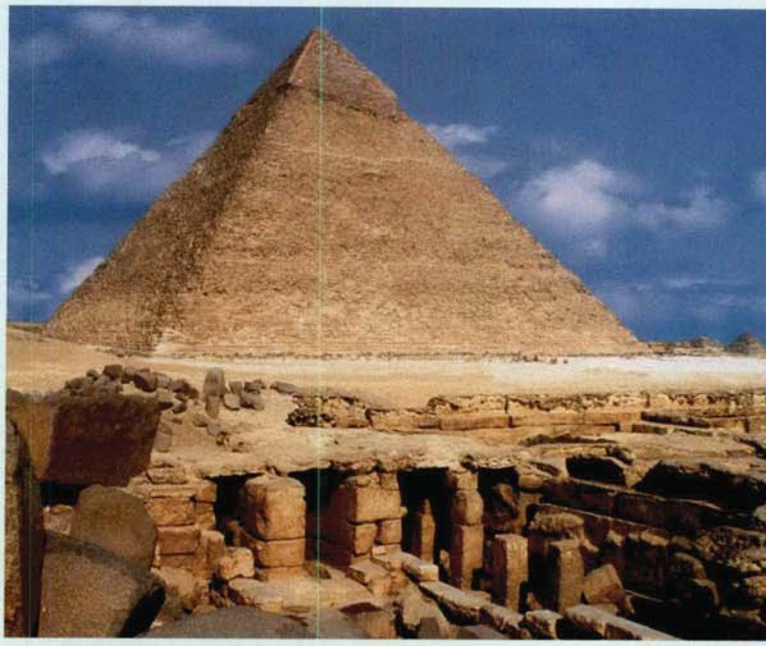
Analytic Geometry in Three Dimensions

11



Larry Lee/CORBIS

The Great Pyramid of Giza, one of the Seven Wonders of the Ancient World, was built about 2600 B.C. to 2500 B.C. It contains more than two million stone blocks.

The Big Picture

In this chapter you will learn how to

- plot points, find distances between points, and find midpoints of line segments connecting points in space.
- write equations of spheres and graph traces of surfaces in space.
- represent vectors and find dot products of and angles between vectors in space.
- find cross products of vectors in space and use geometric properties of the cross product.
- use triple scalar products to find volumes of parallelepipeds.
- find parametric and symmetric equations of lines in space.
- find distances between points and planes in space.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- solid analytic geometry (p. 770)
- three-dimensional coordinate system (p. 770)
- *xy*-plane (p. 770)
- *xz*-plane (p. 770)
- *yz*-plane (p. 770)
- octants (p. 770)
- Distance Formula in Space (p. 771)
- Midpoint Formula in Space (p. 771)
- sphere (p. 772)
- surface in space (p. 774)
- trace (p. 774)
- zero vector in space (p. 777)
- standard unit vector notation in space (p. 777)
- component form in space (p. 777)
- angle between two nonzero vectors in space (p. 778)
- orthogonal vectors in space (p. 778)
- parallel vectors in space (p. 779)
- cross product of two vectors in space (p. 784)
- triple scalar product (p. 788)
- direction vector (p. 791)
- direction numbers (p. 791)
- symmetric equations (p. 791)
- angle between two planes (p. 794)
- distance between a plane and a point (p. 797)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

11.1 The Three-Dimensional Coordinate System

The Three-Dimensional Coordinate System

Recall that the Cartesian plane is determined by two perpendicular number lines called the x-axis and the y-axis. These axes, together with their point of intersection (the origin), allow you to develop a two-dimensional coordinate system for identifying points in a plane. To identify a point in space, you must introduce a third dimension to the model. The geometry of this three-dimensional model is called **solid analytic geometry.**

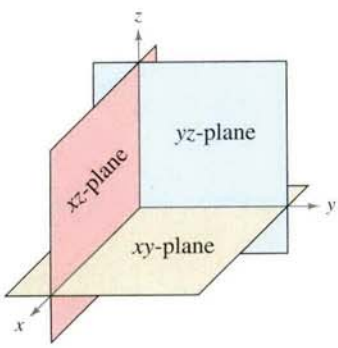


Figure 11.1

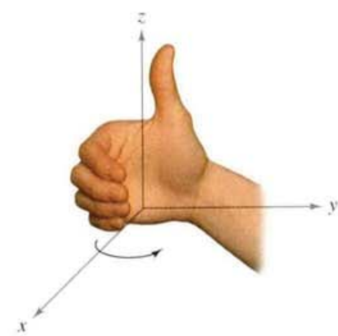


Figure 11.2

You can construct a **three-dimensional coordinate system** by passing a z-axis perpendicular to both the x- and y-axes at the origin. Figure 11.1 shows the positive portion of each coordinate axis. Taken as pairs, the axes determine three **coordinate planes:** the ***xy*-plane,** the ***xz*-plane,** and the ***yz*-plane.** These three coordinate planes separate the three-dimensional coordinate system into eight **octants.** The first octant is the one for which all three coordinates are positive. In this three-dimensional system, a point P in space is determined by an ordered triple (x, y, z), where x, y, and z are as follows.

$x =$ directed distance from yz-plane to P

$y =$ directed distance from xz-plane to P

$z =$ directed distance from xy-plane to P

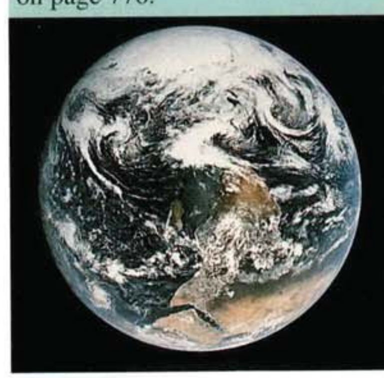
A three-dimensional coordinate system can have either a **left-handed** or a **right-handed** orientation. In this text, you will work exclusively with right-handed systems, as illustrated in Figure 11.2. In a right-handed system, Octants II, III, and IV are found by rotating counterclockwise around the positive z-axis. Octant V is vertically below Octant I. Octants VI, VII, and VIII are then found by rotating counterclockwise around the negative z-axis.

What You Should Learn:

- How to plot points in the three-dimensional coordinate system
- How to find distances between points in space
- How to find midpoints of line segments joining points in space
- How to write equations of spheres in standard form
- How to find traces of surfaces in space

Why You Should Learn It:

The three-dimensional coordinate system can be used to graph equations that model surfaces in space, such as the spherical shape of earth, as shown in Exercise 54 on page 776.



NASA

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 1 Plotting Points in Space

Plot the following points in space.

a. $(2, -3, 3)$ **b.** $(-2, 6, 2)$

c. $(1, 4, 0)$ **d.** $(2, 2, -3)$

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Solution

To plot the point $(2, -3, 3)$, notice that $x = 2$, $y = -3$, and $z = 3$. To help visualize the point, locate the point $(2, -3)$ in the xy-plane (denoted by a cross in Figure 11.3). The point $(2, -3, 3)$ lies three units above the cross. The other three points are also shown in Figure 11.3.

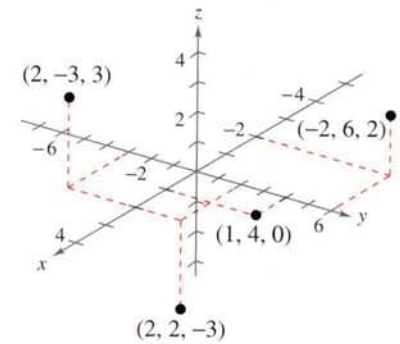


Figure 11.3

The Distance and Midpoint Formulas

Many of the formulas established for the two-dimensional coordinate system can be extended to three dimensions. For example, to find the distance between two points in space, you can use the Pythagorean Theorem twice, as shown in Figure 11.4. By doing this, you will obtain the formula for the distance between two points in space.

Distance Formula in Space

The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) given by the **Distance Formula in Space** is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

EXAMPLE 2 Finding the Distance Between Two Points in Space

Find the distance between $(1, 0, 2)$ and $(2, 4, -3)$.

Solution

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} && \text{Distance Formula in Space} \\ &= \sqrt{(2 - 1)^2 + (4 - 0)^2 + (-3 - 2)^2} && \text{Substitute.} \\ &= \sqrt{1 + 16 + 25} && \text{Simplify.} \\ &= \sqrt{42} && \text{Simplify.} \end{aligned}$$

Notice the similarity between the Distance Formulas in the plane and in space. The Midpoint Formulas in the plane and in space are also similar.

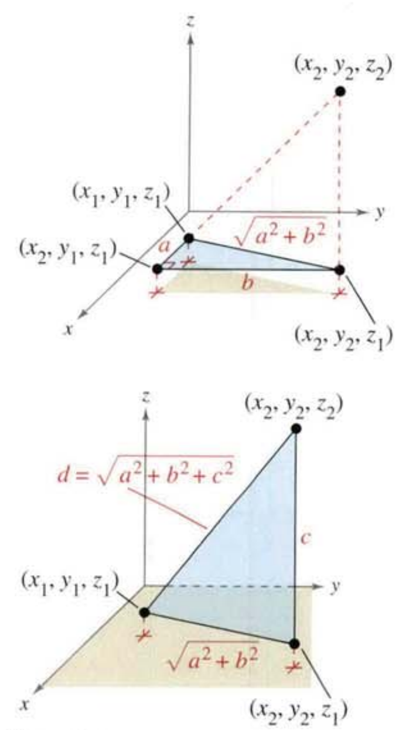


Figure 11.4

Midpoint Formula in Space

The midpoint of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) given by the **Midpoint Formula in Space** is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

EXAMPLE 3 Using the Midpoint Formula in Space

Find the midpoint of the line segment joining $(5, -2, 3)$ and $(0, 4, 4)$.

Solution

Using the Midpoint Formula, the midpoint is

$$\left(\frac{5+0}{2}, \frac{-2+4}{2}, \frac{3+4}{2}\right) = \left(\frac{5}{2}, 1, \frac{7}{2}\right)$$

as shown in Figure 11.5.

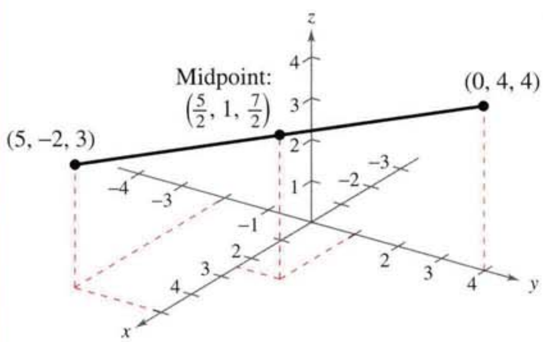


Figure 11.5

The *Interactive* CD-ROM and *Internet* versions of this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

The Equation of a Sphere

A **sphere** with center at (h, k, j) and radius r is defined as the set of all points (x, y, z) such that the distance between (x, y, z) and (h, k, j) is r, as shown in Figure 11.6. Using the Distance Formula, this condition can be written as

$$\sqrt{(x-h)^2 + (y-k)^2 + (z-j)^2} = r.$$

By squaring both sides of this equation, you obtain the standard equation of a sphere.

