. How do the graphs differ from those in parts (b) and (c)?

In Exercises 11–26, sketch the curve represented by the parametric equations (indicate the direction of the curve). Use a graphing utility to confirm your result. Then eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve.

11. $x = t$
$y = -4t$

12. $x = t$
$y = \frac{1}{2}t$

13. $x = 3t + 1$, $y = 2t - 1$

14. $x = 3 - 2t$, $y = 2 + 3t$

15. $x = \frac{1}{4}t$, $y = t^2$

16. $x = t$, $y = t^3$

17. $x = t + 5$, $y = t^2$

18. $x = \sqrt{t}$, $y = 1 - t$

19. $x = 2t$, $y = |t - 2|$

20. $x = |t - 1|$, $y = t + 2$

21. $x = 3\cos\theta$, $y = 3\sin\theta$

22. $x = \cos\theta$, $y = 3\sin\theta$

23. $x = e^{-t}$, $y = e^{3t}$

24. $x = e^{2t}$, $y = e^{t}$

25. $x = t^3$, $y = 3\ln t$

26. $x = \ln 2t$, $y = 2t^2$

In Exercises 27–34, use a graphing utility to graph the curve represented by the parametric equations.

27. $x = 4\sin 2\theta$, $y = 2\cos 2\theta$

28. $x = \cos\theta$, $y = 2\sin 2\theta$

29. $x = 4 + 2\cos\theta$, $y = -1 + \sin\theta$

30. $x = 4 + 2\cos\theta$, $y = -1 + 2\sin\theta$

31. $x = 4\sec\theta$, $y = 3\tan\theta$

32. $x = \sec\theta$, $y = \tan\theta$

33. $x = t/2$, $y = \ln(t^2 + 1)$

34. $x = 10 - 0.01e^t$, $y = 0.4t^2$

In Exercises 35 and 36, determine how the plane curves differ from each other.

35. (a) $x = t$, $y = 2t + 1$ (b) $x = \cos\theta$, $y = 2\cos\theta + 1$
(c) $x = e^{-t}$, $y = 2e^{-t} + 1$ (d) $x = e^t$, $y = 2e^t + 1$

36. (a) $x = t$, $y = t$ (b) $x = t^2$, $y = t^2$
(c) $x = -t$, $y = -t$ (d) $x = t^3$, $y = t^3$

In Exercises 37–40, eliminate the parameter and obtain the standard form of the rectangular equation.

37. Line through (x_1, y_1) and (x_2, y_2):
$x = x_1 + t(x_2 - x_1)$
$y = y_1 + t(y_2 - y_1)$

38. Circle: $x = h + r\cos\theta$
$y = k + r\sin\theta$

39. Ellipse: $x = h + a\cos\theta$
$y = k + b\sin\theta$

40. Hyperbola: $x = h + a\sec\theta$
$y = k + b\tan\theta$

In Exercises 41–46, use the results of Exercises 37–40 to find a set of parametric equations for the line or conic.

41. Line: Passes through $(0, 0)$ and $(5, -2)$
42. Line: Passes through $(1, 4)$ and $(5, -2)$
43. Circle: Center: $(2, 1)$; Radius: 4
44. Circle: Center: $(-3, 1)$; Radius: 3
45. Ellipse: Vertices: $(\pm 5, 0)$; Foci: $(\pm 4, 0)$
46. Hyperbola: Vertices: $(0, \pm 1)$; Foci: $(0, \pm 2)$

In Exercises 47 and 48, find two different sets of parametric equations for the given rectangular equation.

47. $y = 3x - 2$

48. $y = x^2$

In Exercises 49–54, use a graphing utility to obtain a graph of the curve represented by the parametric equations.

49. Cycloid: $x = 2(\theta - \sin\theta)$
$y = 2(1 - \cos\theta)$

50. Prolate cycloid: $x = 2\theta - 4\sin\theta$
$y = 2 - 4\cos\theta$

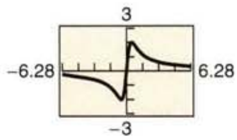
51. Hypocycloid: $x = 3\cos^3\theta$
$y = 3\sin^3\theta$

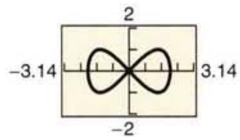
52. Curtate cycloid: $x = 2\theta - \sin\theta$
$y = 2 - \cos\theta$

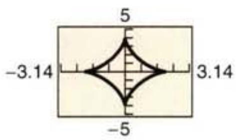
53. Witch of Agnesi: $x = 2\cot\theta$
$y = 2\sin^2\theta$

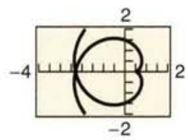
54. Folium of Descartes: $x = \dfrac{3t}{1 + t^3}$
$y = \dfrac{3t^2}{1 + t^3}$

In Exercises 55–58, match the parametric equations with the correct graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)


(b)


(c)


(d)


55. Lissajous curve: $x = 2\cos\theta,\ y = \sin 2\theta$

56. Evolute of ellipse: $x = 2\cos^3\theta,\ y = 4\sin^3\theta$

57. Involute of circle: $x = \frac{1}{2}(\cos\theta + \theta\sin\theta)$

$y = \frac{1}{2}(\sin\theta - \theta\cos\theta)$

58. Serpentine curve: $x = \frac{1}{2}\cot\theta,\ y = 4\sin\theta\cos\theta$

Projectile Motion **In Exercises 59 and 60, consider a projectile launched at a height h feet above the ground at an angle θ with the horizontal. If the initial velocity is v_0 feet per second, the path of the projectile is modeled by the parametric equations**

$$x = (v_0\cos\theta)t \quad \text{and} \quad y = h + (v_0\sin\theta)t - 16t^2.$$

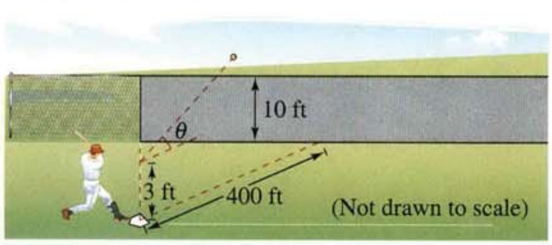
59. ***Baseball*** The center-field fence in a ballpark is 10 feet high and 400 feet from home plate. The baseball is hit 3 feet above the ground. It leaves the bat at an angle of θ degrees with the horizontal at a speed of 100 miles per hour.

(a) Write a set of parametric equations for the path of the baseball.

(b) Use a graphing utility to sketch the path of the baseball for $\theta = 15°$. Is the hit a home run?

(c) Use a graphing utility to sketch the path of the baseball for $\theta = 23°$. Is the hit a home run?

(d) Find the minimum angle required for the hit to be a home run.



60. ***Football*** The quarterback of a football team releases a pass at a height of 7 feet above the playing field, and the football is caught by a receiver at a height of 4 feet, 30 yards directly downfield. The pass is released at an angle of 35° with the horizontal.

(a) Write a set of parametric equations for the path of the football.

(b) Find the speed of the football when it is released.

(c) Use a graphing utility to graph the path of the football and approximate its maximum height.

(d) Find the time the receiver has to position himself after the quarterback releases the football.

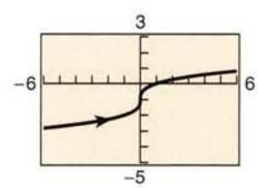
Synthesis

True or False? **In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.**

61. The two sets of parametric equations $x = t$, $y = t^2 + 1$ and $x = 3t$, $y = 9t^2 + 1$ correspond to the same rectangular equation.

62. The graph of the parametric equations $x = t^2$ and $y = t^2$ is the line $y = x$.

63. ***Think About It*** The graph of the parametric equations $x = t^3$ and $y = t - 1$ is shown below. Would the graph change for the equations $x = (-t^3)$ and $y = -t - 1$? If so, how would it change?



Review

In Exercises 64–67, find all solutions of the equation.

64. $5x^2 + 8 = 0$

65. $x^2 - 6x + 4 = 0$

66. $4x^2 + 4x - 11 = 0$

67. $x^4 - 18x^2 + 18 = 0$

In Exercises 68–73, find the sum. Use a graphing utility to verify your result.

68. $\displaystyle\sum_{n=1}^{50} 8n$

69. $\displaystyle\sum_{n=1}^{200} (n - 8)$

70. $\displaystyle\sum_{n=1}^{40} \left(300 - \frac{1}{2}n\right)$

71. $\displaystyle\sum_{n=1}^{70} \frac{7 - 5n}{12}$

72. $\displaystyle\sum_{n=0}^{18} 8\left(\frac{1}{2}\right)^n$

73. $\displaystyle\sum_{n=0}^{10} 10\left(\frac{2}{3}\right)^n$

10.6 Polar Coordinates

Introduction

So far, you have been representing graphs of equations as collections of points (x, y) on the rectangular coordinate system, where x and y represent the directed distances from the coordinate axes to the point (x, y). In this section, you will study a second system called the **polar coordinate system.**

To form the polar coordinate system in the plane, fix a point O, called the **pole** (or **origin**), and construct from O an initial ray called the **polar axis,** as shown in Figure 10.49. Then each point P in the plane can be assigned **polar coordinates** (r, θ) as follows.

1. r = *directed distance* from O to P
2. θ = *directed angle*, counterclockwise from polar axis to segment $\overline{OP}$

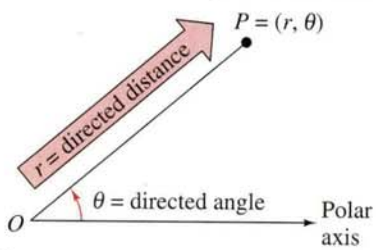


Figure 10.49

What You Should Learn:

- How to plot points and find multiple representations of points in the polar coordinate system
- How to convert points from rectangular to polar form and vice versa
- How to convert equations from rectangular to polar form and vice versa

Why You Should Learn It:

Polar coordinates offer a different mathematical perspective on graphing. For instance, in Exercises 5–12 on page 743, you see that a polar coordinate can be written in more than one way.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 1 Plotting Points in the Polar Coordinate System

a. The point $(r, \theta) = (2, \pi/3)$ lies two units from the pole on the terminal side of the angle $\theta = \pi/3$, as shown in Figure 10.50(a).

b. The point $(r, \theta) = (3, -\pi/6)$ lies three units from the pole on the terminal side of the angle $\theta = -\pi/6$, as shown in Figure 10.50(b).

c. The point $(r, \theta) = (3, 11\pi/6)$ coincides with the point $(3, -\pi/6)$, as shown in Figure 10.50(c).

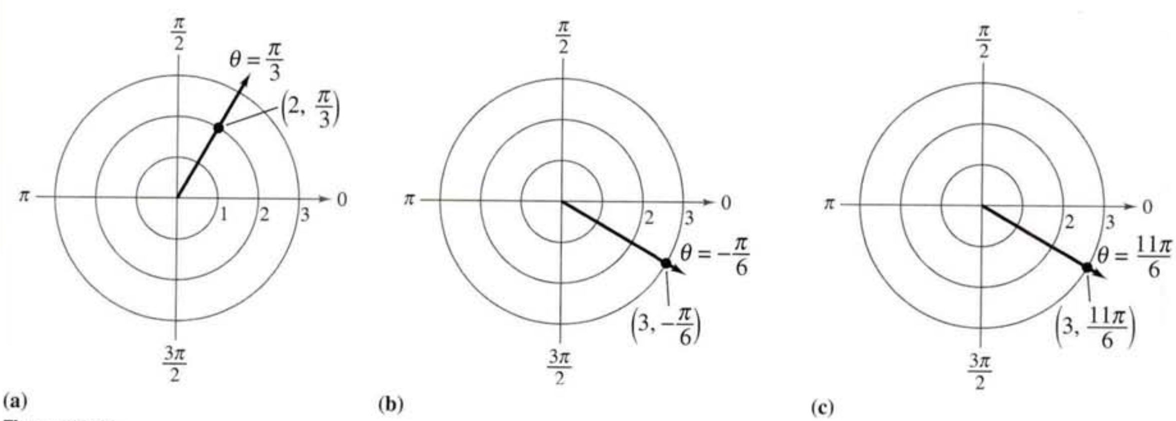


Figure 10.50

In rectangular coordinates, each point (x, y) has a unique representation. This is not true for polar coordinates. For instance, the coordinates (r, θ) and $(r, 2\pi + \theta)$ represent the same point, as illustrated in Example 1. Another way to obtain multiple representations of a point is to use negative values for r. Because r is a *directed distance*, the coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point. In general, the point (r, θ) can be represented as

$$(r, \theta) = (r, \theta \pm 2n\pi) \quad \text{or} \quad (r, \theta) = (-r, \theta \pm (2n + 1)\pi)$$

where n is any integer. Moreover, the pole is represented by $(0, \theta)$, where θ is any angle.