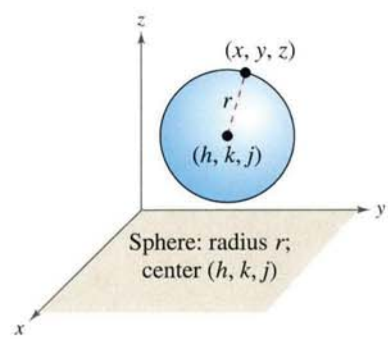


Figure 11.6

Standard Equation of a Sphere

The **standard equation of a sphere** whose center is (h, k, j) and whose radius is r is

$$(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2.$$

Notice the similarity of this formula to the equation of a circle in the plane.

$$(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2 \qquad \text{Equation of sphere in space}$$

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{Equation of circle in the plane}$$

As is true with the equation of a circle, the equation of a sphere is simplified when the center lies at the origin. In this case, the equation is

$$x^2 + y^2 + z^2 = r^2. \qquad \text{Sphere with center at origin}$$

EXAMPLE 4 Finding the Equation of a Sphere

Find the standard equation for the sphere whose center is $(2, 4, 3)$ and whose radius is 3. Does this sphere intersect the xy-plane?

Solution

$$(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2 \qquad \text{Standard equation}$$

$$(x - 2)^2 + (y - 4)^2 + (z - 3)^2 = 3^2 \qquad \text{Substitute.}$$

From the graph shown in Figure 11.7, you can see that the center of the sphere lies three units above the xy-plane. Because the sphere has a radius of 3, you can conclude that it does intersect the xy-plane—at the point $(2, 4, 0)$.

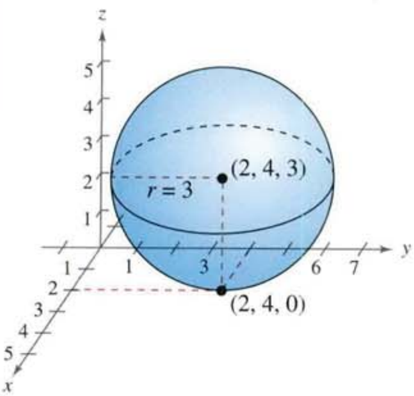


Figure 11.7

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

Exploration

Find the equation of the sphere that has the points $(3, -2, 6)$ and $(-1, 4, 2)$ as endpoints of a diameter. Explain how this problem gives you a chance to use all three formulas discussed so far in this section: the Distance Formula in Space, the Midpoint Formula in Space, and the standard equation of a sphere.

EXAMPLE 5 Finding the Center and Radius of a Sphere

Find the center and radius of the sphere whose equation is

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 8 = 0.$$

Solution

You can obtain the standard equation of this sphere by completing the square, as follows.

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 8 = 0$$

$$(x^2 - 2x + \quad) + (y^2 + 4y + \quad) + (z^2 - 6z + \quad) = -8$$

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) + (z^2 - 6z + 9) = -8 + 1 + 4 + 9$$

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 6$$

So, the center of the sphere is $(1, -2, 3)$, and its radius is $\sqrt{6}$, as shown in Figure 11.8.

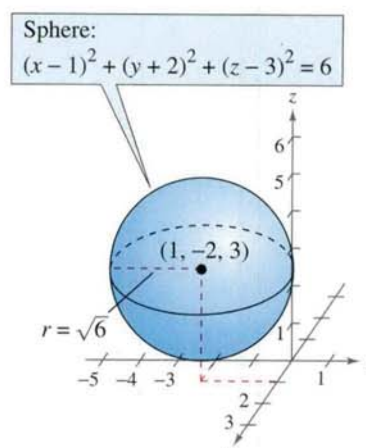


Figure 11.8

Note in Example 5 that the points satisfying the equation of the sphere are "surface points," not "interior points." In general, the collection of points satisfying an equation involving x, y, and z is called a **surface in space.**

Finding the intersection of a surface with one of the three coordinate planes (or with a plane parallel to one of the three coordinate planes) helps one visualize the surface. Such an intersection is called a **trace** of the surface. For example, the xy-trace of a surface consists of all points that are common to both the surface *and* the xy-plane. Similarly, the xz-trace of a surface consists of all points that are common to both the surface and xz-plane.

A computer animation of this example appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 6 Finding a Trace of a Surface

Sketch the xy-trace of the sphere whose equation is

$$(x-3)^2+(y-2)^2+(z+4)^2=5^2.$$

Solution

To find the xy-trace of this surface, use the fact that every point in the xy-plane has a z-coordinate of zero. This means that if you substitute $z = 0$ into the given equation, the resulting equation will represent the intersection of the surface with the xy-plane.

$$\begin{aligned}(x-3)^2+(y-2)^2+(0+4)^2&=25\\(x-3)^2+(y-2)^2+16&=25\\(x-3)^2+(y-2)^2&=9\\(x-3)^2+(y-2)^2&=3^2\end{aligned}$$

From this form, you can see that the xy-trace is a circle of radius 3, as shown in Figure 11.9.

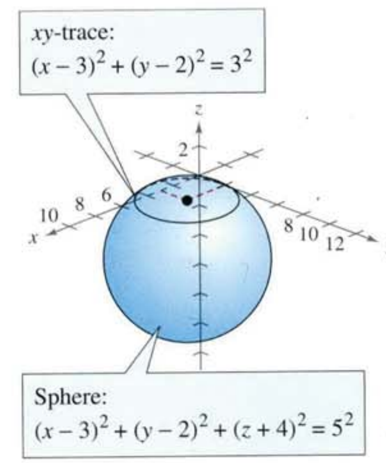


Figure 11.9

Most three-dimensional graphing utilities represent surfaces by sketching several traces of the surface. The traces are usually taken in equally spaced parallel planes.

To sketch the graph of an equation involving x, y, and z with a three-dimensional "function grapher," you must first solve the equation for z. After entering the equation(s), you need to specify a rectangular viewing cube (the three-dimensional analog of a viewing window). Consult your user's manual for instructions.

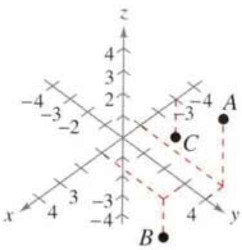
Writing About Math *Comparing Two and Three Dimensions*

In this section, you saw similarities between formulas in two-dimensional coordinate geometry and three-dimensional coordinate geometry. In two-dimensional coordinate geometry, the graph of the equation $ax + by + c = 0$ is a line. In three-dimensional coordinate geometry, what is the graph of the equation $ax + by + cz = 0$? Is it a line? Write a short paragraph explaining your reasoning.

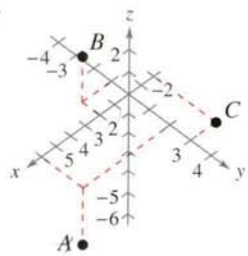
11.1 Exercises

In Exercises 1 and 2, approximate the coordinates of the points.

1.



2.



In Exercises 3–6, plot the points in the same three-dimensional coordinate system.

3. (a) $(2, 1, 3)$ (b) $(-1, 2, 1)$

4. (a) $(1, -2, 4)$ (b) $(2, 4, -2)$

5. (a) $(5, -1, 2)$ (b) $(5, -2, -2)$

6. (a) $(0, 4, -3)$ (b) $(4, 0, 4)$

In Exercises 7–10, find the coordinates of the point.

7. The point is located three units behind the yz-plane, three units to the right of the xz-plane, and four units above the xy-plane.

8. The point is located six units in front of the yz-plane, one unit to the left of the xz-plane, and one unit below the xy-plane.

9. The point is located on the x-axis, 12 units in front of the yz-plane.

10. The point is located in the yz-plane, four units to the right of the xz-plane, and three units above the xy-plane.

In Exercises 11–16, determine the octant(s) in which (x, y, z) is located so that the condition(s) is (are) satisfied.

11. $x > 0, y < 0, z > 0$

12. $x < 0, y > 0, z < 0$

13. $z > 0$

14. $y < 0$

15. $xy < 0$

16. $yz > 0$

In Exercises 17–22, find the distance between the indicated points.

17. $(3, 2, 5)$ and $(7, 4, 8)$

18. $(4, 1, 9)$ and $(2, 1, 6)$

19. $(-1, 4, -2)$ and $(5, -6, 2)$

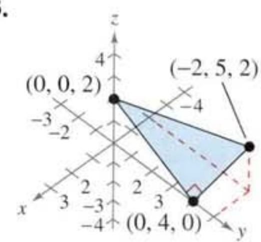
20. $(1, 1, -7)$ and $(-2, -3, -7)$

21. $(0, -3, 0)$ and $(1, 0, -10)$

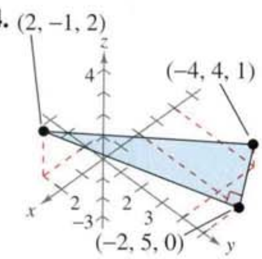
22. $(2, -4, 0)$ and $(0, 6, -3)$

In Exercises 23 and 24, find the lengths of the sides of the right triangle. Show that these lengths satisfy the Pythagorean Theorem.

23.



24.



In Exercises 25 and 26, find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.

25. $(1, -3, -2), (5, -1, 2), (-1, 1, 2)$

26. $(5, 3, 4), (7, 1, 3), (3, 5, 3)$

In Exercises 27–32, find the coordinates of the midpoint of the line segment joining the points.

27. $(3, -6, 10), (-3, 4, 4)$

28. $(2, -2, -8), (4, 4, 16)$

29. $(6, -2, 5), (-4, 2, 6)$

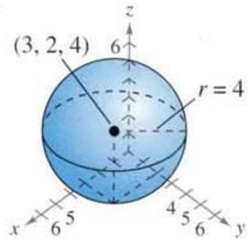
30. $(-3, 5, 5), (-6, 4, 8)$

31. $(-2, 8, 10), (7, -4, 2)$

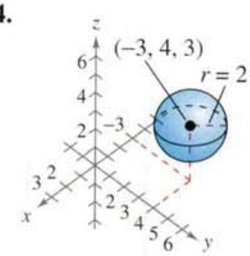
32. $(9, -5, 1), (9, -2, -4)$

In Exercises 33–40, find the standard form of the equation of the sphere.

33.



34.



35. Center: $(0, 4, 3)$; Radius: 3

The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

36. Center: $(1, -2, 3)$; Radius: 5

37. Center: $(-3, 7, 5)$; Diameter: 10

38. Center: $(0, 5, -9)$; Diameter: 8

39. Endpoints of a diameter: $(3, 0, 0)$, $(0, 0, 6)$

40. Endpoints of a diameter: $(2, -2, 2)$, $(-1, 4, 6)$

In Exercises 41–46, find the center and radius of the sphere.

41. $x^2 + y^2 + z^2 - 4x + 2y - 6z + 10 = 0$

42. $x^2 + y^2 + z^2 - 6x + 4y + 9 = 0$

43. $x^2 + y^2 + z^2 + 4x - 8z + 19 = 0$

44. $x^2 + y^2 + z^2 - 8y - 6z + 13 = 0$

45. $9x^2 + 9y^2 + 9z^2 - 18x - 6y - 72z + 73 = 0$

46. $2x^2 + 2y^2 + 2z^2 - 2x - 6y - 4z + 5 = 0$

In Exercises 47–50, sketch the graph of the equation and sketch the specified traces.

47. $(x - 1)^2 + y^2 + z^2 = 36$; xz-trace

48. $x^2 + (y + 3)^2 + z^2 = 25$; yz-trace

49. $(x + 2)^2 + (y - 3)^2 + z^2 = 9$; yz-trace

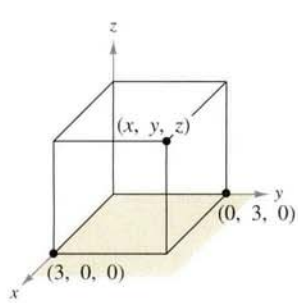
50. $x^2 + (y - 1)^2 + (z + 1)^2 = 4$; xy-trace

In Exercises 51 and 52, use a three-dimensional graphing utility to graph the sphere.