EXAMPLE 2 Multiple Representation of Points

Plot the point $(3, -3\pi/4)$ and find three additional polar representations of this point, using $-2\pi < \theta < 2\pi$.

Solution

The point is shown in Figure 10.51. Three other representations are as follows.

$$\left(3, -\frac{3\pi}{4} + 2\pi\right) = \left(3, \frac{5\pi}{4}\right)$$ Add 2π to θ.

$$\left(-3, -\frac{3\pi}{4} - \pi\right) = \left(-3, -\frac{7\pi}{4}\right)$$ Replace r by $-r$; subtract π from θ.

$$\left(-3, -\frac{3\pi}{4} + \pi\right) = \left(-3, \frac{\pi}{4}\right)$$ Replace r by $-r$; add π to θ.

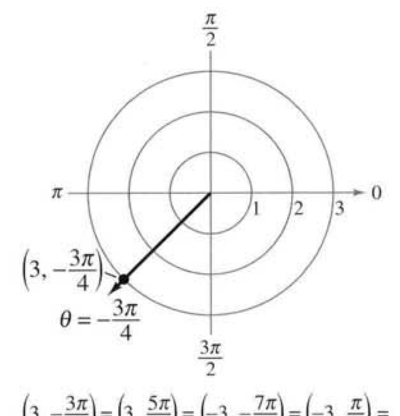


Figure 10.51

Coordinate Conversion

To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin, as shown in Figure 10.52. Because (x, y) lies on a circle of radius r, it follows that $r^2 = x^2 + y^2$. Moreover, for $r > 0$, the definitions of the trigonometric functions imply that

$$\tan \theta = \frac{y}{x}, \qquad \cos \theta = \frac{x}{r}, \qquad \text{and} \qquad \sin \theta = \frac{y}{r}.$$

You can show that the same relationships hold for $r < 0$.

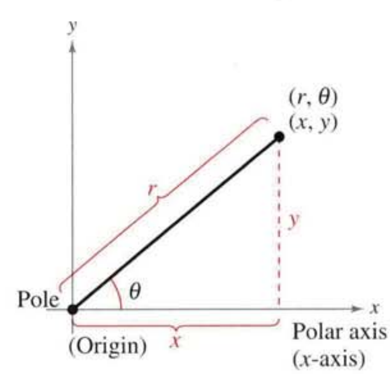


Figure 10.52

Coordinate Conversion

The polar coordinates (r, θ) are related to the rectangular coordinates (x, y) as follows.

$$x = r\cos\theta \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

$$y = r\sin\theta \qquad\qquad r^2 = x^2 + y^2$$

EXAMPLE 3 Polar-to-Rectangular Conversion

Convert each point to rectangular coordinates. (See Figure 10.53.)

a. $(2, \pi)$ **b.** $\left(\sqrt{3}, \frac{\pi}{6}\right)$

Solution

a. For the point $(r, \theta) = (2, \pi)$, you have

$$x = r\cos\theta = 2\cos\pi = -2$$

and

$$y = r\sin\theta = 2\sin\pi = 0.$$

The rectangular coordinates are $(x, y) = (-2, 0)$.

b. For the point $(r, \theta) = \left(\sqrt{3}, \pi/6\right)$, you have

$$x = \sqrt{3}\cos\frac{\pi}{6} = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2}$$

and

$$y = \sqrt{3}\sin\frac{\pi}{6} = \sqrt{3}\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}.$$

The rectangular coordinates are $(x, y) = \left(3/2, \sqrt{3}/2\right)$.

Exploration

Set your graphing utility to *polar* mode. Then graph the equation $r = 3$. (Use a viewing window of $0 \le \theta \le 2\pi$, $-6 \le x \le 6$, and $-4 \le y \le 4$.) You should obtain a circle of radius 3.

a. Use the *trace* feature to cursor around the circle. Can you locate the point $(3, 5\pi/4)$?

b. Can you locate other names for the point $(3, 5\pi/4)$? If so, explain how you did it.

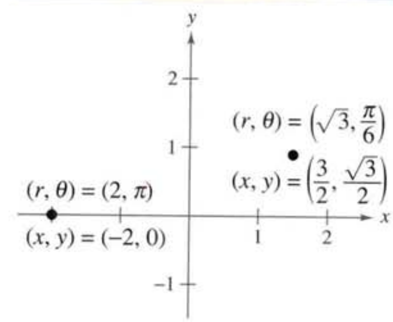


Figure 10.53

EXAMPLE 4 Rectangular-to-Polar Conversion

Convert each point to polar coordinates.

a. $(-1, 1)$ **b.** $(0, 2)$

Solution

a. For the second-quadrant point $(x, y) = (-1, 1)$, you have

$$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$$

$$\theta = \frac{3\pi}{4}.$$

Because θ lies in the same quadrant as (x, y), use positive r.

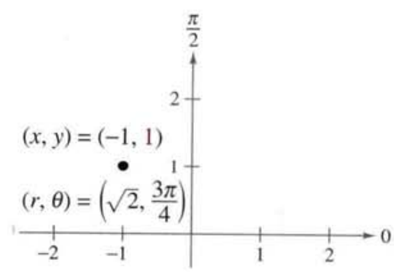
$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

So, *one* set of polar coordinates is $(r, \theta) = \left(\sqrt{2}, 3\pi/4\right)$, as shown in Figure 10.54(a).

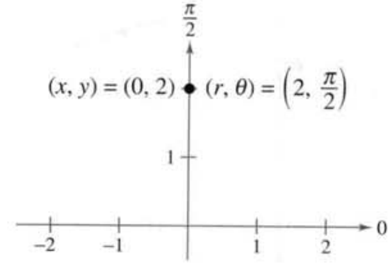
b. Because the point $(x, y) = (0, 2)$ lies on the positive y-axis, choose

$$\theta = \pi/2 \quad \text{and} \quad r = 2.$$

This implies that one set of polar coordinates is $(r, \theta) = (2, \pi/2)$, as shown in Figure 10.54(b).



(a)



(b)

Figure 10.54

Equation Conversion

By comparing Examples 3 and 4, you see that point conversion from the polar to the rectangular system is straightforward, whereas point conversion from the rectangular to the polar system is more involved. For equations, the opposite is true. To convert a rectangular equation to polar form, you simply replace x by $r\cos\theta$ and y by $r\sin\theta$. For instance, the rectangular equation $y = x^2$ can be written in polar form as follows.

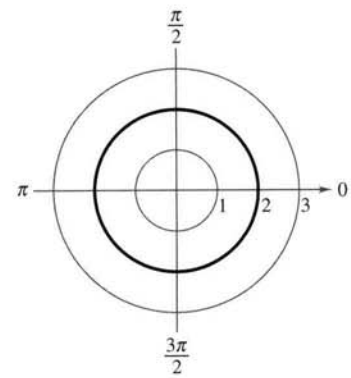
$$y = x^2 \qquad \text{Rectangular equation}$$

$$r\sin\theta = (r\cos\theta)^2 \qquad \text{Polar equation}$$

$$r = \sec\theta\tan\theta \qquad \text{Simplest form}$$

On the other hand, converting a polar equation to rectangular form requires considerable ingenuity.

Example 5 demonstrates several polar-to-rectangular conversions that enable you to sketch the graphs of some polar equations.



(a)

EXAMPLE 5 Converting Polar Equations to Rectangular Form

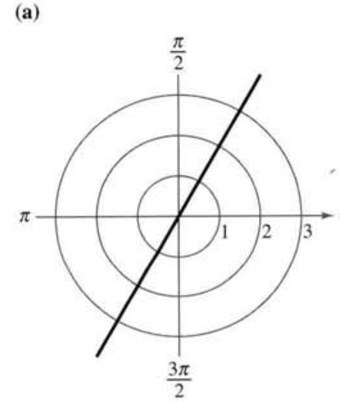
Describe the graph of each polar equation and find the corresponding rectangular equation.

a. $r = 2$ **b.** $\theta = \dfrac{\pi}{3}$ **c.** $r = \sec\theta$

Solution

a. The graph of the polar equation $r = 2$ consists of all points that are two units from the pole. In other words, this graph is a circle centered at the origin with a radius of 2, as shown in Figure 10.55(a). You can confirm this by converting to rectangular form, using the relationship $r^2 = x^2 + y^2$.

$$\underbrace{r = 2}_{\text{Polar equation}} \quad \Rightarrow \quad r^2 = 2^2 \quad \Rightarrow \quad \underbrace{x^2 + y^2 = 2^2}_{\text{Rectangular equation}}$$



(b)

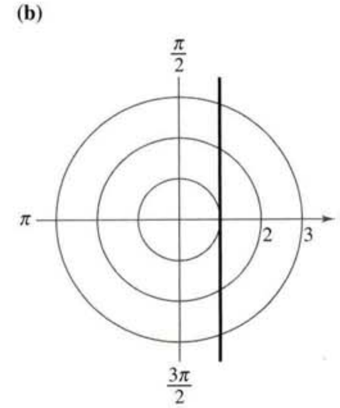
b. The graph of the polar equation $\theta = \pi/3$ consists of all points on the line that make an angle of $\pi/3$ with the positive x-axis, as shown in Figure 10.55(b). To convert to rectangular form, you make use of the relationship $\tan\theta = y/x$.

$$\underbrace{\theta = \frac{\pi}{3}}_{\text{Polar equation}} \quad \Rightarrow \quad \tan\theta = \sqrt{3} \quad \Rightarrow \quad \underbrace{y = \sqrt{3}x}_{\text{Rectangular equation}}$$

c. The graph of the polar equation $r = \sec\theta$ is not evident by simple inspection, so you convert to rectangular form by using the relationship $r\cos\theta = x$.

$$\underbrace{r = \sec\theta}_{\text{Polar equation}} \quad \Rightarrow \quad r\cos\theta = 1 \quad \Rightarrow \quad \underbrace{x = 1}_{\text{Rectangular equation}}$$

Now you see that the graph is a vertical line, as shown in Figure 10.55(c).



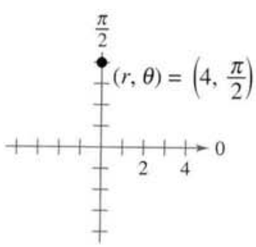
(c)

Figure 10.55

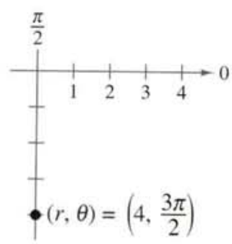
10.6 Exercises

In Exercises 1–4, a point in polar coordinates is given. Find the corresponding rectangular coordinates for the point.

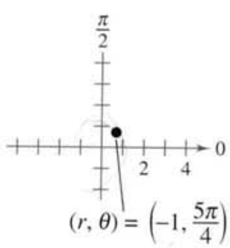
1. $\left(4, \frac{\pi}{2}\right)$



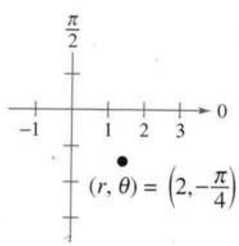
2. $\left(4, \frac{3\pi}{2}\right)$



3. $\left(-1, \frac{5\pi}{4}\right)$



4. $\left(2, -\frac{\pi}{4}\right)$



In Exercises 5–12, plot the point given in polar coordinates and find three additional polar representations of the point, using $-2\pi < \theta < 2\pi$.

5. $\left(4, \frac{2\pi}{3}\right)$
6. $\left(1, \frac{7\pi}{4}\right)$
7. $\left(5, -\frac{5\pi}{3}\right)$
8. $\left(-3, -\frac{7\pi}{6}\right)$
9. $\left(\sqrt{3}, \frac{5\pi}{6}\right)$
10. $\left(5\sqrt{2}, -\frac{11\pi}{6}\right)$
11. $\left(\frac{3}{2}, -\frac{3\pi}{2}\right)$
12. $\left(-\frac{7}{8}, -\frac{\pi}{6}\right)$

In Exercises 13–22, plot the point given in polar coordinates and find the corresponding rectangular coordinates for the point.