51. $x^2 + y^2 + z^2 - 6x - 8y - 10z + 46 = 0$

52. $x^2 + y^2 + z^2 + 6y - 8z + 21 = 0$

53. ***Crystals*** Crystals are classified according to their symmetry. Crystals shaped like cubes are classified as isometric. Suppose you have mapped the vertices of a crystal onto a three-dimensional coordinate system. Determine (x, y, z) if the crystal is isometric.



54. ***Earth*** Assume that earth is a sphere with a radius of 3963 miles. If the center of earth is placed at the origin of a three-dimensional coordinate system, what is the equation of the sphere? Lines of longitude that run north–south could be represented by what trace(s)? What shape would each of these traces form? Lines of latitude that run east–west could be represented by what trace(s)? What shape would each of these traces form?

Synthesis

True or False? **In Exercises 55 and 56, determine whether the statement is true or false. Justify your answer.**

55. In the ordered triple (x, y, z) that represents point P in space, x is the directed distance from the xy-plane to P.

56. The surface consisting of all points (x, y, z) in space that are the same distance r from the point (h, j, k) has a circle as its xy-trace.

57. ***Think About It*** What is the z-coordinate of any point in the xy-plane? What is the y-coordinate of any point in the xz-plane? What is the x-coordinate of any point in the yz-plane?

58. A sphere intersects the yz-plane. Describe the trace.

59. A plane intersects the xy-plane. Describe the trace.

60. A line segment has (x_1, y_1, z_1) as one endpoint and (x_m, y_m, z_m) as its midpoint. Find the other endpoint (x_2, y_2, z_2) of the line segment in terms of x_1, y_1, z_1, x_m, y_m, and z_m.

61. Use the result of Exercise 60 to find the coordinates of the endpoint of a line segment if the coordinates of the other endpoint and the midpoint are $(3, 0, 2)$ and $(5, 8, 7)$, respectively.

Review

In Exercises 62–67, find the standard form of the equation of the conic.

62. Parabola: Vertex: $(4, 1)$; Focus: $(1, 1)$

63. Parabola: Vertex: $(-2, 5)$; Focus: $(-2, 0)$

64. Ellipse: Vertices: $(0, 3)$, $(6, 3)$;
Minor axis of length 4

65. Ellipse: Foci: $(0, 0)$, $(0, 6)$; Major axis of length 9

66. Hyperbola: Vertices: $(4, 0)$, $(8, 0)$;
Foci: $(0, 0)$, $(12, 0)$

67. Hyperbola: Vertices: $(3, 1)$, $(3, 9)$;
Foci: $(3, 0)$, $(3, 10)$

11.2 Vectors in Space

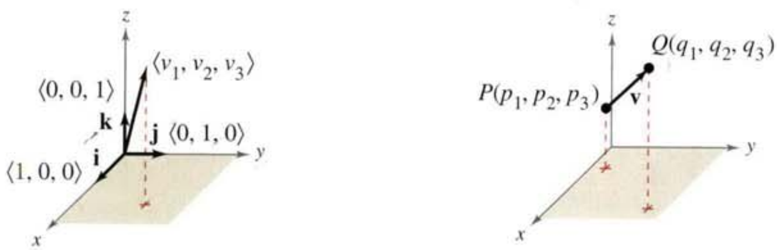
Vectors in Space

Physical forces and velocities are not confined to the plane, so it is natural to extend the concept of vectors from two-dimensional space to three-dimensional space. In space, vectors are denoted by ordered triples

$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$ Component form

The **zero vector** is denoted by $\mathbf{0} = \langle 0, 0, 0 \rangle$. Using the unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ in the direction of the positive z-axis, the **standard unit vector notation** for $\mathbf{v}$ shown in Figure 11.10(a) is

$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$ Unit vector form



(a) (b)

Figure 11.10

If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, as shown in Figure 11.10(b), the **component form** of $\mathbf{v}$ is produced by subtracting the coordinates of the initial point from the coordinates of the terminal point

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle.$$

Vectors in Space

1. Two vectors are equal if and only if their corresponding components are equal.
2. The magnitude (or length) of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.
3. A unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$, $\mathbf{v} \neq \mathbf{0}$.
4. The sum of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

   $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle.$ Vector addition
5. The scalar multiple of the real number c and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is

   $c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle.$ Scalar multiplication
6. The dot product of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

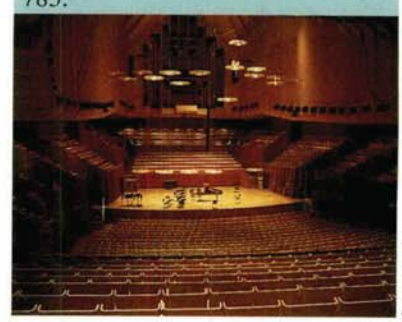
   $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$ Dot product

What You Should Learn:

- How to find the component form, the unit vector in the same direction, and magnitude of vectors in space
- How to find dot products of and angles between vectors in space
- How to determine whether vectors in space are parallel or orthogonal
- How to use vectors in space to solve real-life problems

Why You Should Learn It:

Vectors in space can be used to represent many physical forces, such as tension in the wires used to support auditorium lights, as shown in Exercise 49 on page 783.



Superstock

EXAMPLE 1 Finding the Component Form of a Vector

Find the component form and length of the vector **v** having initial point (3, 4, 2) and terminal point (3, 6, 4). Then find a unit vector in the direction of **v**.

Solution

The component form of **v** is

$$\mathbf{v} = \langle 3 - 3, 6 - 4, 4 - 2 \rangle = \langle 0, 2, 2 \rangle$$

which implies that its length is

$$\|\mathbf{v}\| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

The unit vector in the direction of **v** is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2\sqrt{2}}\langle 0, 2, 2 \rangle = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

EXAMPLE 2 Finding the Dot Product of Two Vectors

Find the dot product of $\langle 0, 3, -2 \rangle$ and $\langle 4, -2, 3 \rangle$.

Solution

$$\begin{aligned}\langle 0, 3, -2 \rangle \cdot \langle 4, -2, 3 \rangle &= 0(4) + 3(-2) + (-2)(3) \\ &= 0 - 6 - 6 \\ &= -12\end{aligned}$$

Note that the dot product of two vectors is a real number, not a vector.

STUDY TIP

Some graphing utilities have the capability to perform vector operations, such as the dot product. Consult your user's manual for instructions.

As was discussed in Section 6.4, the **angle between two nonzero vectors** is the angle θ, $0 \le \theta \le \pi$, between its respective standard position vectors. (See Figure 11.11.) This angle can be found using the dot product. (Note that the angle between the zero vector and another vector is not defined.)

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.

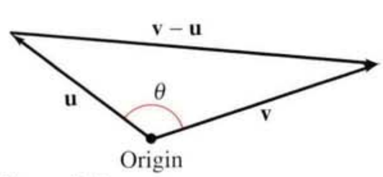


Figure 11.11

Angle Between Two Vectors

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

If the dot product of two nonzero vectors is zero, the angle between the vectors is 90°. Such vectors are called **orthogonal.** For instance, the standard unit vectors **i**, **j**, and **k** are orthogonal to each other.

EXAMPLE 3 Finding the Angle Between Two Vectors

Find the angle between $\mathbf{u} = \langle 1, 0, 2 \rangle$ and $\mathbf{v} = \langle 3, 1, 0 \rangle$.

Solution

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\langle 1, 0, 2\rangle \cdot \langle 3, 1, 0\rangle}{\|\langle 1, 0, 2\rangle\|\,\|\langle 3, 1, 0\rangle\|} = \frac{3}{\sqrt{50}}$$

This implies that the angle between the two vectors is

$$\theta = \arccos\frac{3}{\sqrt{50}}$$

$$\approx 64.9^\circ$$

as shown in Figure 11.12.

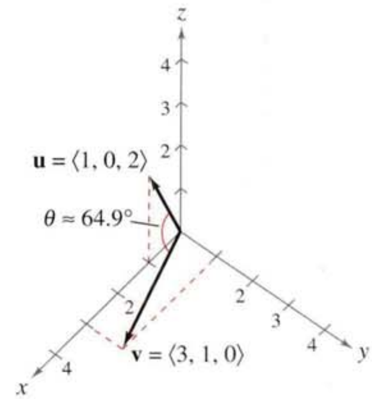


Figure 11.12

Parallel Vectors

Recall from the definition of scalar multiplication that positive scalar multiples of a nonzero vector $\mathbf{v}$ have the same direction as $\mathbf{v}$, whereas negative multiples have the direction opposite that of $\mathbf{v}$. In general, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if there is some scalar c such that $\mathbf{u} = c\mathbf{v}$. For example, in Figure 11.13, the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are parallel because $\mathbf{u} = 2\mathbf{v}$ and $\mathbf{w} = -\mathbf{v}$.

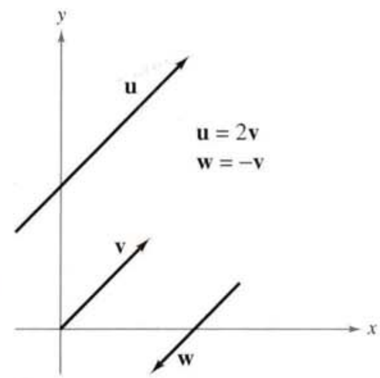


Figure 11.13

EXAMPLE 4 Parallel Vectors

Vector $\mathbf{w}$ has initial point $(1, -2, 0)$ and terminal point $(3, 2, 1)$. Which of the following vectors is parallel to $\mathbf{w}$?

a. $\mathbf{u} = \langle 4, 8, 2 \rangle$

b. $\mathbf{v} = \langle 4, 8, 4 \rangle$

Solution

Begin by writing $\mathbf{w}$ in component form.

$$\mathbf{w} = \langle 3 - 1, 2 - (-2), 1 - 0 \rangle$$

$$= \langle 2, 4, 1 \rangle$$

a. The vector $\mathbf{u}$ is parallel to $\mathbf{w}$ because

$$\mathbf{u} = \langle 4, 8, 2 \rangle$$

$$= 2\langle 2, 4, 1 \rangle$$

$$= 2\mathbf{w}.$$

b. In this case, you need to find a scalar c such that

$$\langle 4, 8, 4 \rangle = c\langle 2, 4, 1 \rangle.$$

However, equating corresponding components produces $c = 2$ for the first two components and $c = 4$ for the third. So, the equation has no solution, and the vectors $\mathbf{v}$ and $\mathbf{w}$ are not parallel.

You can use vectors to determine whether three points are collinear (lie on the same line). The points P, Q, and R are collinear if and only if the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel.

EXAMPLE 5 Using Vectors to Determine Collinear Points

Determine whether the following points lie on the same line.

$$P(2, -1, 4), \qquad Q(5, 4, 6), \qquad \text{and} \qquad R(-4, -11, 0)$$

Solution

The component forms of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \langle 5 - 2, 4 - (-1), 6 - 4 \rangle = \langle 3, 5, 2 \rangle$$

and

$$\overrightarrow{PR} = \langle -4 - 2, -11 - (-1), 0 - 4 \rangle = \langle -6, -10, -4 \rangle.$$

Because $\overrightarrow{PR} = -2\overrightarrow{PQ}$, you can conclude that they are parallel. Therefore, the points P, Q, and R lie on the same line, as shown in Figure 11.14.

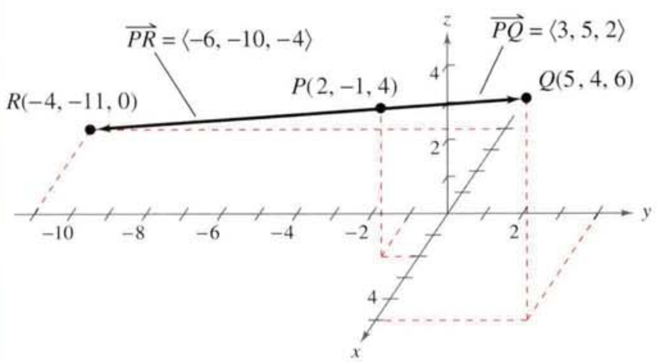


Figure 11.14

EXAMPLE 6 Finding the Terminal Point of a Vector

The initial point of the vector $\mathbf{v} = \langle 4, 2, -1 \rangle$ is $P(3, -1, 6)$. What is the terminal point of this vector?

Solution

Using the component form of the vector whose initial point is P and whose terminal point is Q, you can write

$$\begin{aligned}\overrightarrow{PQ} &= \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle \\ &= \langle q_1 - 3, q_2 + 1, q_3 - 6 \rangle \\ &= \langle 4, 2, -1 \rangle.\end{aligned}$$

This implies that $q_1 - 3 = 4$, $q_2 + 1 = 2$, and $q_3 - 6 = -1$. The solutions of these three equations are

$$q_1 = 7, \qquad q_2 = 1, \qquad \text{and} \qquad q_3 = 5.$$

So, the terminal point is $Q(7, 1, 5)$.

Application

In Section 6.3, you saw how to use vectors to solve an equilibrium problem in a plane. The next example shows how to use vectors to solve an equilibrium problem in space.

EXAMPLE 7 Solving an Equilibrium Problem

A weight of 480 pounds is supported by three ropes. As shown in Figure 11.15, the weight is located at $S(0, 2, -1)$. The ropes are tied to the points $P(2, 0, 0)$, $Q(0, 4, 0)$, and $R(-2, 0, 0)$. Find the force (or tension) on each rope.

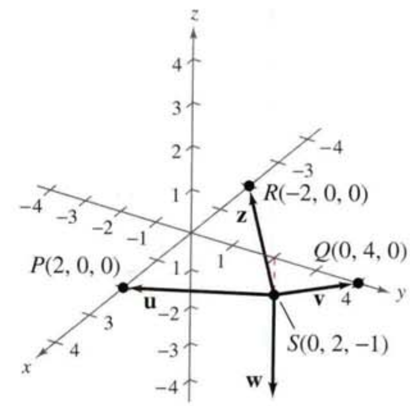


Figure 11.15

Solution

The (downward) force of the weight is represented by the vector

$$\mathbf{w} = \langle 0, 0, -480 \rangle.$$

The force vectors corresponding to the ropes are as follows.

$$\mathbf{u} = \|\mathbf{u}\| \frac{\overrightarrow{SP}}{\|\overrightarrow{SP}\|} = \|\mathbf{u}\| \frac{\langle 2-0, 0-2, 0-(-1)\rangle}{3} = \|\mathbf{u}\| \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

$$\mathbf{v} = \|\mathbf{v}\| \frac{\overrightarrow{SQ}}{\|\overrightarrow{SQ}\|} = \|\mathbf{v}\| \frac{\langle 0-0, 4-2, 0-(-1)\rangle}{\sqrt{5}} = \|\mathbf{v}\| \left\langle 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$\mathbf{z} = \|\mathbf{z}\| \frac{\overrightarrow{SR}}{\|\overrightarrow{SR}\|} = \|\mathbf{z}\| \frac{\langle -2-0, 0-2, 0-(-1)\rangle}{3} = \|\mathbf{z}\| \left\langle -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

For the system to be in equilibrium, it must be true that

$$\mathbf{u} + \mathbf{v} + \mathbf{z} + \mathbf{w} = \mathbf{0}$$

or

$$\mathbf{u} + \mathbf{v} + \mathbf{z} = -\mathbf{w}.$$

This yields the following system of linear equations.

$$\begin{cases} \frac{2}{3}\|\mathbf{u}\| \qquad\qquad\quad - \frac{2}{3}\|\mathbf{z}\| = 0 \\ -\frac{2}{3}\|\mathbf{u}\| + \frac{2}{\sqrt{5}}\|\mathbf{v}\| - \frac{2}{3}\|\mathbf{z}\| = 0 \\ \frac{1}{3}\|\mathbf{u}\| + \frac{1}{\sqrt{5}}\|\mathbf{v}\| + \frac{1}{3}\|\mathbf{z}\| = 480 \end{cases}$$

The solution of this system is

$$\|\mathbf{u}\| = 360.0$$

$$\|\mathbf{v}\| \approx 536.7$$

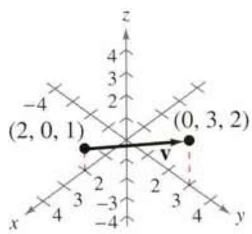
$$\|\mathbf{z}\| = 360.0.$$

So, the rope attached at point P has 360 pounds of tension, the rope attached at point Q has about 536.7 pounds of tension, and the rope attached at point R has 360 pounds of tension.

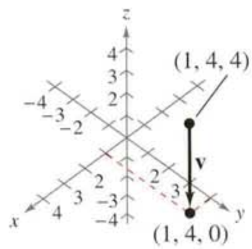
11.2 Exercises

In Exercises 1 and 2, (a) find the component form of the vector v and (b) sketch the vector with its initial point at the origin.

1.



2.



In Exercises 3 and 4, (a) write the component form of the vector v, (b) find the length of v, and (c) find a unit vector in the direction of v.

3. Initial point of $\mathbf{v}$: $(-1, -2, 1)$
Terminal point of $\mathbf{v}$: $(3, 2, 5)$

4. Initial point of $\mathbf{v}$: $(-4, 5, 5)$
Terminal point of $\mathbf{v}$: $(4, 0, 0)$

In Exercises 5 and 6 sketch each scalar multiple of v.

5. $\mathbf{v} = \langle 1, 2, 2 \rangle$
(a) $2\mathbf{v}$ (b) $-\mathbf{v}$ (c) $\frac{3}{2}\mathbf{v}$ (d) $0\mathbf{v}$

6. $\mathbf{v} = \langle 2, -2, 1 \rangle$
(a) $-\mathbf{v}$ (b) $2\mathbf{v}$ (c) $\frac{1}{2}\mathbf{v}$ (d) $\frac{5}{2}\mathbf{v}$

In Exercises 7–10, find the vector z, given $\mathbf{u} = \langle -1, 3, 2 \rangle$, $\mathbf{v} = \langle 1, -2, -2 \rangle$, and $\mathbf{w} = \langle 5, 0, -5 \rangle$. Use a graphing utility to verify your answer.

7. $\mathbf{z} = \mathbf{u} - 2\mathbf{v}$

8. $\mathbf{z} = -7\mathbf{u} + \mathbf{v} - \frac{1}{5}\mathbf{w}$

9. $2\mathbf{z} - 4\mathbf{u} = \mathbf{w}$

10. $\mathbf{u} + \mathbf{v} - 2\mathbf{w} + \mathbf{z} = \mathbf{0}$

In Exercises 11–16, find the magnitude of v.

11. $\mathbf{v} = \langle 4, 1, 4 \rangle$

12. $\mathbf{v} = \langle -2, 0, -5 \rangle$

13. $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$

14. $\mathbf{v} = -\mathbf{i} + 4\mathbf{j} - 2\mathbf{j}$

15. Initial point of $\mathbf{v}$: $(1, -3, 4)$
Terminal point of $\mathbf{v}$: $(1, 0, -1)$

16. Initial point of $\mathbf{v}$: $(0, -1, 0)$
Terminal point of $\mathbf{v}$: $(1, 2, -2)$

In Exercises 17 and 18, find a unit vector (a) in the direction of u and (b) in the direction opposite of u.

17. $\mathbf{u} = 8\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

18. $\mathbf{u} = -3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$

In Exercises 19–24, use a graphing utility to determine the specified quantity where $\mathbf{u} = \langle -1, 3, 4 \rangle$ and $\mathbf{v} = \langle 5, 4.5, -6 \rangle$.

19. $6\mathbf{u} - 4\mathbf{v}$

20. $2\mathbf{u} + \frac{5}{2}\mathbf{v}$

21. $\|\mathbf{u} + \mathbf{v}\|$

22. $\|\mathbf{u} - \mathbf{v}\|$

23. $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

24. $\dfrac{\mathbf{u}}{\|\mathbf{u}\|}$

In Exercises 25–28, find the dot product of u and v. Use a graphing utility to verify your work.

25. $\mathbf{u} = \langle 4, 4, -1 \rangle$
$\mathbf{v} = \langle 2, -5, -8 \rangle$

26. $\mathbf{u} = \langle 0, -6, 6 \rangle$
$\mathbf{v} = \langle 12, -1, -2 \rangle$

27. $\mathbf{u} = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$
$\mathbf{v} = 9\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

28. $\mathbf{u} = 3\mathbf{j} - 6\mathbf{k}$
$\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$

In Exercises 29–32, find the angle θ between the two vectors.

29. $\mathbf{u} = \langle 0, 2, 2 \rangle$
$\mathbf{v} = \langle 3, 0, -4 \rangle$

30. $\mathbf{u} = \langle -1, 3, 0 \rangle$
$\mathbf{v} = \langle 1, 2, -1 \rangle$

31. $\mathbf{u} = 10\mathbf{i} + 40\mathbf{j}$
$\mathbf{v} = -3\mathbf{j} + 8\mathbf{k}$

32. $\mathbf{u} = 8\mathbf{j} - 20\mathbf{k}$
$\mathbf{v} = 10\mathbf{i} - 5\mathbf{k}$

In Exercises 33–36, determine whether u and v are orthogonal, parallel, or neither. Use a graphing utility to verify your answer.

33. $\mathbf{u} = \langle -12, 6, 15 \rangle$
$\mathbf{v} = \langle 8, -4, -10 \rangle$

34. $\mathbf{u} = \langle -1, 3, -1 \rangle$
$\mathbf{v} = \langle 2, -1, 5 \rangle$

35. $\mathbf{u} = \frac{3}{4}\mathbf{i} - \frac{1}{2}\mathbf{j} + 2\mathbf{k}$
$\mathbf{v} = 4\mathbf{i} + 10\mathbf{j} + \mathbf{k}$

36. $\mathbf{u} = -\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$
$\mathbf{v} = 8\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$

In Exercises 37–40, use vectors to determine whether the points lie in a straight line.

37. $(5, 4, 1), (7, 3, -1), (4, 5, 3)$

38. $(-2, 7, 4), (-4, 8, 1), (0, 6, 7)$

39. $(1, 3, 2), (-1, 2, 5), (3, 4, -1)$

40. $(0, 4, 4), (-1, 5, 6), (-2, 6, 7)$

In Exercises 41–44, the vector v and its initial point are given. Find the terminal point.