13. $\left(4, -\frac{\pi}{3}\right)$
14. $\left(2, \frac{7\pi}{6}\right)$
15. $\left(-1, -\frac{3\pi}{4}\right)$
16. $\left(-3, -\frac{2\pi}{3}\right)$
17. $\left(0, -\frac{7\pi}{6}\right)$
18. $\left(0, \frac{5\pi}{4}\right)$
19. $\left(32, \frac{5\pi}{2}\right)$
20. $\left(18, -\frac{3\pi}{2}\right)$
21. $\left(\sqrt{2}, 2.36\right)$
22. $(-3, -1.57)$

In Exercises 23–26, use a graphing utility to find the rectangular coordinates for the point given in polar coordinates.

23. $\left(2, \frac{3\pi}{4}\right)$
24. $\left(-2, \frac{7\pi}{6}\right)$
25. $(-4.5, 1.3)$
26. $(8.25, 3.5)$

In Exercises 27–36, the rectangular coordinates of a point are given. Plot the point and find *two* sets of polar coordinates for the point for $0 \le \theta < 2\pi$.

27. $(-7, 0)$
28. $(0, -5)$
29. $(1, 1)$
30. $(-3, -3)$
31. $(-3, 4)$
32. $(3, -1)$
33. $\left(-\sqrt{3}, -\sqrt{3}\right)$
34. $(2, -2)$
35. $(4, 6)$
36. $(5, 12)$

In Exercises 37–44, use a graphing utility to find one set of polar coordinates for the point given in rectangular coordinates.

37. $(3, -2)$
38. $(-4, 1)$
39. $\left(\sqrt{3}, 2\right)$
40. $\left(3\sqrt{2}, 3\sqrt{2}\right)$
41. $\left(\frac{5}{2}, \frac{4}{3}\right)$
42. $\left(\frac{11}{4}, -\frac{5}{8}\right)$
43. $(0, -5)$
44. $(-8, 0)$

In Exercises 45–54, convert the rectangular equation to polar form. Assume $a \ge 0$.

45. (a) $x^2 + y^2 = 49$ (b) $x^2 + y^2 = a^2$
46. (a) $x^2 + y^2 - 6x = 0$ (b) $x^2 + y^2 - 8y = 0$
47. (a) $x^2 + y^2 - 2ax = 0$ (b) $x^2 + y^2 - 2ay = 0$
48. (a) $y = 4$ (b) $y = b$
49. (a) $x = 12$ (b) $x = a$
50. (a) $3x - 6y + 2 = 0$ (b) $4x + 7y - 2 = 0$

51. (a) $xy = 4$ (b) $2xy = 1$

52. (a) $y = x$ (b) $y^2 = 2x$

53. (a) $y^2 = x^3$ (b) $x^2 = y^3$

54. (a) $(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$
(b) $y^2 - 8x - 16 = 0$

In Exercises 55–70, convert the polar equation to rectangular form.

55. $r = 4 \sin \theta$

56. $r = 4 \cos \theta$

57. $\theta = \dfrac{\pi}{6}$

58. $\theta = \dfrac{5\pi}{3}$

59. $r = 4$

60. $r = 10$

61. $r = -3 \csc \theta$

62. $r = 2 \sec \theta$

63. $r^2 = \cos \theta$

64. $r^2 = \sin 2\theta$

65. $r = 2 \sin 3\theta$

66. $r = 3 \cos 2\theta$

67. $r = \dfrac{1}{1 - \cos \theta}$

68. $r = \dfrac{2}{1 + \sin \theta}$

69. $r = \dfrac{6}{2 - 3 \sin \theta}$

70. $r = \dfrac{6}{2 \cos \theta - 3 \sin \theta}$

In Exercises 71–76, describe the graph of the polar equation and find the corresponding rectangular equation. Sketch its graph.

71. $r = 3$

72. $r = 8$

73. $\theta = \dfrac{\pi}{4}$

74. $\theta = \dfrac{5\pi}{6}$

75. $r = 3 \sec \theta$

76. $r = 2 \csc \theta$

Synthesis

True or False? **In Exercises 77 and 78, determine whether the statement is true or false. Justify your answer.**

77. If (r_1, θ_1) and (r_2, θ_2) represent the same point in the polar coordinate system, then $|r_1| = |r_2|$.

78. If (r, θ_1) and (r, θ_2) represent the same point in the polar coordinate system, then $\theta_1 = \theta_2 + 2\pi n$ for some integer n.

79. ***Think About It***

(a) Show that the distance between the points (r_1, θ_1) and (r_2, θ_2) is

$$\sqrt{r_1^{\,2} + r_2^{\,2} - 2r_1r_2 \cos(\theta_1 - \theta_2)}.$$

(b) Describe the position of the points relative to each other if $\theta_1 = \theta_2$. Simplify the distance formula for this case. Is the simplification what you expected? Explain.

(c) Simplify the distance formula if $\theta_1 - \theta_2 = 90°$. Is the simplification what you expected? Explain.

(d) Choose two points on the polar coordinate system and find the distance between them. Then choose different polar representations of the same two points and apply the distance formula again. Discuss the result.

80. ***Exploration***

(a) Set the viewing window of your graphing utility to rectangular coordinates and locate the cursor at any position off the coordinate axes. Move the cursor horizontally and observe any changes in the displayed coordinates of the points. Explain the changes. Now repeat the process moving the cursor vertically.

(b) Set the viewing window of your graphing utility to polar coordinates and locate the cursor at any position off the coordinate axes. Move the cursor horizontally and observe any changes in the displayed coordinates of the points. Explain the changes. Now repeat the process moving the cursor vertically.

(c) Explain why the results of parts (a) and (b) are not the same.

Review

In Exercises 81–84 use determinants to solve the system of equations.

81. $\begin{cases} 5x - 7y = -11 \\ -3x + y = -3 \end{cases}$

82. $\begin{cases} 3x + 5y = 10 \\ 4x - 2y = -5 \end{cases}$

83. $\begin{cases} 3a - 2b + c = 0 \\ 2a + b - 3c = 0 \\ a - 3b + 9c = 0 \end{cases}$

84. $\begin{cases} 5u + 7v + 9w = 15 \\ u - 2v - 3w = 7 \\ 8u - 2v + w = 0 \end{cases}$

In Exercises 85–88, find the coefficient *a* of the given term in the expansion of the binomial.

	Binomial	*Term*
85.	$(x + 5)^8$	ax^3
86.	$(x^2 - 3)^{10}$	ax^8
87.	$(2x - y)^{12}$	ax^7y^5
88.	$(3x - 2y)^7$	ax^3y^4

10.7 Graphs of Polar Equations

Introduction

In previous chapters you spent a lot of time learning how to sketch graphs in rectangular coordinates. You began with the basic point-plotting method. Then you used sketching aids such as a graphing utility, symmetry, intercepts, asymptotes, periods, and shifts to further investigate the nature of the graph. This section approaches curve sketching in the polar coordinate system similarly.

What You Should Learn:

- How to graph polar equations by point plotting
- How to use symmetry, zeros, and maximum r-values as graphing aids
- How to recognize special polar graphs

Why You Should Learn It:

Several common figures, such as the circle in Exercise 3 on page 752, are easier to graph in the polar coordinate system than in the rectangular coordinate system.

EXAMPLE 1 Graphing a Polar Equation by Point Plotting

Sketch the graph of the polar equation $r = 4 \sin \theta$.

Solution

The sine function is periodic, so you can get a full range of r-values by considering values of θ in the interval $0 \le \theta \le 2\pi$, as shown in the table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	0	2	$2\sqrt{3}$	4	$2\sqrt{3}$	2	0	-2	-4	-2	0

If you plot these points as shown in Figure 10.56, it appears that the graph is a circle of radius 2 whose center is at the point $(x, y) = (0, 2)$.

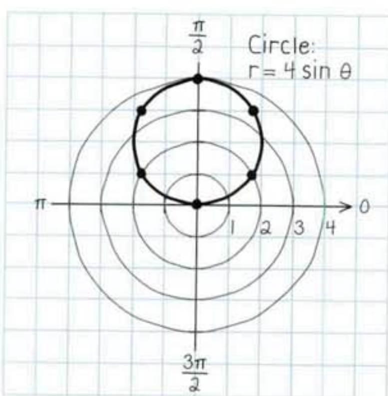


Figure 10.56

You can confirm the graph found in Example 1 in three ways.

1. *Convert to Rectangular Form* Multiply both sides of the polar equation by r and convert the result to rectangular form.
2. *Use a Polar Coordinate Mode* Set your graphing utility to *polar* mode and graph the polar equation. (Use $0 \le \theta \le 2\pi$, $-6 \le x \le 6$, and $-4 \le y \le 4$.)
3. *Use a Parametric Mode* Set your graphing utility to *parametric* mode and graph $x = (4 \sin t) \cos t$ and $y = (4 \sin t) \sin t$.

Most graphing utilities have a *polar-coordinate* graphing mode. If yours doesn't, you can use the following parametric conversion to graph a polar equation.

Polar Equations in Parametric Form

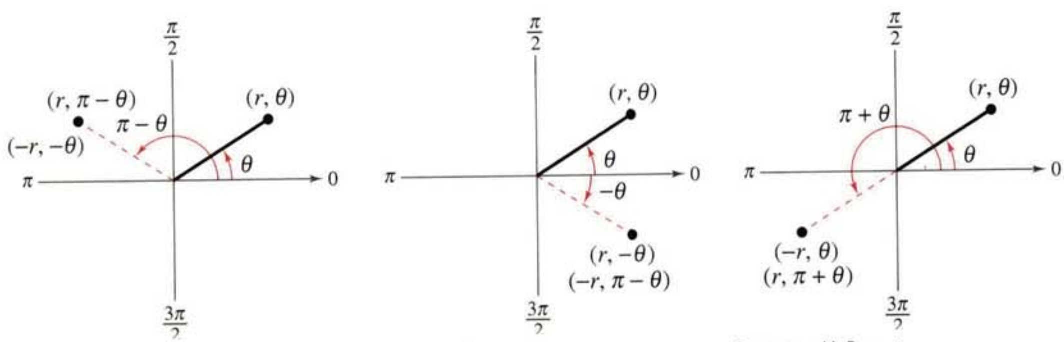
The graph of the polar equation $r = f(\theta)$ can be written in parametric form, using t as a parameter, as follows.

$$x = f(t)\cos t \qquad \text{and} \qquad y = f(t)\sin t.$$

Symmetry

In Figure 10.56, note that as θ increases from 0 to 2π the graph is traced out twice. Moreover, note that the graph is *symmetric with respect to the line* $\theta = \pi/2$. Had you known about this symmetry and retracing ahead of time, you could have used fewer points.

Symmetry with respect to the line $\theta = \pi/2$ is one of three important types of symmetry to consider in polar curve sketching. (See Figure 10.57.)



Symmetry with Respect to the Line $\theta = \pi/2$

Figure 10.57

Symmetry with Respect to the Polar Axis

Symmetry with Respect to the Pole

Tests for Symmetry on Polar Coordinates

The graph of a polar equation is symmetric with respect to the following if the given substitution yields an equivalent equation.

1. The line $\theta = \pi/2$: Replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$.
2. The polar axis: Replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$.
3. The pole: Replace (r, θ) by $(r, \pi + \theta)$ or $(-r, \theta)$.

EXAMPLE 2 Using Symmetry to Sketch a Polar Graph

Use symmetry to sketch the graph of $r = 3 + 2\cos\theta$.

Solution

Replacing (r, θ) by $(r, -\theta)$ produces

$$r = 3 + 2\cos(-\theta)$$
$$= 3 + 2\cos\theta.$$

So, you can conclude that the curve is symmetric with respect to the polar axis. Plotting the points in the table and using polar axis symmetry, you obtain the graph shown in Figure 10.58. This graph is called a **limaçon.**

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	5	$3 + \sqrt{3}$	4	3	2	$3 - \sqrt{3}$	1

Use a graphing utility to confirm this graph.