41. $\mathbf{v} = \langle 2, -4, 7 \rangle$
Initial point: $(1, 5, 0)$

42. $\mathbf{v} = \langle 4, -1, -1 \rangle$
Initial point: $(-1, 3, 2)$

43. $\mathbf{v} = \left\langle 4, \frac{3}{2}, -\frac{1}{4} \right\rangle$
Initial point: $\left(2, 1, -\frac{3}{2}\right)$

44. $\mathbf{v} = \left\langle \frac{5}{2}, -\frac{1}{2}, 4 \right\rangle$
Initial point: $\left(3, 2, -\frac{1}{2}\right)$

45. Determine the values of c such that $\|c\mathbf{u}\| = 3$ where $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

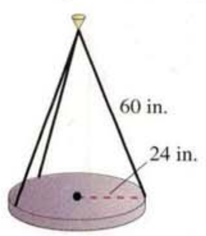
46. Determine the values of c such that $\|c\mathbf{u}\| = 12$ where $\mathbf{u} = -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

In Exercises 47 and 48, write the component form of v.

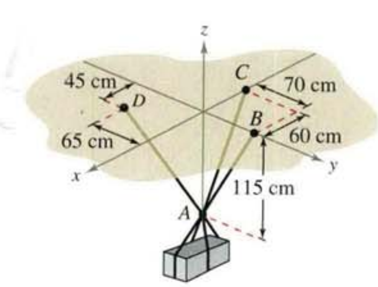
47. $\mathbf{v}$ lies in the yz-plane, has magnitude 4, and makes an angle of 45° with the positive y-axis.

48. $\mathbf{v}$ lies in the xz-plane, has magnitude 10, and makes an angle of 60° with the positive z-axis.

49. ***Light Installation*** The lights in an auditorium are 30-pound disks of radius 24 inches. Each disk is supported by three equally spaced 60-inch wires to the ceiling. Find the tension in each wire.



50. ***Load Supports*** Find the tension in each of the supporting cables in the figure if the weight of the crate is 500 newtons.



Synthesis

True or False? **In Exercises 51 and 52, determine whether the statement is true or false. Justify your answer.**

51. If the dot product of two nonzero vectors is zero, then the angle between the vectors is a right angle.

52. If AB and AC are parallel vectors, then points A, B, and C are collinear.

53. ***Exploration*** Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$.

(a) Sketch $\mathbf{u}$ and $\mathbf{v}$.

(b) If $\mathbf{w} = \mathbf{0}$, show that a and b must both be zero.

(c) Find a and b such that $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(d) Show that no choice of a and b yields $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

54. ***Think About It*** The initial and terminal points of the vector $\mathbf{v}$ are (x_1, y_1, z_1) and (x, y, z), respectively. Describe the set of all points (x, y, z) such that $\|\mathbf{v}\| = 4$.

55. What is known about the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ if $\mathbf{u} \cdot \mathbf{v} < 0$? Explain.

56. ***Writing*** Consider the two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. Describe the geometric figure generated by the terminal points of vectors $t\mathbf{v}$, $\mathbf{u} + t\mathbf{v}$, and $s\mathbf{u} + t\mathbf{v}$, where s and t represent real numbers.

Review

In Exercises 57–60, sketch the curve represented by the parametric equations (indicate the direction of the curve). Then eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve.

57. $x = 2t - 1$
$y = -t + 3$

58. $x = 4t - 1$
$y = 2t + 1$

59. $x = t - 2$
$y = 2t^2$

60. $x = -t + 1$
$y = t^3$

In Exercises 61 and 62, find the determinant of the matrix.

61. $\begin{bmatrix} 12 & 4 & -1 \\ -2 & 3 & 2 \\ 5 & 8 & 1 \end{bmatrix}$

62. $\begin{bmatrix} 8 & 2 & 9 \\ 12 & 3 & 9 \\ 3 & 13 & 4 \end{bmatrix}$

11.3 The Cross Product of Two Vectors

The Cross Product

Many applications in physics, engineering, and geometry involve finding a vector in space that is orthogonal to two given vectors. In this section you will study a product that will yield such a vector. It is called the **cross product,** and it is most conveniently defined and calculated using the standard unit vector form.

Definition of Cross Product of Two Vectors in Space

Let

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

be vectors in space. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

It is important to note that this definition applies only to three-dimensional vectors. The cross product is not defined for two-dimensional vectors.

A convenient way to calculate $\mathbf{u} \times \mathbf{v}$ is to use the following *determinant form* with cofactor expansion. (This 3×3 determinant form is used simply to help remember the formula for the cross product—it is technically not a determinant because not all the entries of the corresponding matrix are real numbers.)

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{array}{l} \\ \Leftarrow \text{Put } \mathbf{u} \text{ in Row 2.} \\ \Leftarrow \text{Put } \mathbf{v} \text{ in Row 3.} \end{array}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Note the minus sign in front of the $\mathbf{j}$-component.

Exploration

Try finding the following cross products. What can you conclude?

a. $\mathbf{i} \times \mathbf{j}$ **b.** $\mathbf{i} \times \mathbf{k}$ **c.** $\mathbf{j} \times \mathbf{k}$

What You Should Learn:

- How to find cross products of vectors in space
- How to use geometric properties of cross products of vectors in space
- How to use triple scalar products to find volumes of parallelepipeds

Why You Should Learn It:

The cross product of two vectors in space has many applications in physics and engineering. For instance, in Exercise 47 on page 790, the cross product is used to find the torque on the crank of a bicycle's brake.

Bob Daemmrich/Tony Stone Images

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 1 Finding Cross Products

Given $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, find the following.

a. $\mathbf{u} \times \mathbf{v}$ **b.** $\mathbf{v} \times \mathbf{u}$ **c.** $\mathbf{v} \times \mathbf{v}$

Solution

a. $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{k}$$

$$= (4 - 1)\mathbf{i} - (2 - 3)\mathbf{j} + (1 - 6)\mathbf{k}$$

$$= 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$$

b. $$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{k}$$

$$= (1 - 4)\mathbf{i} - (3 - 2)\mathbf{j} + (6 - 1)\mathbf{k}$$

$$= -3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

Note that this result is the negative of that in part (a).

c. $$\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{vmatrix} = \mathbf{0}$$

The results obtained in Example 1 suggest some interesting algebraic properties of the cross product. For instance,

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \qquad \text{and} \qquad \mathbf{v} \times \mathbf{v} = \mathbf{0}.$$

These properties, and several others, are summarized in the following list.

Algebraic Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space and let c be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

STUDY TIP

Some graphing utilities have the capability to perform vector operations, such as the cross product. Consult your user's manual for instructions.

The Granger Collection

American physicist **Josiah Willard Gibbs (1839–1903)** developed three-dimensional vector analysis. He introduced the dot product and the cross product.

Geometric Properties of the Cross Product

The first property listed on the previous page indicates that the cross product is *not commutative*. In particular, this property indicates that the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but opposite directions. The following list gives some other *geometric* properties of the cross product of two vectors.

Geometric Properties of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta$.
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples.
4. $\|\mathbf{u} \times \mathbf{v}\|$ = area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

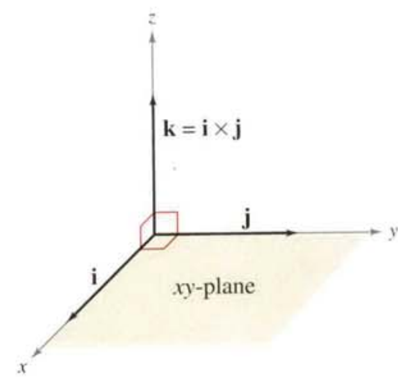


Figure 11.16

Both $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ are perpendicular to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. One way to remember the orientation of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is to compare them with the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, as shown in Figure 11.16. The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a *right-handed system*.

EXAMPLE 2 Using the Cross Product

Find a unit vector that is orthogonal to both

$$\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \mathbf{v} = -3\mathbf{i} + 6\mathbf{j}.$$

Solution

The cross product $\mathbf{u} \times \mathbf{v}$, as shown in Figure 11.17, is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 1 \\ -3 & 6 & 0 \end{vmatrix}$$

$$= -6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$$

Because

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-6)^2 + (-3)^2 + 6^2}$$
$$= \sqrt{81}$$
$$= 9$$

a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

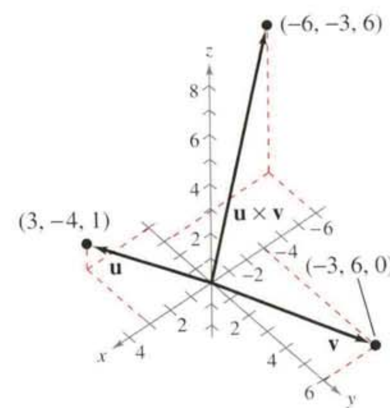


Figure 11.17

In Example 2, note that you could have used the cross product $\mathbf{v} \times \mathbf{u}$ to form a unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. With that choice, you would have obtained the *negative* of the unit vector found in the example.

The fourth geometric property of the cross product states that $\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides. A simple example of this is given by the unit square with adjacent sides of $\mathbf{i}$ and $\mathbf{j}$. Because

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

and $\|\mathbf{k}\| = 1$, it follows that the square has an area of 1. This geometric property of the cross product is illustrated further in the next example.

EXAMPLE 3 Geometric Application of the Cross Product

Show that the quadrilateral with vertices at the following points is a parallelogram. Then find the area of the parallelogram. Is the parallelogram a rectangle?

$$A(5, 2, 0), \qquad B(2, 6, 1), \qquad C(2, 4, 7), \qquad D(5, 0, 6)$$

Solution

From Figure 11.18 you can see that the sides of the quadrilateral correspond to the following four vectors.

$$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

$$\overrightarrow{CD} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} = -\overrightarrow{AB}$$

$$\overrightarrow{AD} = 0\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

$$\overrightarrow{CB} = 0\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -\overrightarrow{AD}$$

Because $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AD}$ is parallel to $\overrightarrow{CB}$, it follows that the quadrilateral is a parallelogram with $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as adjacent sides. Moreover, because

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} = 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$$

the area of the parallelogram is

$$\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{26^2 + 18^2 + 6^2} = \sqrt{1036} \approx 32.19.$$

You can tell whether the parallelogram is a rectangle by finding the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

$$\sin\theta = \frac{\|\overrightarrow{AB} \times \overrightarrow{AD}\|}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AD}\|}$$

$$= \frac{\sqrt{1036}}{\sqrt{26}\sqrt{40}} \approx 0.998$$

$$\theta \approx 86.4°$$

So, because $\theta \neq 90°$, the parallelogram is not a rectangle.

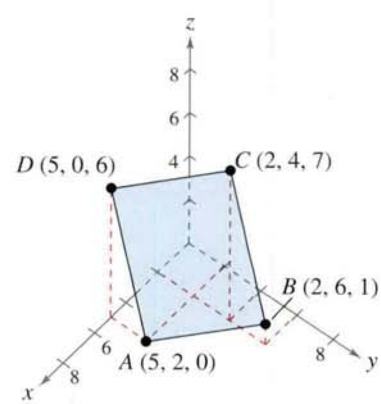


Figure 11.18

The Triple Scalar Product

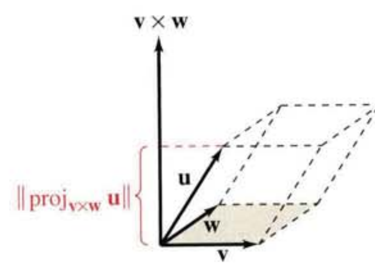
For vectors **u**, **v**, and **w** in space, the dot product of **u** and $\mathbf{v} \times \mathbf{w}$ is called the **triple scalar product** of **u**, **v**, and **w**.

The Triple Scalar Product

The **triple scalar product** of **u**, **v**, and **w** is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

If the vectors **u**, **v**, and **w** do not lie in the same plane, the triple scalar product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ can be used to determine the volume of the parallelepiped with **u**, **v**, and **w** as adjacent edges, as shown in Figure 11.19.