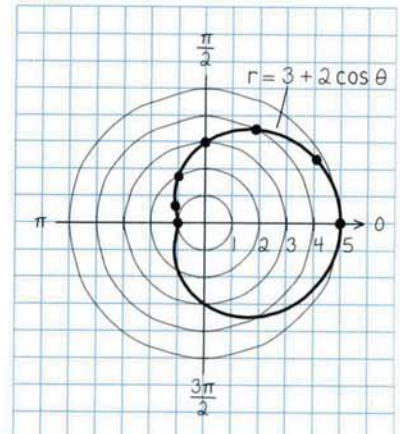


Figure 10.58

STUDY T!P

The *table* feature of a graphing utility is very useful in constructing tables of values for polar equations. Set your graphing utility to *polar* mode and enter the polar equation in Example 2. You can verify the table of values in Example 2 by starting the table at $\theta = 0$ and incrementing the value of θ by $\pi/6$.

The three tests for symmetry in polar coordinates on page 746 are sufficient to guarantee symmetry, but they are not necessary. For instance, Figure 10.59 shows the graph of

$$r = \theta + 2\pi.$$ Spiral of Archimedes

From the figure, you can see that the graph is symmetric with respect to the line $\theta = \pi/2$. Yet the tests on page 746 fail to indicate symmetry because neither of the following replacements yields an equivalent equation.

Original Equation	*Replacement*	*New Equation*
$r = \theta + 2\pi$	(r, θ) by $(-r, -\theta)$	$-r = -\theta + 2\pi$
$r = \theta + 2\pi$	(r, θ) by $(r, \pi - \theta)$	$r = -\theta + 3\pi$

The equations discussed in Examples 1 and 2 are of the form

$$r = 4\sin\theta = f(\sin\theta)$$

and

$$r = 3 + 2\cos\theta = g(\cos\theta).$$

The graph of the first equation is symmetric with respect to the line $\theta = \pi/2$, and the graph of the second equation is symmetric with respect to the polar axis. This observation can be generalized to yield the following *quick test for symmetry.*

1. The graph of $r = f(\sin\theta)$ is symmetric with respect to the line $\theta = \pi/2$.
2. The graph of $r = g(\cos\theta)$ is symmetric with respect to the polar axis.

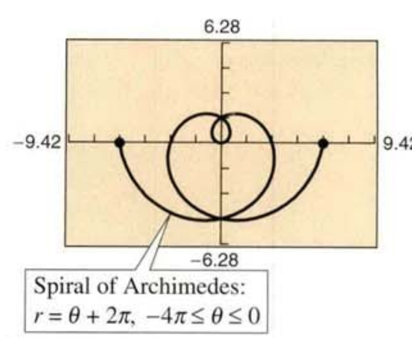


Figure 10.59

Zeros and Maximum r-Values

Two additional aids to sketching graphs of polar equations involve knowing the θ-values for which $|r|$ is maximum and knowing the θ-values for which $r = 0$. In Example 1, the maximum value of $|r|$ for $r = 4 \sin \theta$ is $|r| = 4$, and this occurs when $\theta = \pi/2$ (see Figure 10.56). Moreover, $r = 0$ when $\theta = 0$.

EXAMPLE 3 Finding Maximum r-Values of a Polar Graph

Find the maximum value of r for the graph of $r = 1 - 2 \cos \theta$.

Graphical Solution

Because the polar equation is of the form

$$r = 1 - 2 \cos \theta = g(\cos \theta)$$

you know the graph is symmetric with respect to the polar axis. You can confirm this by graphing the polar equation, as shown in Figure 10.60. (In the graph, θ varies from 0 to 2π.) To find the maximum r-value for the graph, use your graphing utility's *trace* feature. When you do this, you should find that the graph has a maximum r-value of 3. This value of r occurs when $\theta = \pi$. In the graph, note that the point $(3, \pi)$ is farthest from the pole.

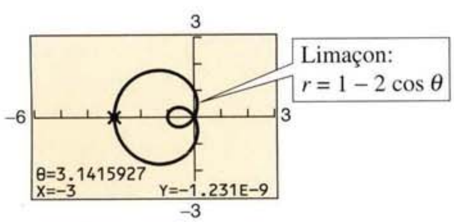


Figure 10.60

Numerical Solution

To approximate the maximum value of r for the graph of $r = 1 - 2 \cos \theta$, use the *table* feature of a graphing utility to create a table that begins at $\theta = 0$ and increments by $\pi/12$, as shown in Figure 10.61. From the table, the maximum value of r appears to be 3 when $\theta = 3.1416 \approx \pi$.

θ	r1
2.0944	2
2.3562	2.4142
2.618	2.7321
2.8798	2.9319
3.1416	3
3.4034	2.9319
3.6652	2.7321
θ=3.14159265359	

Figure 10.61

By creating a second table that begins at $\theta = \pi/2$ and increments by $\pi/24$, as shown in Figure 10.62, the maximum value of r still appears to be 3 when $\theta = 3.1416 \approx \pi$.

θ	r1
2.7489	2.8478
2.8798	2.9319
3.0107	2.9829
3.1416	3
3.2725	2.9829
3.4034	2.9319
3.5343	2.8478
θ=3.14159265359	

Figure 10.62

Note how the negative r-values determine the *inner loop* of the graph in Figure 10.60. This type of graph is a limaçon.

Exploration

The graph of the polar equation $r = e^{\cos\theta} - 2\cos 4\theta + \sin^5(\theta/12)$ is called *the butterfly curve*, as shown in Figure 10.63.

a. The graph at the right was produced using $0 \le \theta \le 2\pi$. Does this show the entire graph? Explain your reasoning.

b. Use the *trace* feature of your graphing calculator to approximate the maximum r-value of the graph. Does this value change if you use $0 \le \theta \le 4\pi$ instead of $0 \le \theta \le 2\pi$? Explain.

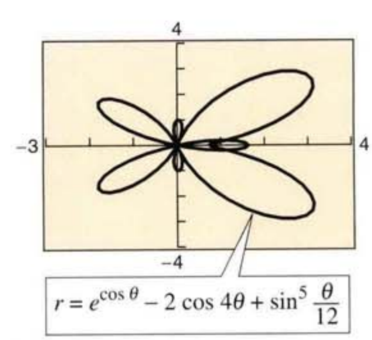


Figure 10.63

Some curves reach their zeros and maximum r-values at more than one point. Example 4 shows how to handle this situation.

EXAMPLE 4 Analyzing a Polar Graph

Analyze the graph of $r = 2 \cos 3\theta$.

Solution

Symmetry With respect to the polar axis

Maximum value of $|r|$ $|r| = 2$ when $3\theta = 0, \pi, 2\pi, 3\pi$

or $\theta = 0, \pi/3, 2\pi/3, \pi$

Zeros of r $r = 0$ when $3\theta = \pi/2, 3\pi/2, 5\pi/2$

or $\theta = \pi/6, \pi/2, 5\pi/6$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0

By plotting these points and using the specified symmetry, zeros, and maximum values, you can obtain the graph shown in Figure 10.64. This graph is called a **rose curve,** and each loop on the graph is called a *petal*. Note how the entire curve is generated as θ increases from 0 to π.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

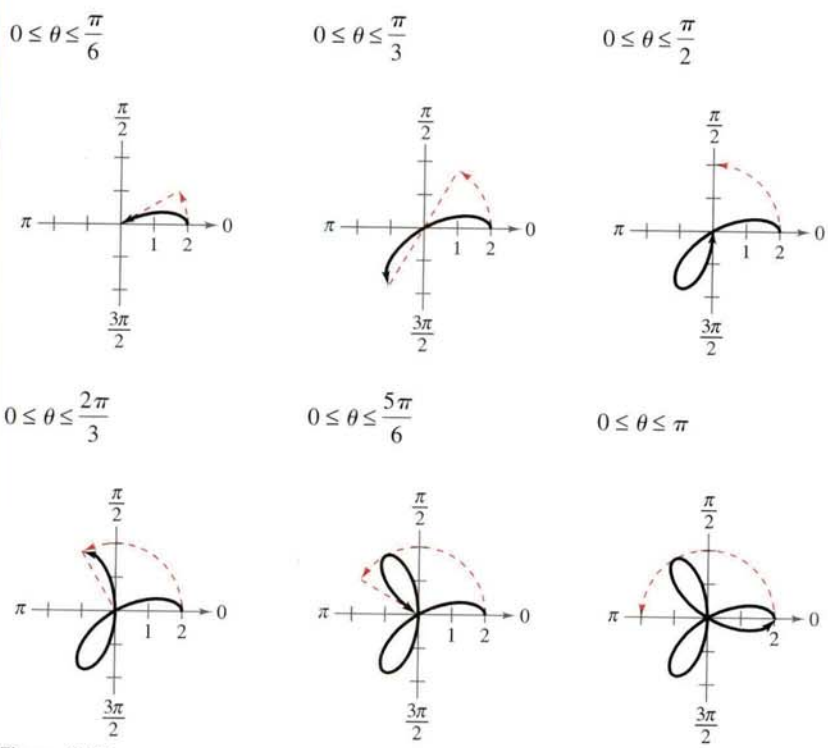


Figure 10.64

Exploration

Notice that the rose curve in Example 4 has three petals. How many petals does the rose curve $r = 2 \cos 4\theta$ have? Experiment with other rose curves and determine the number of petals for the curves $r = 2 \cos n\theta$ and $r = 2 \sin n\theta$, where n is a positive integer.

Special Polar Graphs

Several important types of graphs have equations that are simpler in polar form than in rectangular form. For example, the circle

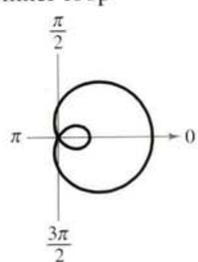
$$r = 4 \sin \theta$$

in Example 1 has the more complicated rectangular equation

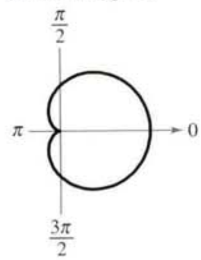
$$x^2 + (y - 2)^2 = 4.$$

The following list gives several other types of graphs that have simple polar equations.

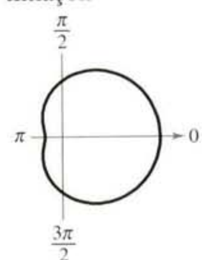
$\frac{a}{b} < 1$
Limaçon with inner loop



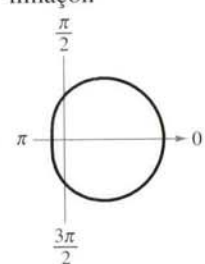
$\frac{a}{b} = 1$
Cardioid (heart-shaped)



$1 < \frac{a}{b} < 2$
Dimpled limaçon

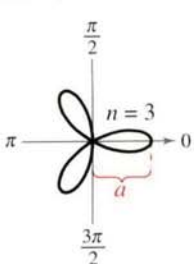


$\frac{a}{b} \geq 2$
Convex limaçon

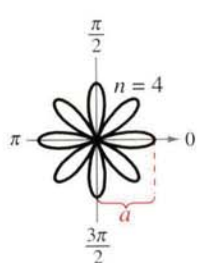


Limaçons
$r = a \pm b \cos \theta$
$r = a \pm b \sin \theta$
$(a > 0, b > 0)$

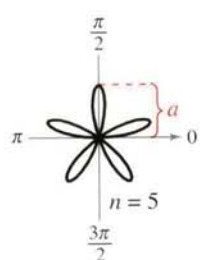
$r = a \cos n\theta$
Rose curve



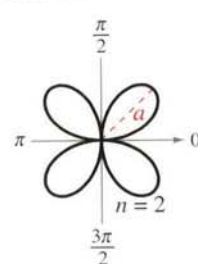
$r = a \cos n\theta$
Rose curve



$r = a \sin n\theta$
Rose curve

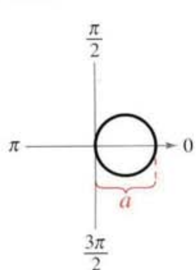


$r = a \sin n\theta$
Rose curve

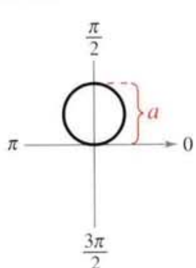


Rose Curves
n petals if n is odd
$2n$ petals if n is even
$(n \geq 2)$

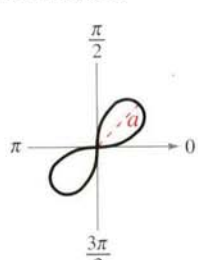
$r = a \cos \theta$
Circle



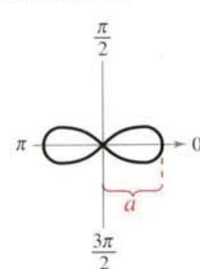
$r = a \sin \theta$
Circle



$r^2 = a^2 \sin 2\theta$
Lemniscate



$r^2 = a^2 \cos 2\theta$
Lemniscate



Circles and Lemniscates

EXAMPLE 5 Analyzing a Rose Curve

Analyze the graph of

$$r = 3 \cos 2\theta.$$

Solution

Begin with an analysis of the basic features of the graph.