Area of base $= \|\mathbf{v} \times \mathbf{w}\|$
Volume of parallelepiped $= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$
Figure 11.19

Geometric Property of Triple Scalar Product

The volume V of a parallelepiped with vectors **u**, **v**, and **w** as adjacent edges is

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

EXAMPLE 4 Volume by the Triple Scalar Product

Find the volume of the parallelepiped having

$$\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}, \qquad \mathbf{v} = 2\mathbf{j} - 2\mathbf{k}, \qquad \text{and} \qquad \mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$$

as adjacent edges, as shown in Figure 11.20.

Solution

The value of the triple scalar product is

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\
&= 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\
&= 3(4) + 5(6) + 1(-6) \\
&= 36.
\end{aligned}$$

So, the volume of the parallelepiped is

$$\begin{aligned}
|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| &= |36| \\
&= 36.
\end{aligned}$$

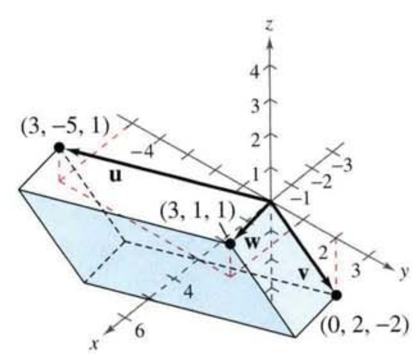


Figure 11.20

11.3 Exercises

In Exercises 1–4, find the cross product of the unit vectors and sketch the result.

1. $\mathbf{j} \times \mathbf{i}$
2. $\mathbf{k} \times \mathbf{j}$
3. $\mathbf{i} \times \mathbf{k}$
4. $\mathbf{k} \times \mathbf{i}$

In Exercises 5–12, find u × v and show that it is orthogonal to both u and v.

5. $\mathbf{u} = \langle 1, -4, 0 \rangle$, $\mathbf{v} = \langle 2, 6, 0 \rangle$
6. $\mathbf{u} = \langle -3, 2, 3 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$
7. $\mathbf{u} = \langle 7, -5, 2 \rangle$, $\mathbf{v} = \langle -1, 4, -1 \rangle$
8. $\mathbf{u} = \langle -5, 5, 11 \rangle$, $\mathbf{v} = \langle 2, 2, 3 \rangle$
9. $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
10. $\mathbf{u} = \mathbf{i} + \frac{3}{2}\mathbf{j} - \frac{5}{2}\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{4}\mathbf{k}$
11. $\mathbf{u} = 6\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
12. $\mathbf{u} = \frac{2}{3}\mathbf{i}$, $\mathbf{v} = \frac{1}{3}\mathbf{j} - 3\mathbf{k}$

In Exercises 13–18, use a graphing utility to find u × v.

13. $\mathbf{u} = \langle 2, 4, 3 \rangle$, $\mathbf{v} = \langle 0, -2, 1 \rangle$
14. $\mathbf{u} = \langle 4, -2, 6 \rangle$, $\mathbf{v} = \langle -1, 5, 7 \rangle$
15. $\mathbf{u} = 6\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$
16. $\mathbf{u} = 8\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{3}{4}\mathbf{j} - \frac{1}{4}\mathbf{k}$
17. $\mathbf{u} = -\mathbf{i} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} - 2\mathbf{k}$
18. $\mathbf{u} = \mathbf{i} - 2\mathbf{k}$, $\mathbf{v} = -\mathbf{j} + \mathbf{k}$

In Exercises 19–24, find a unit vector orthogonal to u and v.

19. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
20. $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{k}$
21. $\mathbf{u} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$
22. $\mathbf{u} = 7\mathbf{i} - 14\mathbf{j} + 5\mathbf{k}$, $\mathbf{v} = 14\mathbf{i} + 28\mathbf{j} - 15\mathbf{k}$
23. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
24. $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

In Exercises 25–30, find the area of the parallelogram that has the vectors as adjacent sides.

25. $\mathbf{u} = \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{k}$
26. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{k}$
27. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$
28. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$
29. $\mathbf{u} = \langle 2, 2, -3 \rangle$, $\mathbf{v} = \langle 0, 2, 3 \rangle$
30. $\mathbf{u} = \langle 4, -3, 2 \rangle$, $\mathbf{v} = \langle 5, 0, 1 \rangle$

In Exercises 31 and 32, (a) verify that the points are the vertices of a parallelogram, (b) find its area, and (c) decide whether the parallelogram is a rectangle.

31. $A(2, -1, 4)$, $B(3, 1, 2)$, $C(0, 5, 6)$, $D(-1, 3, 8)$
32. $A(3, 5, 0)$, $B(-1, 8, 5)$, $C(1, 3, 11)$, $D(5, 0, 6)$

In Exercises 33–36, find the area of the triangle with the given vertices. (The area of the triangle having u and v as adjacent sides is $\frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|$.)

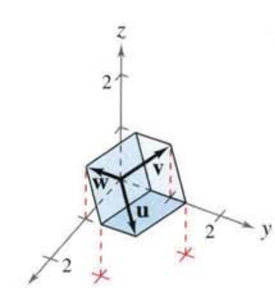
33. $(0, 0, 0)$, $(4, -2, 6)$, $(-4, 0, 3)$
34. $(1, -4, 3)$, $(2, 0, 2)$, $(-2, 2, 0)$
35. $(2, 3, -5)$, $(-2, -2, 0)$, $(3, 0, 6)$
36. $(2, 4, 0)$, $(-2, -4, 0)$, $(0, 0, 4)$

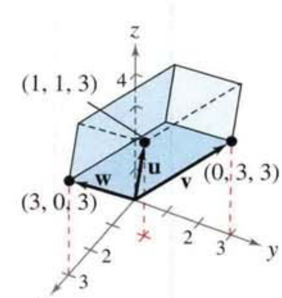
In Exercises 37–40, find the triple scalar product.

37. $\mathbf{u} = \langle 2, 3, 3 \rangle$, $\mathbf{v} = \langle 4, 4, 0 \rangle$, $\mathbf{w} = \langle 0, 0, 4 \rangle$
38. $\mathbf{u} = \langle 20, 10, 10 \rangle$, $\mathbf{v} = \langle 1, 4, 4 \rangle$, $\mathbf{w} = \langle 0, 2, 2 \rangle$
39. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$, $\mathbf{w} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
40. $\mathbf{u} = \mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 4\mathbf{k}$, $\mathbf{w} = -3\mathbf{j} + 6\mathbf{k}$

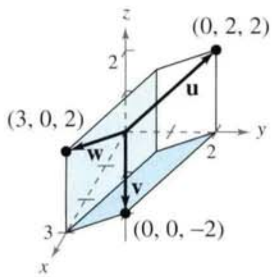
In Exercises 41–44, use the triple scalar product to find the volume of the parallelepiped having adjacent edges u, v, and w.

41. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{k}$
42. $\mathbf{u} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 3\mathbf{j} + 3\mathbf{k}$, $\mathbf{w} = 3\mathbf{i} + 3\mathbf{k}$

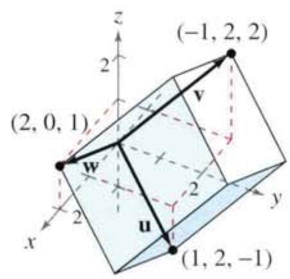




43. $\mathbf{u} = \langle 0, 2, 2\rangle$
$\mathbf{v} = \langle 0, 0, -2\rangle$
$\mathbf{w} = \langle 3, 0, 2\rangle$



44. $\mathbf{u} = \langle 1, 2, -1\rangle$
$\mathbf{v} = \langle -1, 2, 2\rangle$
$\mathbf{w} = \langle 2, 0, 1\rangle$

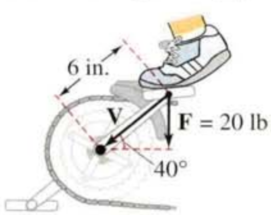


In Exercises 45 and 46, find the volume of the parallelepiped with the given vertices.

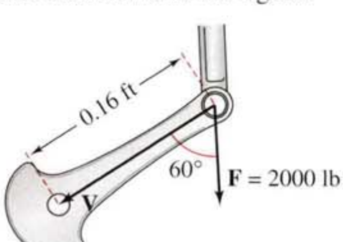
45. $A(0, 0, 0)$, $B(4, 0, 0)$, $C(4, -2, 3)$, $D(0, -2, 3)$, $E(4, 5, 3)$, $F(0, 5, 3)$, $G(0, 3, 6)$, $H(4, 3, 6)$

46. $A(3, 0, 0)$, $B(4, 1, 2)$, $C(3, -1, 4)$, $D(2, -2, 2)$, $E(-1, 5, 4)$, $F(0, 6, 6)$, $G(-1, 4, 8)$, $H(-2, 3, 6)$

47. ***Torque*** A child applies the brakes on a bicycle by applying a downward force of 20 pounds on the pedal when the 6-inch crank makes a 40° angle with the horizontal. Vectors representing the crank and the force are $\mathbf{V} = \frac{1}{2}(\cos 40°\,\mathbf{j} + \sin 40°\,\mathbf{k})$ and $\mathbf{F} = -20\mathbf{k}$, respectively. Find the torque on the crank if it is given by $\|\mathbf{V} \times \mathbf{F}\|$.



48. ***Torque*** Both the magnitude and direction of the force on a crankshaft change as the crankshaft rotates. Use the technique shown in Exercise 47 to find the torque on the crankshaft using the position and data shown in the figure.



Synthesis

True or False? **In Exercises 49 and 50, determine whether the statement is true or false. Justify your answer.**

49. The cross product is not defined for vectors in the plane.

50. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in space that are nonzero and not parallel, then $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

In Exercises 51 and 52, prove the property of the cross product where $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$.

51. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

52. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

53. Consider the vectors $\mathbf{u} = \langle \cos\alpha, \sin\alpha, 0\rangle$ and $\mathbf{v} = \langle \cos\beta, \sin\beta, 0\rangle$ where $\alpha > \beta$. Find the cross product of the vectors and use the result to prove the identity $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

Review

In Exercises 54–61, evaluate the expression without using a calculator.

54. $\cos 480°$ **55.** $\tan 300°$

56. $\sin 690°$ **57.** $\cos 930°$

58. $\sin\dfrac{19\pi}{6}$ **59.** $\cos\dfrac{17\pi}{6}$

60. $\tan\dfrac{15\pi}{4}$ **61.** $\tan\dfrac{10\pi}{3}$

In Exercises 62 and 63, sketch the constraint region. Then find the minimum and maximum values of the objective function, and where they occur, subject to the constraints.

62. Objective function:
$z = 6x + 4y$
Constraints:
$x \geq 0$
$y \geq 0$
$x + 6y \leq 30$
$6x + y \leq 40$

63. Objective function:
$z = 6x + 7y$
Constraints:
$x \geq 0$
$y \geq 0$
$4x + 3y \geq 24$
$x + 3y \geq 15$

11.4 Lines and Planes in Space

What You Should Learn:

- How to find parametric and symmetric equations of lines in space
- How to find equations of planes in space
- How to sketch planes in space
- How to find distances between points and planes in space

Why You Should Learn It:

Normal vectors to a plane are important in modeling and solving real-life problems. For instance, in Exercise 50 on page 799, normal vectors are used to find the angles between two adjacent sides of a chute on a grain elevator of a combine.