Type of curve	Rose curve with $2n = 4$ petals
Symmetry	With respect to polar axis, the line $\theta = \pi/2$, and the pole
Maximum value of $\lvert r \rvert$	$\lvert r \rvert = 3$ when $\theta = 0, \pi/2, \pi, 3\pi/2$
Zeros of r	$r = 0$ when $\theta = \pi/4, 3\pi/4$

Using a graphing utility (with $0 \le \theta \le 2\pi$), you can obtain the graph shown in Figure 10.65.

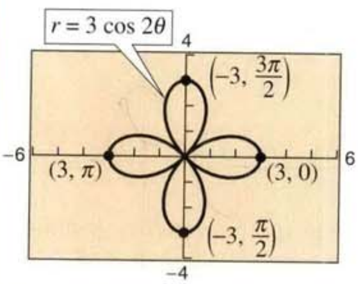


Figure 10.65

EXAMPLE 6 Analyzing a Lemniscate

Analyze the graph

$$r^2 = 9 \sin 2\theta.$$

Solution

Begin with an analysis of the basic features of the graph.

Type of curve	Lemniscate
Symmetry	With respect to pole
Maximum value of $\lvert r \rvert$	$\lvert r \rvert = 3$ when $\theta = \pi/4$
Zeros of r	$r = 0$ when $\theta = 0, \pi/2$

Using a graphing utility (with $r = \sqrt{9 \sin 2\theta}$ and $0 \le \theta \le 2\pi$), you can obtain the graph shown in Figure 10.66.

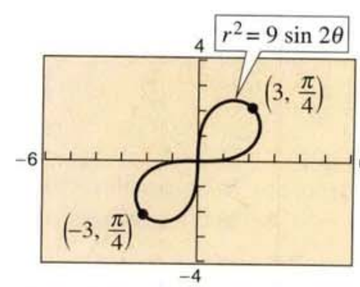


Figure 10.66

Writing About Math *Heart to Bell*

Use a graphing utility to graph the polar equation

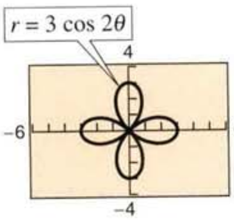
$$r = \cos 5\theta + n \cos \theta$$

for $0 \le \theta < \pi$ for the integers $n = -5$ to $n = 5$. As you graph these equations, you should see the graph change shape from a heart to a bell. Write a short paragraph explaining what values of n produce the heart portion of the curve and what values of n produce the bell.

10.7 Exercises

In Exercises 1–6, identify the type of polar graph.

1.



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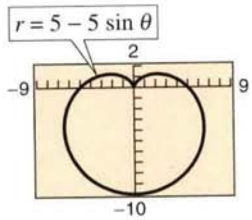

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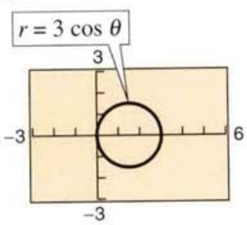

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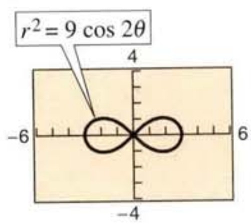
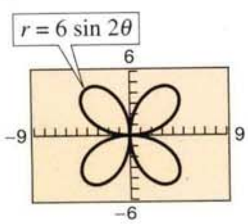


5.



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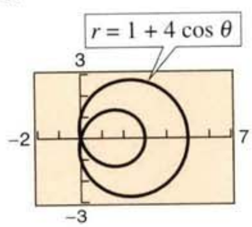


In Exercises 7–16, test for symmetry with respect to $\theta = \pi/2$, the polar axis, and the pole.

7. $r = 10 + 4\cos\theta$

8. $r = 16\cos 3\theta$

9. $r = \dfrac{6}{1 + \sin\theta}$

10. $r = \dfrac{4}{1 - \cos\theta}$

11. $r = 6\sin\theta$

12. $r = 4 - \sin\theta$

13. $r = 4\sec\theta\csc\theta$

14. $r = 2\csc\theta\cos\theta$

15. $r^2 = 25\sin 2\theta$

16. $r^2 = 25\cos 4\theta$

In Exercises 17–20, find the maximum value of $|r|$ and any zeros of r. Verify your answers numerically.

17. $r = 10 - 10\sin\theta$

18. $r = 6 + 12\cos\theta$

19. $r = 4\cos 3\theta$

20. $r = 5\sin 2\theta$

In Exercises 21–38, sketch the graph of the polar equation. Use a graphing utility to confirm your graph.

21. $r = 5$

22. $r = 2$

23. $\theta = \dfrac{\pi}{6}$

24. $\theta = -\dfrac{5\pi}{3}$

25. $r = 3\sin\theta$

26. $r = 3\cos\theta$

27. $r = 3(1 - \cos\theta)$

28. $r = 2(1 - \sin\theta)$

29. $r = 3 - 4\cos\theta$

30. $r = 5 - 4\sin\theta$

31. $r = 6 + \sin\theta$

32. $r = 4 + 5\cos\theta$

33. $r = 5\cos 3\theta$

34. $r = -\sin 5\theta$

35. $r = 7\sin 2\theta$

36. $r = 3\cos 5\theta$

37. $r = \dfrac{\theta}{2}$

38. $r = \theta$

In Exercises 39–54, use a graphing utility to graph the polar equation. Describe your viewing window.

39. $r = \dfrac{\theta}{4}$

40. $r = -\dfrac{\theta}{3}$

41. $r = 6\cos\theta$

42. $r = \cos 2\theta$

43. $r = 2(3 - \sin\theta)$

44. $r = 6 - 4\sin\theta$

45. $r = 3 - 6\cos\theta$

46. $r = 2(3 - 2\sin\theta)$

47. $r = \dfrac{3}{\sin\theta - 2\cos\theta}$

48. $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$

49. $r^2 = 4\cos 2\theta$

50. $r^2 = 4\sin\theta$

51. $r = 4\sin\theta\cos^2\theta$

52. $r = 2\cos(3\theta - 2)$

53. $r = 4\csc\theta + 5$

54. $r = 4 - \sec\theta$

In Exercises 55–62, use a graphing utility to graph the polar equation. Find an interval for θ for which the graph is traced *only once*.

55. $r = 3 - 2\cos\theta$

56. $r = 2(1 - 2\sin\theta)$

57. $r = 2 + \sin\theta$

58. $r = 4 + 3\cos\theta$

59. $r = 2\cos\left(\dfrac{3\theta}{2}\right)$

60. $r = 3\sin\left(\dfrac{5\theta}{2}\right)$

61. $r^2 = 4\sin 2\theta$

62. $r^2 = \dfrac{1}{\theta}$

In Exercises 63–66, use a graphing utility to graph the polar equation and show that the indicated line is an asymptote of the graph.

Name of Graph	*Polar Equation*	*Asymptote*
63. Conchoid	$r = 2 - \sec\theta$	$x = -1$
64. Conchoid	$r = 2 + \csc\theta$	$y = 1$
65. Hyperbolic spiral	$r = \dfrac{2}{\theta}$	$y = 2$
66. Strophoid	$r = 2\cos 2\theta \sec\theta$	$x = -2$

Synthesis

True or False? **In Exercises 67–70, determine whether the statement is true or false. Justify your answer.**

67. The point with polar coordinate $\left(6, \dfrac{11\pi}{6}\right)$ lies on the graph of $r = 2\sin\theta + 5$.

68. The graph of $r = 4\cos 8\theta$ is a rose curve with 8 petals.

69. The graph of $r = 10\sin 5\theta$ is a rose curve with 10 petals.

70. A rose curve will always have symmetry with respect to the line $\theta = \pi/2$.

71. ***Graphical Reasoning*** Use a graphing utility to graph the polar equation

$$r = 6[1 + \cos(\theta - \phi)]$$

for (a) $\phi = 0$, (b) $\phi = \pi/4$, and (c) $\phi = \pi/2$. Use the graphs to describe the effect of the angle ϕ. Write the equation as a function of $\sin\theta$ for part (c).

72. The graph of $r = f(\theta)$ is rotated about the pole through an angle ϕ. Show that the equation of the rotated graph is $r = f(\theta - \phi)$.

73. Consider the graph of $r = f(\sin\theta)$.

(a) Show that if the graph is rotated counterclockwise $\pi/2$ radians about the pole, the equation of the rotated graph is $r = f(-\cos\theta)$.

(b) Show that if the graph is rotated counterclockwise π radians about the pole, the equation of the rotated graph is $r = f(-\sin\theta)$.

(c) Show that if the graph is rotated counterclockwise $3\pi/2$ radians about the pole, the equation of the rotated graph is $r = f(\cos\theta)$.

In Exercises 74–76, use the results of Exercise 72 and 73.

74. Write an equation for the limaçon $r = 2 - \sin\theta$ after it has been rotated through the given angle.

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) π (d) $\dfrac{3\pi}{2}$

75. Write an equation for the rose curve $r = 2\sin 2\theta$ after it has been rotated through the given angle.

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{2\pi}{3}$ (d) π

76. Sketch the graph of each equation.

(a) $r = 1 - \sin\theta$ (b) $r = 1 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

77. ***Exploration*** Use a graphing utility to graph the polar equation $r = 2 + k\cos\theta$ for $k = 0$, $k = 1$, $k = 2$, and $k = 3$. Identify each graph.

78. ***Exploration*** Consider the polar equation $r = 3\sin k\theta$.

(a) Use a graphing utility to graph the equation for $k = 1.5$. Find the interval for θ for which the graph is traced only once.

(b) Use a graphing utility to graph the equation for $k = 2.5$. Find the interval for θ for which the graph is traced only once.

(c) Is it possible to find an interval for θ for which the graph is traced only once for any rational number k? Explain.

Review

In Exercises 79–82, write the first five terms of the arithmetic sequence. Find the common difference and write the *n*th term of the sequence as a function of *n*.

79. $a_1 = 2, \quad a_8 = 23$

80. $a_1 = \frac{5}{2}, \quad a_3 = \frac{11}{6}$

81. $a_1 = 150, \quad a_{k+1} = a_k - 18$

82. $a_1 = 0.525, \quad a_{k+1} = a_k + 0.75$

In Exercises 83–88, find the sum.

83. $\displaystyle\sum_{n=1}^{20} 4n$

84. $\displaystyle\sum_{n=1}^{50} 8n$

85. $\displaystyle\sum_{n=1}^{120} (n + 5)$

86. $\displaystyle\sum_{n=1}^{200} (300 - n)$

87. $\displaystyle\sum_{n=0}^{\infty} \left(\tfrac{1}{8}\right)^n$

88. $\displaystyle\sum_{n=1}^{\infty} 6(0.4)^{n-1}$

10.8 Polar Equations of Conics

Alternative Definition of Conics

In Sections 10.2 and 10.3, you learned that the rectangular equations of ellipses and hyperbolas take simple forms when the origin lies at the *center*. As it happens, there are many important applications of conics in which it is more convenient to use one of the *foci* as the origin for the coordinate system. For example, the sun lies at one focus of the earth's orbit. Similarly the light source of a parabolic reflector lies at its focus. In this section you will learn that polar equations of conics take simple forms if one of the foci lies at the pole.