Paul A. Souders/CORBIS

Lines in Space

In the plane, *slope* is used to determine an equation of a line. In space, it is more convenient to use *vectors* to determine the equation of a line. In Figure 11.21, consider the line L through the point $P = (x_1, y_1, z_1)$ and parallel to the vector

$\mathbf{v} = \langle a, b, c \rangle$. Direction vector for L

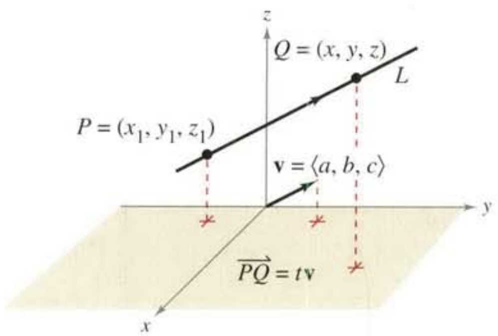


Figure 11.21

The vector $\mathbf{v}$ is the **direction vector** for the line L, and a, b, and c are the **direction numbers.** One way of describing the line L is to say that it consists of all points $Q = (x, y, z)$ for which the vector $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$. This means that $\overrightarrow{PQ}$ is a scalar multiple of $\mathbf{v}$, and you can write $\overrightarrow{PQ} = t\mathbf{v}$, where t is a scalar.

$$\begin{aligned}\overrightarrow{PQ} &= \langle x - x_1, y - y_1, z - z_1 \rangle \\ &= \langle at, bt, ct \rangle \\ &= t\mathbf{v}\end{aligned}$$

By equating corresponding components, you can obtain the **parametric equations of a line in space.**

Parametric Equations of a Line in Space

A line L parallel to the **vector** $\mathbf{v} = \langle a, b, c \rangle$ and passing through the point $P = (x_1, y_1, z_1)$ is represented by the parametric equations

$$x = x_1 + at \qquad y = y_1 + bt \qquad z = z_1 + ct.$$

If the direction numbers a, b, and c are all nonzero, you can eliminate the parameter t to obtain the **symmetric equations** of a line.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \text{Symmetric equations}$$

EXAMPLE 1 Finding Parametric and Symmetric Equations

Find parametric and symmetric equations of the line L that passes through the point $(1, -2, 4)$ and is parallel to $\mathbf{v} = \langle 2, 4, -4 \rangle$.

Solution

To find a set of parametric equations of the line, use the coordinates $x_1 = 1$, $y_1 = -2$, and $z_1 = 4$ and direction numbers $a = 2$, $b = 4$, and $c = -4$ (see Figure 11.22).

$$x = 1 + 2t, \qquad y = -2 + 4t, \qquad z = 4 - 4t \qquad \text{Parametric equations}$$

Because a, b, and c are all nonzero, a set of symmetric equations is

$$\frac{x-1}{2} = \frac{y+2}{4} = \frac{z-4}{-4}. \qquad \text{Symmetric equations}$$

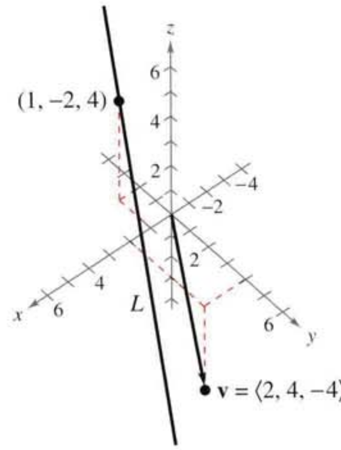


Figure 11.22

Neither the parametric equations nor the symmetric equations of a given line are unique. For instance, in Example 1, by letting $t = 1$ in the parametric equations you would obtain the point $(3, 2, 0)$. Using this point with the direction numbers $a = 2$, $b = 4$, and $c = -4$ produces the parametric equations

$$x = 3 + 2t, \qquad y = 2 + 4t, \qquad z = -4t.$$

EXAMPLE 2 Parametric Equations of a Line Through Two Points

Find a set of parametric equations of the line that passes through the points $(-2, 1, 0)$ and $(1, 3, 5)$.

Solution

Begin by letting $P = (-2, 1, 0)$ and $Q = (1, 3, 5)$. Then a direction vector for the line passing through P and Q is

$$\begin{aligned}\mathbf{v} &= \overrightarrow{PQ}\\ &= \langle 1 - (-2), 3 - 1, 5 - 0 \rangle\\ &= \langle 3, 2, 5 \rangle\\ &= \langle a, b, c \rangle.\end{aligned}$$

Using the direction numbers $a = 3$, $b = 2$, and $c = 5$, with the point $P = (-2, 1, 0)$, you can obtain the parametric equations

$$x = -2 + 3t, \qquad y = 1 + 2t, \qquad z = 5t.$$

You can check the answer to Example 2 by verifying that the two original points lie on the line. To see this, substitute $t = 0$ and $t = 1$ into the parametric equations as follows.

$t = 0$:

$$x = -2 + 3t = -2 + 3(0) = -2$$
$$y = 1 + 2t = 1 + 2(0) = 1$$
$$z = 5t = 5(0) = 0$$

$t = 1$:

$$x = -2 + 3t = -2 + 3(1) = 1$$
$$y = 1 + 2t = 1 + 2(1) = 3$$
$$z = 5t = 5(1) = 5$$

Planes in Space

You have seen how an equation of a line in space can be obtained from a point on the line and a vector *parallel* to it. You will now see that an equation of a plane in space can be obtained from a point in the plane and a vector *normal* (perpendicular) to the plane.

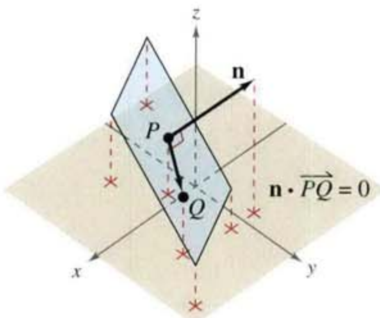


Figure 11.23

Consider the plane containing the point $P = (x_1, y_1, z_1)$ having a nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$, as shown on Figure 11.23. This plane consists of all points $Q = (x, y, z)$ for which the vector $\overrightarrow{PQ}$ is orthogonal to $\mathbf{n}$. Using the dot product, you can write the following.

$$\mathbf{n} \cdot \overrightarrow{PQ} = 0$$

$$\langle a, b, c\rangle \cdot \langle x - x_1, y - y_1, z - z_1\rangle = 0$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

The latter equation of the plane is said to be in **standard form.**

Standard Equation of a Plane in Space

The plane containing the point (x_1, y_1, z_1) and having normal vector $\mathbf{n} = \langle a, b, c\rangle$ can be represented by the **standard form of the equation of a plane**

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

By regrouping terms, you obtain the **general form of the equation of a plane** in space

$ax + by + cz + d = 0.$ General form of equation of plane

Given the general form of the equation of a plane, it is easy to find a normal vector to the plane. Simply use the coefficients of x, y, and z and write $\mathbf{n} = \langle a, b, c\rangle$.

Exploration

Consider the following four planes.

$$2x + 3y - z = 2$$
$$4x + 6y - 2z = 5$$
$$-2x - 3y + z = -2$$
$$-6x - 9y + 3z = 11$$

What are the normal vectors for each plane? What can you say about the relative positions of these planes in space?

EXAMPLE 3 Finding an Equation of a Plane in Three-Space

Find the general equation of the plane containing the points $(2, 1, 1)$, $(0, 4, 1)$, and $(-2, 1, 4)$.

Solution

To find the equation of the plane, you need a point in the plane and a vector that is normal to the plane. There are three choices for the point, but no normal vector is given. To obtain a normal vector, use the cross product of vectors **u** and **v** extending from the point $(2, 1, 1)$ to the points $(0, 4, 1)$ and $(-2, 1, 4)$, as shown in Figure 11.24. The component forms of **u** and **v** are

$$\begin{aligned}\mathbf{u} &= \langle 0-2, 4-1, 1-1\rangle\\ &= \langle -2, 3, 0\rangle\\ \mathbf{v} &= \langle -2-2, 1-1, 4-1\rangle\\ &= \langle -4, 0, 3\rangle\end{aligned}$$

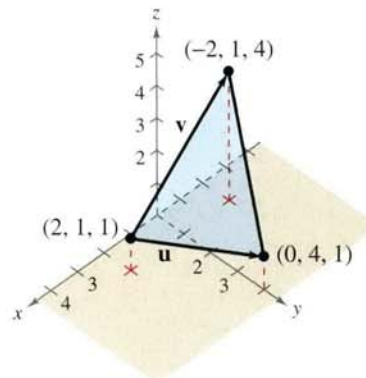


Figure 11.24

and it follows that

$$\begin{aligned}\mathbf{n} = \mathbf{u}\times\mathbf{v} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & 3 & 0\\ -4 & 0 & 3\end{vmatrix}\\ &= 9\mathbf{i} + 6\mathbf{j} + 12\mathbf{k}\\ &= \langle a, b, c\rangle\end{aligned}$$

is normal to the given plane. Using the direction numbers for **n** and the point $(x_1, y_1, z_1) = (2, 1, 1)$, you can determine an equation of the plane to be

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$9(x - 2) + 6(y - 1) + 12(z - 1) = 0 \qquad \text{Standard form}$$

$$9x + 6y + 12z - 36 = 0$$

$$3x + 2y + 4z - 12 = 0. \qquad \text{General form}$$

In Example 3, check to see that each of the three points satisfies the equation $3x + 2y + 4z - 12 = 0$.

Two distinct planes in three-space either are parallel or intersect in a line. If they intersect, you can determine the angle between them from the angle between their normal vectors, as shown in Figure 11.25. Specifically, if vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to two intersecting planes, the angle θ between the normal vectors is equal to the **angle between the two planes** and is

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}. \qquad \text{Angle between two planes}$$

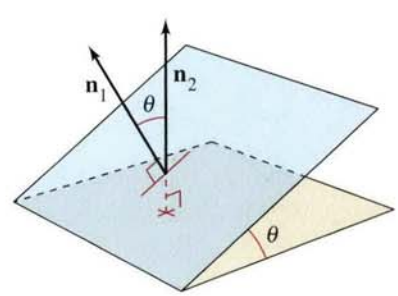


Figure 11.25

Consequently, two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are

1. *perpendicular* if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.
2. *parallel* if $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$.

EXAMPLE 4 Finding the Line of Intersection of Two Planes

Find the angle between the two planes

$$x - 2y + z = 0 \qquad \text{Equation for plane 1}$$

$$2x + 3y - 2z = 0 \qquad \text{Equation for plane 2}$$

and find parametric equations of their line of intersection (see Figure 11.26).