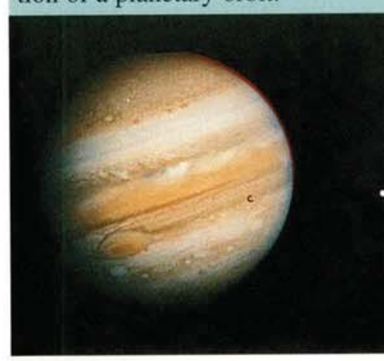
To begin, consider the following alternative definition of a conic that uses the concept of eccentricity.

What You Should Learn:

- How to define conics in terms of eccentricities
- How to write equations of conics in polar form
- How to use equations of conics in polar form to model real-life problems

Why You Should Learn It:

The elliptical orbits of planets and satellites can be modeled with polar equations. Exercise 39 on page 759 shows a polar equation of a planetary orbit.

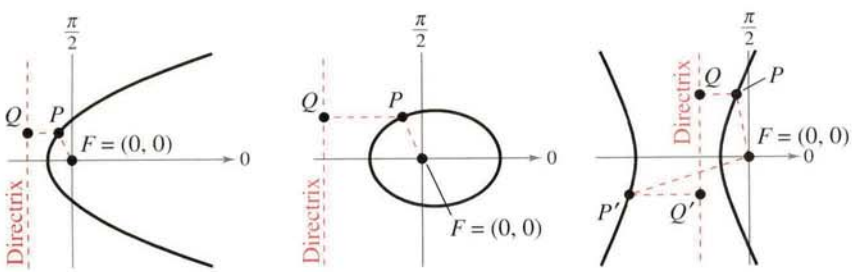


NASA

Alternative Definition of a Conic

The locus of a point in the plane which moves so that its distance from a fixed point (focus) is in constant ratio to its distance from a fixed line (directrix) is a **conic.** The constant ratio is the **eccentricity** of the conic and is denoted by e. Moreover, the conic is an **ellipse** if $e < 1$, a **parabola** if $e = 1$, and a **hyperbola** if $e > 1$.

In Figure 10.67, note that for each type of conic, the pole corresponds to the fixed point (focus) given in the definition.



Parabola: $e = 1$

$\frac{PF}{PQ} = 1$

Ellipse: $0 < e < 1$

$\frac{PF}{PQ} < 1$

Hyperbola: $e > 1$

$\frac{PF}{PQ} = \frac{P'F}{P'Q'} > 1$

Figure 10.67

Polar Equations of Conics

The benefit of locating a focus of a conic at the pole is that the equation of the conic takes on a simpler form. A proof of the polar form is given in Appendix A.

Polar Equations of Conics

The graph of a polar equation of the form

1. $r = \dfrac{ep}{1 \pm e \cos \theta}$ **2.** $r = \dfrac{ep}{1 \pm e \sin \theta}$

is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.

Equations of the form $r = \dfrac{ep}{1 \pm e \cos \theta}$ Vertical directrix

correspond to conics with vertical directrices and equations of the form

$$r = \frac{ep}{1 \pm e \sin \theta}$$ Horizontal directrix

correspond to conics with horizontal directrices. Moreover, the converse is also true—that is, any conic with a focus at the pole and having a horizontal or vertical directrix can be represented by one of the given equations.

A computer simulation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 1 Determining a Conic from Its Equation

Determine the type of conic represented by the equation $r = \dfrac{15}{3 - 2 \cos \theta}$.

Algebraic Solution

To determine the type of conic, rewrite the equation in the form $r = ep/(1 \pm e \cos \theta)$.

$$r = \frac{15}{3 - 2 \cos \theta}$$

$$= \frac{5}{1 - (2/3) \cos \theta}$$ Divide numerator and denominator by 3.

From this form you can conclude that the graph is an ellipse with $e = \frac{2}{3}$.

Graphical Solution

Use a graphing utility in *polar* mode to graph $r = \dfrac{15}{3 - 2 \cos \theta}$. Be sure to use a square setting. From the graph in Figure 10.68, you can see that the conic appears to be an ellipse.

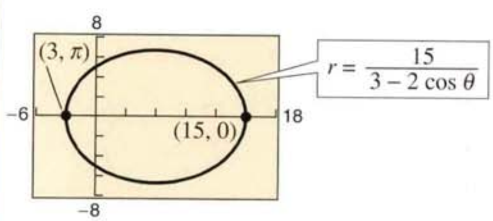


Figure 10.68

For the ellipse in Figure 10.68, the major axis is horizontal and the vertices lie at $(r, \theta) = (15, 0)$ and $(r, \theta) = (3, \pi)$. So, the length of the *major* axis is $2a = 18$. To find the length of the *minor* axis, you can use the equations $e = c/a$ and $b^2 = a^2 - c^2$ to conclude that

$$b^2 = a^2 - c^2 = a^2 - (ea)^2 = a^2(1 - e^2).$$ Ellipse

Because $e = \frac{2}{3}$, you have $b^2 = 9^2[1 - (2/3)^2] = 45$, which implies that $b = \sqrt{45} = 3\sqrt{5}$. So, the length of the minor axis is $2b = 6\sqrt{5}$. A similar analysis for hyperbolas yields

$$b^2 = c^2 - a^2 = (ea)^2 - a^2 = a^2(e^2 - 1).$$ Hyperbola

EXAMPLE 2 Analyzing the Graph of a Polar Equation

Analyze the graph of the polar equation

$$r = \frac{32}{3 + 5 \sin \theta}.$$

Solution

Dividing the numerator and denominator by 3 produces

$$r = \frac{32/3}{1 + (5/3) \sin \theta}.$$

Because $e = 5/3 > 1$, the graph is a hyperbola. The transverse axis of the hyperbola lies on the line $\theta = \pi/2$ and the vertices occur at $(r, \theta) = (4, \pi/2)$ and $(r, \theta) = (-16, 3\pi/2)$. Because the length of the transverse axis is 12, you can see that $a = 6$. To find b, write

$$b^2 = a^2(e^2 - 1) = 6^2\left[\left(\frac{5}{3}\right)^2 - 1\right] = 64.$$

Therefore, $b = 8$. The asymptotes of the hyperbola are $y = 10 \pm \frac{3}{4}x$, as shown in Figure 10.69.

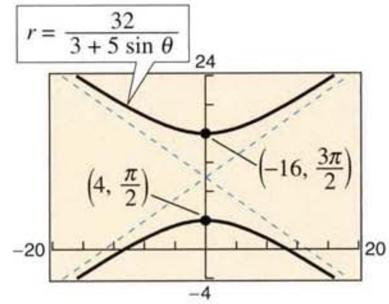


Figure 10.69

Exploration

Try using a graphing utility in *polar* mode to verify the four orientations shown at the left. Remember that e must be positive, but p can be positive or negative.

In the next example, you are asked to find a polar equation for a specified conic. To do this, let p be the distance between the pole and the directrix.

1. Horizontal directrix above the pole: $r = \dfrac{ep}{1 + e \sin \theta}$
2. Horizontal directrix below the pole: $r = \dfrac{ep}{1 - e \sin \theta}$
3. Vertical directrix to the right of the pole: $r = \dfrac{ep}{1 + e \cos \theta}$
4. Vertical directrix to the left of the pole: $r = \dfrac{ep}{1 - e \cos \theta}$

EXAMPLE 3 Finding the Polar Equation of a Conic

Find the polar equation of the parabola whose focus is the pole and whose directrix is the line $y = 3$.

Solution

From Figure 10.70, you can see that the directrix is horizontal. So, you can choose an equation of the form

$$r = \frac{ep}{1 + e \sin \theta}.$$

Moreover, because the eccentricity of a parabola is $e = 1$ and the distance between the pole and the directrix is $p = 3$, you have the equation

$$r = \frac{3}{1 + \sin \theta}.$$

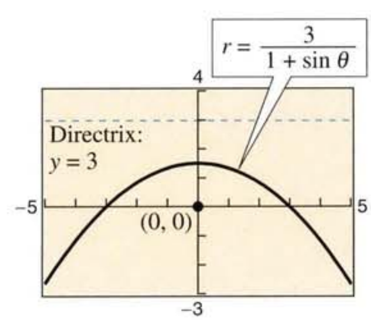


Figure 10.70

Application

Kepler's Laws (listed below), named after the German astronomer Johannes Kepler (1571–1630), can be used to describe the orbits of the planets about the sun.

1. Each planet moves in an elliptical orbit with the sun as a focus.
2. A ray from the sun to the planet sweeps out equal areas of the ellipse in equal times.
3. The square of the period is proportional to the cube of the mean distance between the planet and the sun.

Although Kepler simply stated these laws on the basis of observation, they were later validated by Isaac Newton (1642–1727). In fact, Newton was able to show that each law can be deduced from a set of universal laws of motion and gravitation that govern the movement of all heavenly bodies, including comets and satellites. This is illustrated in the next example, which involves the comet named after the English mathematician and physicist Edmund Halley (1656–1742).

If you use earth as a reference with a period of 1 year and a distance of 1 astronomical unit, the proportionality constant in Kepler's third law is 1. For example, because Mars has a mean distance to the sun of $d = 1.523$ AU, its period P is given by $d^3 = P^2$. So, the period for Mars is $P = 1.88$ years.

EXAMPLE 4 Halley's Comet

Halley's comet has an elliptical orbit with an eccentricity of $e \approx 0.97$. The length of the major axis of the orbit is approximately 36.18 astronomical units. (An *astronomical unit* is defined as the mean distance between earth and the sun, 93 million miles.) Find a polar equation for the orbit. How close does Halley's comet come to the sun?

Solution

Using a vertical axis, as shown in Figure 10.71, choose an equation of the form $r = ep/(1 + e\sin\theta)$. Because the vertices of the ellipse occur when $\theta = \pi/2$ and $\theta = 3\pi/2$, you can determine the length of the major axis to be the sum of the r-values of the vertices. That is,

$$2a = \frac{0.97p}{1 + 0.97} + \frac{0.97p}{1 - 0.97} \approx 32.83p \approx 36.18.$$

So, $p \approx 1.102$ and $ep \approx (0.97)(1.102) \approx 1.069$. Using this value in the equation, you have

$$r = \frac{1.069}{1 + 0.97\sin\theta}$$

where r is measured in astronomical units. To find the closest point to the sun (the focus), substitute $\theta = \pi/2$ into this equation to obtain

$$r = \frac{1.069}{1 + 0.97\sin(\pi/2)} \approx 0.54 \text{ astronomical units} \approx 50{,}000{,}000 \text{ miles.}$$

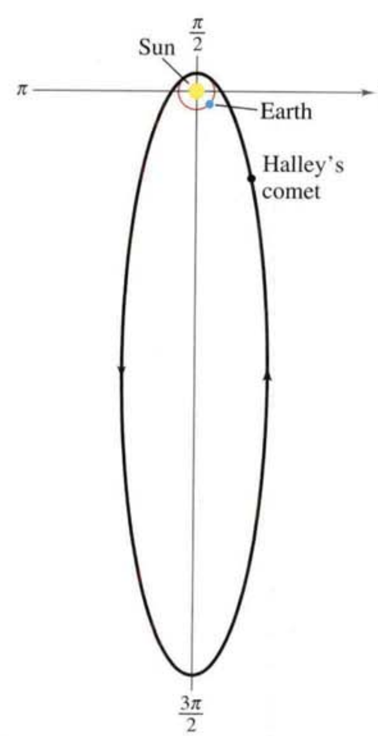


Figure 10.71

10.8 Exercises

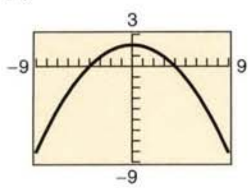
Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the polar equation when (a) $e = 1$, (b) $e = 0.5$, and (c) $e = 1.5$.**

1. $r = \dfrac{2e}{1 + e\cos\theta}$ **2.** $r = \dfrac{2e}{1 - e\cos\theta}$

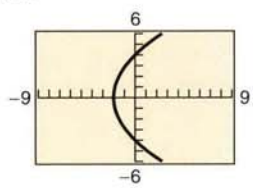
3. $r = \dfrac{2e}{1 - e\sin\theta}$ **4.** $r = \dfrac{2e}{1 + e\sin\theta}$

In Exercises 5–8, match the polar equation with the correct graph. [The graphs are labeled (a), (b), (c), and (d).]

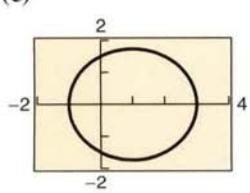
(a)



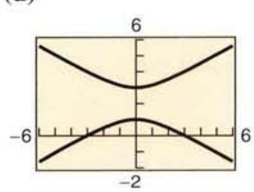
(b)



(c)



(d)



5. $r = \dfrac{4}{1 - \cos\theta}$ **6.** $r = \dfrac{3}{2 - \cos\theta}$

7. $r = \dfrac{3}{1 + 2\sin\theta}$ **8.** $r = \dfrac{4}{1 + \sin\theta}$

In Exercises 9–18, determine the type of conic represented by the equation algebraically. Use a graphing utility to confirm your result graphically.

9. $r = \dfrac{6}{1 - \cos\theta}$ **10.** $r = \dfrac{6}{1 + \sin\theta}$

11. $r = \dfrac{4}{4 - \cos\theta}$ **12.** $r = \dfrac{7}{7 + \sin\theta}$

13. $r = \dfrac{8}{4 + 3\sin\theta}$ **14.** $r = \dfrac{6}{3 - 2\cos\theta}$

15. $r = \dfrac{4}{2 + 3\sin\theta}$ **16.** $r = \dfrac{5}{-1 + 2\cos\theta}$

17. $r = \dfrac{3}{4 - 8\cos\theta}$ **18.** $r = \dfrac{10}{3 + 9\sin\theta}$

In Exercises 19–22, use a graphing utility to graph the polar equation. Identify the graph.

19. $r = \dfrac{-5}{1 - \sin\theta}$ **20.** $r = \dfrac{-3}{2 + 4\sin\theta}$

21. $r = \dfrac{12}{2 - \cos\theta}$ **22.** $r = \dfrac{14}{14 + 17\sin\theta}$

In Exercises 23–26, use a graphing utility to graph the rotated conic.

23. $r = \dfrac{6}{1 - \cos(\theta - \pi/4)}$ (See Exercise 9.)

24. $r = \dfrac{7}{7 + \sin(\theta - \pi/3)}$ (See Exercise 12.)

25. $r = \dfrac{8}{4 + 3\sin(\theta + \pi/6)}$ (See Exercise 13.)

26. $r = \dfrac{5}{-1 + 2\cos(\theta + 2\pi/3)}$ (See Exercise 16.)

In Exercises 27–38, find a polar equation of the conic with its focus at the pole.

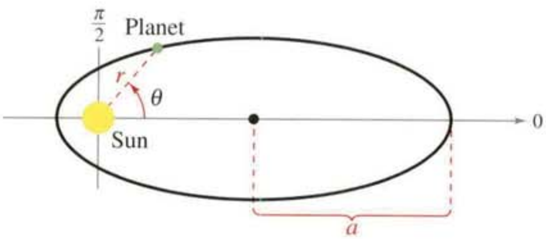
	Conic	*Eccentricity*	*Directrix*
27.	Parabola	$e = 1$	$x = -1$
28.	Parabola	$e = 1$	$y = -4$
29.	Ellipse	$e = \frac{1}{2}$	$y = 1$
30.	Ellipse	$e = \frac{3}{4}$	$y = -4$
31.	Hyperbola	$e = 2$	$x = 1$
32.	Hyperbola	$e = \frac{3}{2}$	$x = -1$

	Conic	*Vertex or Vertices*
33.	Parabola	$\left(1, -\dfrac{\pi}{2}\right)$
34.	Parabola	$\left(10, \dfrac{\pi}{2}\right)$
35.	Ellipse	$(2, 0), (8, \pi)$
36.	Ellipse	$\left(2, \dfrac{\pi}{2}\right), \left(4, \dfrac{3\pi}{2}\right)$

	Conic	*Vertex or Vertices*
37.	Hyperbola	$\left(1, \frac{3\pi}{2}\right), \left(9, \frac{3\pi}{2}\right)$
38.	Hyperbola	$\left(4, \frac{\pi}{2}\right), \left(-1, \frac{3\pi}{2}\right)$

39. ***Planetary Motion*** The planets travel in elliptical orbits with the sun as a focus. Assume that the focus is at the pole, the major axis lies on the polar axis, and the length of the major axis is $2a$. Show that the polar equation of the orbit is

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta}, \text{ where } e \text{ is the eccentricity.}$$



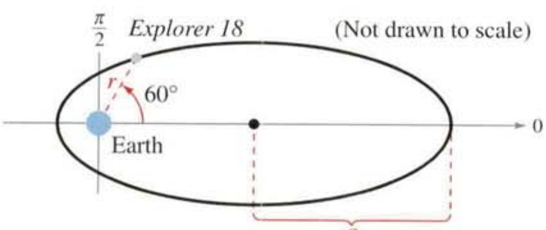
40. ***Planetary Motion*** Use the result of Exercise 39 to show that the minimum distance (*perihelion*) from the sun to the planet is $r = a(1 - e)$ and that the maximum distance (*aphelion*) is $r = a(1 + e)$.

In Exercises 41 and 42, use the results of Exercises 39 and 40 to find the polar equation of the planet and the perihelion and aphelion distances.

41. Earth $\quad a = 92.957 \times 10^6$ miles, $e = 0.0167$

42. Pluto $\quad a = 5.900 \times 10^9$ kilometers, $e = 0.2444$

43. ***Explorer 18*** On November 27, 1963, the United States launched *Explorer 18*. Its low and high points over the surface of earth were 119 miles and 122,800 miles, respectively. The center of earth is the focus of the orbit. Find the polar equation for the orbit and then find the distance between the surface of earth (assume a radius of 4000 miles) and the satellite when $\theta = 60°$.



44. ***Explorer 18*** In Exercise 43, find the distance between the surface of earth and the satellite when $\theta = 30°$.

Synthesis

True or False? **In Exercises 45 and 46, determine whether the statement is true or false. Justify your answer.**

45. The graph of $r = 4/(-3 - 3\sin\theta)$ has a horizontal directrix above the pole.

46. The conic represented by the following equation is an ellipse.

$$r^2 = \frac{16}{9 - 4\cos\left(\theta + \frac{\pi}{4}\right)}$$

47. Show that the polar equation for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{is} \quad r^2 = \frac{b^2}{1 - e^2\cos^2\theta}.$$

48. Show that the polar equation for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{is} \quad r^2 = \frac{-b^2}{1 - e^2\cos^2\theta}.$$

In Exercises 49 and 50, use the results of Exercises 47 and 48 to write the polar form of the equation of the conic.

49. $\frac{x^2}{169} + \frac{y^2}{144} = 1$ **50.** $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Review

In Exercises 51–54, find the number of distinguishable permutations of the group of letters.

51. M, A, M, M, A, L **52.** B, A, R, B, E, C, U, E

53. C, L, E, V, E, L, A, N, D

54. C, I, N, C, I, N, N, A, T, I

In Exercises 55–58, evaluate the expression. Do not use a calculator.

55. ${}_{12}C_9$ **56.** ${}_{18}C_{16}$

57. ${}_{10}P_3$ **58.** ${}_{29}P_2$

10 Chapter Summary

What did you learn?

Section 10.1

	Review Exercises
☐ How to recognize a conic as the intersection of a plane and a double-napped cone	1, 2
☐ How to write equations of parabolas in standard form	3–6
☐ How to use the reflective property of parabolas to solve real-life problems	7–10

Section 10.2

☐ How to write equations of ellipses in standard form	11–14
☐ How to use properties of ellipses to model and solve real-life problems	15, 16
☐ How to find eccentricities of ellipses	17–20

Section 10.3

☐ How to write equations of hyperbolas in standard form	21–24
☐ How to find asymptotes of hyperbolas	25–28
☐ How to use properties of hyperbolas to solve real-life problems	29, 30
☐ How to classify conics from their general equations	31, 32

Section 10.4

☐ How to rotate the coordinate axes to eliminate the xy-term in equations of conics	33–36
☐ How to use the discriminant to classify conics	37–40
☐ How to solve systems of quadratic equations	41, 42

Section 10.5

☐ How to evaluate sets of parametric equations for given values of the parameter	43–46
☐ How to graph curves that are represented by sets of parametric equations and rewrite sets of parametric equations as single rectangular equations	47–54, 57, 58
☐ How to find sets of parametric equations for graphs	55, 56

Section 10.6

☐ How to plot points in the polar coordinate system	59–64
☐ How to convert points from rectangular to polar form and vice versa	65–72
☐ How to convert equations from rectangular to polar form and vice versa	73–88

Section 10.7

☐ How to graph a polar equation by point plotting	89–100
☐ How to use symmetry, zeros, and maximum r-values as graphing aids	101–106
☐ How to recognize special polar graphs	107–114

Section 10.8

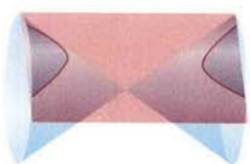
☐ How to define conics in terms of eccentricities	115–120
☐ How to write equations of conics in polar form	121–124
☐ How to use equations of conics in polar form to model real-life problems	125

X.T.C.

10 Review Exercises

10.1 In Exercises 1 and 2, state what type of conic is formed by the intersection of the plane and the double-napped cone.

1.



2.

In Exercises 3–6, find the standard form of the equation of the parabola.

3. Vertex: $(4, 2)$
Focus: $(4, 0)$

4. Vertex: $(2, 0)$
Focus: $(0, 0)$

5. Vertex: $(0, 2)$
Directrix: $x = -3$

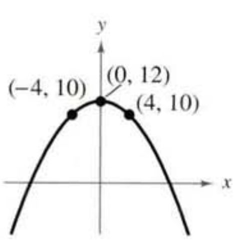
6. Vertex: $(2, 2)$
Directrix: $y = 0$

In Exercises 7 and 8, find an equation of a tangent line to the parabola at the given point and find the x-intercept of the line.

7. $x^2 = -2y,\ (2, -2)$

8. $x^2 = -2y,\ (-4, -8)$

9. ***Parabolic Archway*** A parabolic archway is 12 meters high at the vertex. At a height of 10 meters, the width of the archway is 8 meters. How wide is the archway at ground level?



10. ***Flashlight*** The light bulb in a flashlight is at the focus of its parabolic reflector, 1.5 centimeters from the vertex of the reflector. Write an equation for a cross section of the flashlight's reflector with its focus on the positive x-axis and its vertex at the origin.

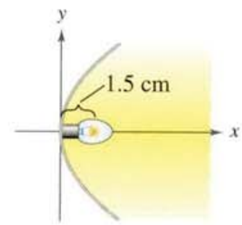


FIGURE FOR 10

10.2 In Exercises 11–14, find the standard form of the equation of the ellipse.

11. Vertices: $(-3, 0), (7, 0)$; Foci: $(0, 0), (4, 0)$

12. Vertices: $(2, 0), (2, 4)$; Foci: $(2, 1), (2, 3)$

13. Vertices: $(0, 1), (4, 1)$;
Endpoints of the minor axis: $(2, 0), (2, 2)$

14. Vertices: $(-4, -1), (-4, 11)$;
Endpoints of the minor axis: $(-6, 5), (-2, 5)$

15. ***Semielliptical Archway*** A semielliptical archway is set on pillars that are 10 feet apart. Its height (atop the pillars) is 4 feet. Where should the foci be placed in order to sketch the semielliptical arch?

16. ***Wading Pool*** You are building a wading pool that is in the shape of an ellipse. Your plans give an equation for the elliptical shape of the pool measured in feet as

$$\frac{x^2}{324} + \frac{y^2}{196} = 1.$$

Find the longest distance across the pool, the shortest distance, and the distance between the foci.

In Exercises 17–20, find the center, vertices, foci, and eccentricity of the ellipse.

17. $16x^2 + 9y^2 - 32x + 72y + 16 = 0$

18. $4x^2 + 25y^2 + 16x - 150y + 141 = 0$

19. $\dfrac{(x + 2)^2}{81} + \dfrac{(y - 1)^3}{100} = 1$

20. $\dfrac{(x - 5)^2}{1} + \dfrac{(y + 3)^2}{36} = 1$

10.3 In Exercises 21–24, find the standard form of the equation of the hyperbola.

21. Vertices: $(-10, 3), (6, 3)$; Foci: $(-12, 3), (8, 3)$
22. Vertices: $(2, 2), (-2, 2)$; Foci: $(4, 2), (-4, 2)$
23. Foci: $(0, 0), (8, 0)$; Asymptotes: $y = \pm 2(x - 4)$
24. Foci: $(3, \pm 2)$; Asymptotes: $y = \pm 2(x - 3)$

In Exercises 25–28, find the center, vertices, foci, and the equations of the asymptotes of the hyperbola. Then sketch its graph.

25. $9x^2 - 16y^2 - 18x - 32y - 151 = 0$
26. $-4x^2 + 25y^2 - 8x + 150y + 121 = 0$
27. $\dfrac{(x-3)^2}{16} - \dfrac{(y+5)^2}{4} = 1$
28. $\dfrac{(y-1)^2}{4} - x^2 = 1$
29. *Loran* A radio transmitting station A is located 200 miles east of transmitting station B. A ship is in an area to the north and 40 miles west of the station A. Synchronized radio pulses transmitted at 186,000 miles per second by the two stations are received 0.0005 seconds sooner from station A than from station B. How far north is the ship?
30. *Locating an Explosion* Two of your friends live 4 miles apart and on the same "east-west" street, and you live halfway between them. You are having a three-way phone conversation when you hear an explosion. Six seconds later your friend to the east hears the explosion, and your friend to the west hears it 8 seconds after you do. Find equations of two hyperbolas that would locate the explosion. (Sound travels at a rate of 1100 feet per second.)

In Exercises 31 and 32, classify the conic from its general equation.

31. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$
32. $4x^2 - 4y^2 - 4x + 8y - 11 = 0$

10.4 In Exercises 33–36, rotate the axes to eliminate the xy-term in the equation. Then write the equation in standard form. Sketch the graph of the equation, showing both sets of axes.

33. $xy - 4 = 0$
34. $x^2 - 10xy + y^2 + 1 = 0$
35. $5x^2 - 2xy + 5y^2 - 12 = 0$
36. $4x^2 + 8xy + 4y^2 + 7\sqrt{2}x + 9\sqrt{2}y = 0$

In Exercises 37–40, (a) use the discriminant to classify the graph, (b) use the quadratic formula to solve for y, and then (c) use a graphing utility to graph the equation.

37. $16x^2 - 8xy + y^2 - 10x + 5y = 0$
38. $13x^2 - 8xy + 7y^2 - 45 = 0$
39. $x^2 + y^2 + 2xy + 2\sqrt{2}x - 2\sqrt{2}y + 2 = 0$
40. $x^2 - 4xy - 2y^2 - 6 = 0$

In Exercises 41 and 42, use any method to solve the system of quadratic equations algebraically. Then verify your results by using a graphing utility to graph the equations and find any points of intersection of the graphs.

41. $\begin{cases} -4x^2 - y^2 - 32x + 24y - 64 = 0 \\ 4x^2 + y^2 + 56x - 24y + 304 = 0 \end{cases}$
42. $\begin{cases} x^2 + y^2 - 25 = 0 \\ 9x - 4y^2 = 0 \end{cases}$

10.5 In Exercises 43–46, evaluate the parametric equations $x = 3 \cos \theta$ and $y = 2 \sin^2 \theta$ for the given value of θ.

43. $\theta = 0$
44. $\theta = \dfrac{\pi}{3}$
45. $\theta = \dfrac{\pi}{6}$
46. $\theta = -\dfrac{\pi}{4}$

In Exercises 47–54, sketch the curve represented by the parametric equations and, where possible, write the corresponding rectangular equation by eliminating the parameter. Verify your result with a graphing utility.

47. $x = 1 + 4t$
$y = 2 - 3t$
48. $x = t + 4$
$y = t^2$
49. $x = \dfrac{1}{t}$
$y = t^2$
50. $x = \dfrac{1}{t}$
$y = 2t + 3$
51. $x = 6 \cos \theta$
$y = 6 \sin \theta$
52. $x = 3 + 3 \cos \theta$
$y = 2 + 5 \sin \theta$
53. $x = \sec \theta$
$y = \tan \theta$
54. $x = 2\theta - \sin \theta$
$y = 2 - \cos \theta$

55. Find a parametric representation of the ellipse with center at $(-3, 4)$, major axis horizontal and eight units in length, and minor axis six units in length.

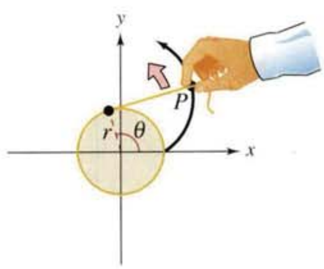
56. Find a parametric representation of the hyperbola with vertices $(0, \pm 4)$ and foci $(0, \pm 5)$.

57. ***Rotary Engine*** The rotary engine was developed by Felix Wankel in the 1950s. It features a rotor that is basically a modified equilateral triangle. The rotor moves in a chamber that, in two dimensions, is an epitrochoid. Use a graphing utility to graph the chamber modeled by the parametric equations $x = \cos 3\theta + 5 \cos \theta$ and $y = \sin 3\theta + 5 \sin \theta$.

58. ***Involute of a Circle*** The involute of a circle is described by the endpoint P of a string that is held taut as it is unwound from a spool. The spool does not rotate. Use a graphing utility to graph the involute of a circle modeled by the parametric equations. Use $r = 2$. Describe your viewing window.

$x = r(\cos \theta + \theta \sin \theta)$

$y = r(\sin \theta - \theta \cos \theta)$



10.6 **In Exercises 59–64, plot the point in the polar coordinate system and find three additional polar representations of the point, using $-2\pi < \theta < 2\pi$.**

59. $\left(1, \frac{5\pi}{4}\right)$ **60.** $\left(-2, \frac{7\pi}{6}\right)$ **61.** $\left(-\frac{5}{2}, \frac{\pi}{6}\right)$

62. $\left(\frac{3}{4}, -\frac{\pi}{3}\right)$ **63.** $\left(\sqrt{5}, \frac{4\pi}{3}\right)$ **64.** $\left(\sqrt{10}, \frac{3\pi}{4}\right)$

In Exercises 65–68, plot the points given in polar coordinates and find the corresponding rectangular coordinates for the point.

65. $\left(5, -\frac{7\pi}{6}\right)$ **66.** $\left(-3, \frac{2\pi}{3}\right)$

67. $\left(12, -\frac{\pi}{2}\right)$ **68.** $\left(\sqrt{3}, 1.78\right)$

In Exercises 69–72, the rectangular coordinates of a point are given. Find two sets of polar coordinates for the point for $0 \le \theta < 2\pi$.

69. $(0, -9)$ **70.** $(-6, 8)$

71. $(5, -5)$ **72.** $\left(-3, -\sqrt{3}\right)$

In Exercises 73–80, convert the polar equation to rectangular form.

73. $r = 5$ **74.** $r = 12$

75. $r = 3 \cos \theta$ **76.** $r = 8 \sin \theta$

77. $r^2 = \cos 2\theta$ **78.** $r^2 = \sin \theta$

79. $r = \dfrac{2}{2 - \sin \theta}$ **80.** $r = \dfrac{10}{4 - 7 \cos \theta}$

In Exercises 81–88, convert the rectangular equation to polar form.

81. $x^2 + y^2 = 9$ **82.** $x^2 + y^2 = 20$

83. $y = 6$ **84.** $x = 14$

85. $x^2 + y^2 - 4x = 0$ **86.** $x^2 + y^2 - 6y = 0$

87. $xy = 5$ **88.** $xy = -2$

10.7 **In Exercises 89–96, sketch the graph of the polar equation. Use a graphing utility to verify your graph.**

89. $r = 5$ **90.** $r = 11$

91. $\theta = \dfrac{\pi}{2}$ **92.** $\theta = -\dfrac{5\pi}{6}$

93. $r = 5 \cos \theta$ **94.** $r = 2 \sin \theta$

95. $r = -2(1 + \cos \theta)$ **96.** $r = 4 - 3 \cos \theta$

In Exercises 97–100, use a graphing utility to graph the polar equation. Describe the graph.

97. $r^2 = 4 \sin^2 2\theta$ **98.** $r^2 = 9 \cos^2 2\theta$

99. $r = \dfrac{3}{\cos\left(\theta - \frac{\pi}{4}\right)}$ **100.** $r = \dfrac{-4}{\sin\left(\theta + \frac{2\pi}{3}\right)}$

In Exercises 101–106, sketch the graph of the polar equation. Identify any symmetry, maximum r-values, and zeros of r. Use a graphing utility to verify your graph.

101. $r = 5 + 4 \cos \theta$ **102.** $r = 3 - 5 \sin \theta$

103. $r = -3 \cos 2\theta$ **104.** $r = \cos 5\theta$

105. $r^2 = \cos 2\theta$ **106.** $r^2 = 5 \sin 2\theta$

In Exercises 107–114, identify and sketch the graph of the polar equation. Use a graphing utility to verify your graph.

107. $r = 6 - \cos\theta$

108. $r = 7(1 - \sin\theta)$

109. $r = 10 + 15\sin\theta$

110. $r = 8 + 3\cos\theta$

111. $r = 4\sin 5\theta$

112. $r = 2\cos 3\theta$

113. $r^2 = 8\cos 2\theta$

114. $r^2 = 36\sin 2\theta$

10.7 In Exercises 115–120, identify and sketch the graph of the polar equation. Use a graphing utility to verify your graph.

115. $r = \dfrac{2}{1 - \sin\theta}$

116. $r = \dfrac{1}{1 + 2\sin\theta}$

117. $r = \dfrac{4}{5 - 3\cos\theta}$

118. $r = \dfrac{6}{-1 + 4\cos\theta}$

119. $r = \dfrac{5}{6 + 2\sin\theta}$

120. $r = \dfrac{3}{4 - 4\cos\theta}$

In Exercises 121–125, find a polar equation for the conic.

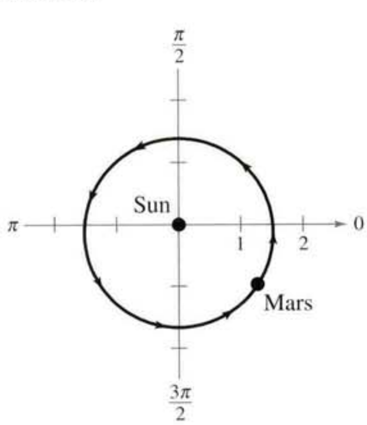
121. Circle — Center: $(8, \pi/2)$; Solution point: $(0, 0)$

122. Parabola — Vertex: $(2, \pi/2)$; Focus: $(0, 0)$

123. Ellipse — Vertices: $(5, 0), (1, \pi)$; One focus: $(0, 0)$

124. Hyperbola — Vertices: $(1, 0), (7, 0)$; One focus: $(0, 0)$

125. ***Planetary Motion*** The planet Mars has an elliptical orbit with an eccentricity of $e \approx 0.092$. The length of the major axis of the orbit is approximately 3.05 astronomical units. Find a polar equation for the orbit and its perihelion and aphelion distances.



Synthesis

True or False? **In Exercises 126–128, determine whether the statement is true or false. Justify your answer.**

126. The graph of $x^2/4 - y^4 = 1$ is a hyperbola.

127. There is only one set of parametric equations that represents the line $y = 3 - 2x$.

128. There is a unique polar coordinate representation of each point in the plane.

Writing **In Exercises 129 and 130, an equation and four variations are given. In your own words, describe how the graph of each of the variations differs from the graph of the first equation.**

129. $y^2 = 8x$

(a) $(y - 2)^2 = 8x$ (b) $y^2 = 8(x + 1)$

(c) $y^2 = -8x$ (d) $y^2 = 4x$

130. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

(a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ (b) $\dfrac{x^2}{4} + \dfrac{y^2}{4} = 1$

(c) $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$ (d) $\dfrac{(x - 3)^2}{4} + \dfrac{y^2}{9} = 1$

131. Consider an ellipse whose major axis is horizontal and 10 units in length. The number b in the standard form of the ellipse must be less than what real number? Explain the change in the shape of the ellipse as b approaches this number.

132. The curve represented by the parametric equations $x = 2\sec t$ and $y = 3\tan t$ is shown in the graph. Would the curve change for the equations $x = 2\sec(-t)$ and $y = 3\tan(-t)$? If so, how would the graph change?