Solution

The normal vectors for the planes are $\mathbf{n}_1 = \langle 1, -2, 1\rangle$ and $\mathbf{n}_2 = \langle 2, 3, -2\rangle$. Consequently, the angle between two planes is determined as follows.

$$\begin{aligned}\cos\theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\\ &= \frac{|-6|}{\sqrt{6}\sqrt{17}}\\ &= \frac{6}{\sqrt{102}} \approx 0.59409.\end{aligned}$$

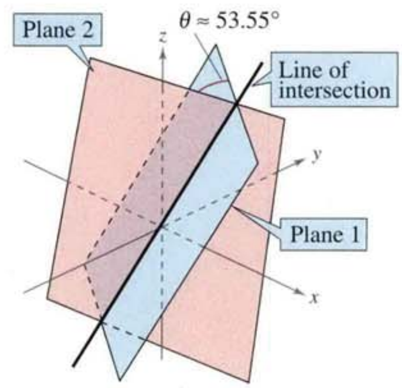


Figure 11.26

This implies that the angle between the two planes is $\theta \approx 53.55°$. You can find the line of intersection of the two planes by simultaneously solving the two linear equations representing the planes. One way to do this is to multiply the first equation by -2 and add the result to the second equation.

$$\begin{array}{lcr} x - 2y + z = 0 & \Rightarrow & -2x + 4y - 2z = 0\\ 2x + 3y - 2z = 0 & & \underline{2x + 3y - 2z = 0}\\ & & 7y - 4z = 0 \quad \Rightarrow \quad y = \dfrac{4z}{7}\end{array}$$

Substituting $y = 4z/7$ back into one of the original equations, you can determine that $x = z/7$. Finally, by letting $t = z/7$, you obtain the parametric equations

$$x = t = x_1 + at, \qquad y = 4t = y_1 + bt, \qquad z = 7t = z_1 + ct.$$

Because $(x_1, y_1, z_1) = (0, 0, 0)$ lies in both planes, you can substitute for x_1, y_1, and z_1 in these parametric equations, which indicates that $a = 1$, $b = 4$, and $c = 7$ are direction numbers for the line of intersection.

Note that the direction numbers in Example 4 can be obtained from the cross product of the two normal vectors as follows.

$$\begin{aligned}\mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 1\\ 2 & 3 & -2\end{vmatrix}\\ &= \begin{vmatrix}-2 & 1\\ 3 & -2\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 1\\ 2 & -2\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & -2\\ 2 & 3\end{vmatrix}\mathbf{k}\\ &= \mathbf{i} + 4\mathbf{j} + 7\mathbf{k}\end{aligned}$$

This means that the line of intersection of the two planes is parallel to the cross product of their normal vectors.

Sketching Planes in Space

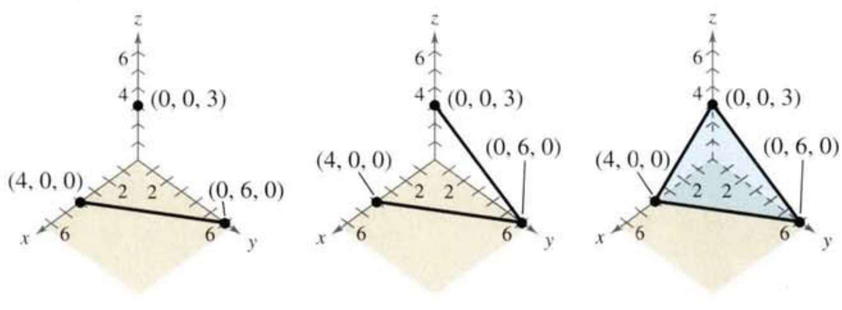
If a plane in space intersects one of the coordinate planes, the line of intersection is called the *trace* of the given plane in the coordinate plane. To sketch a plane in space, it is helpful to find its points of intersection with the coordinate axes and its traces in the coordinate planes. For example, consider the plane

$$3x + 2y + 4z = 12. \qquad \text{Equation of plane}$$

You can find the xy-trace by letting $z = 0$ and sketching the line

$$3x + 2y = 12 \qquad xy\text{-trace}$$

in the xy-plane. This line intersects the x-axis at $(4, 0, 0)$ and the y-axis at $(0, 6, 0)$. In Figure 11.27, this process is continued by finding the yz-trace and the xz-trace, and then shading in the triangular region lying in the first octant.



(a) ***xy*-trace($z = 0$):** $3x + 2y = 12$ **(b)** ***yz*-trace($x = 0$):** $2y + 4z = 12$ **(c)** ***xz*-trace($y = 0$):** $3x + 4z = 12$

Figure 11.27

STUDY TIP

Some graphing utilities can be used to graph a plane in space. To graph the plane at the left use the following steps. Consult your user's manual for instructions on how to do each step.

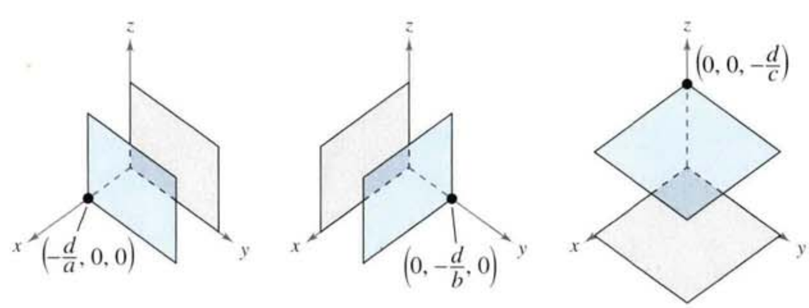
1. Set the graphing mode to three-dimensional.
2. Solve for z and enter the equation.
3. Use the following viewing cube.

   angle of rotation from positive x-axis $= 20°$,
   angle of rotation from positive z-axis $= 70°$,
   $-10 \le x \le 10$,
   $-10 \le x \le 10$,
   $-10 \le z \le 20$

4. Graph the equation.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

If the equation of a plane has a missing variable such as $2x + z = 1$, the plane must be *parallel to the axis* represented by the missing variable, as shown in Figure 11.28. If two variables are missing from the equation of a plane, then it is *parallel to the coordinate plane* represented by the missing variables, as shown in Figure 11.29.



(a) Plane $ax + d = 0$ is parallel to yz-plane. **(b) Plane $by + d = 0$ is parallel to xz-plane.** **(c) Plane $cz + d = 0$ is parallel to xy-plane.**

Figure 11.29

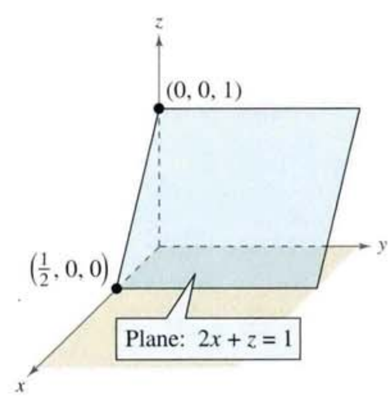
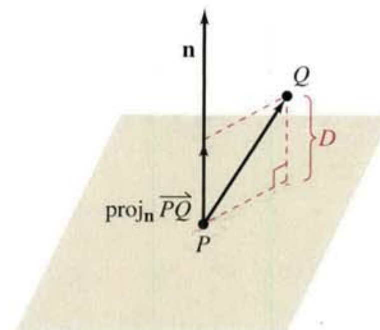


Figure 11.28

Distance Between a Point and a Plane

The distance D between a point Q and a plane is the length of the shortest line segment connecting Q to the plane, as shown in Figure 11.30. If P is *any* point in the plane, you can find this distance by projecting the vector $\overrightarrow{PQ}$ onto the normal vector $\mathbf{n}$. The length of this projection is the desired distance.



$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\|$

Figure 11.30

Distance Between a Point and a Plane

The **distance between a plane and a point** Q (not in the plane) is

$$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

where P is a point in the plane and $\mathbf{n}$ is normal to the plane.

To find a point in the plane given by $ax + by + cz + d = 0$, where $a \neq 0$, let $y = 0$ and $z = 0$. Then, from the equation $ax + d = 0$, you can conclude that the point $(-d/a, 0, 0)$ lies in the plane.

EXAMPLE 5 Finding the Distance Between a Point and a Plane

Find the distance between the point $Q = (1, 5, -4)$ and the plane $3x - y + 2z = 6$.

Solution

You know that $\mathbf{n} = \langle 3, -1, 2\rangle$ is normal to the given plane. To find a point in the plane, let $y = 0$ and $z = 0$, and obtain the point $P = (2, 0, 0)$. The vector from P to Q is

$$\begin{aligned}\overrightarrow{PQ} &= \langle 1 - 2, 5 - 0, -4 - 0\rangle \\ &= \langle -1, 5, -4\rangle.\end{aligned}$$

The formula for the distance between a point and a plane produces

$$\begin{aligned}D &= \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \\ &= \frac{|\langle -1, 5, -4\rangle \cdot \langle 3, -1, 2\rangle|}{\sqrt{9 + 1 + 4}} \\ &= \frac{|-3 - 5 - 8|}{\sqrt{14}} \\ &= \frac{16}{\sqrt{14}}.\end{aligned}$$

The choice of the point P in Example 5 is arbitrary. Try choosing a different point to verify that you obtain the same distance.

11.4 Exercises

In Exercises 1–6, find a set of (a) parametric equations and (b) symmetric equations for the line through the point and parallel to the specified vector or line. (For each line, express the direction numbers as integers.)

	Point	*Parallel to*
1.	$(-1, 4, 0)$	$\mathbf{v} = \langle -2, 4, 1 \rangle$
2.	$(3, -5, 1)$	$\mathbf{v} = \langle 3, -7, -10 \rangle$
3.	$(-4, 1, 0)$	$\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{4}{3}\mathbf{j} - \mathbf{k}$ $3i + 8j - 6k$
4.	$(5, 0, 10)$	$\mathbf{v} = 4\mathbf{i} + 3\mathbf{k}$
5.	$(2, -3, 5)$	$x = 5 + 2t$, $y = 7 - 3t$, $z = -2 + t$
6.	$(1, 0, 1)$	$x = 3 + 3t$, $y = 5 - 2t$, $z = -7 + t$

In Exercises 7–10, find a set of (a) parametric equations and (b) symmetric equations of the line that passes through the given points. Express the direction numbers as integers.

7. $(6, 0, 3), (2, 1, 8)$ **8.** $(4, -1, -1), (-1, 0, 5)$

9. $(-3, 8, 15), (1, -2, 16)$

10. $\left(-\frac{3}{2}, \frac{3}{2}, 2\right), (3, -5, -4)$

11. Determine which of the points lie on the line that passes through the point $(-4, -1, 7)$ and is parallel to the vector $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$.

(a) $(-4, -1, 0)$ (b) $(-1, -2, 7)$

(c) $(-10, 1, 7)$ (d) $(4, 1, -7)$

12. Determine which of the points lie on the line that passes through the point $(-2, 3, 1)$ and is parallel to the vector $\mathbf{v} = 4\mathbf{i} - \mathbf{k}$.

(a) $(2, 3, 0)$ (b) $(-6, 3, 2)$

(c) $(2, 1, 0)$ (d) $(6, 3, -2)$

In Exercises 13 and 14, sketch a graph of the line.

13. $x = 2t, y = 2 + t, z = 1 + \frac{1}{2}t$ **14.** $x = 5 - 2t, y = 1 + t, z = 5 - \frac{1}{2}t$

In Exercises 15–20, find an equation of the plane passing through the point and perpendicular to the specified vector or line.

	Point	*Perpendicular to*
15.	$(3, 4, -2)$	$\mathbf{n} = \mathbf{j}$
16.	$(2, 3, 5)$	$\mathbf{n} = \mathbf{k}$
17.	$(5, 6, 3)$	$\mathbf{n} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
18.	$(0, 0, 0)$	$\mathbf{n} = -3\mathbf{j} + 5\mathbf{k}$
19.	$(2, 0, 0)$	$x = 3 - t$, $y = 2 - 2t$, $z = 4 + t$
20.	$(0, 0, 6)$	$x = 1 - t$, $y = 2 + t$, $z = 4 - 2t$

In Exercises 21–24, find an equation of the plane passing through the three points.

21. $(0, 0, 0), (2, 1, 3), (-2, 1, 3)$

22. $(4, -1, 3), (2, 5, 1), (-1, 2, 1)$

23. $(0, -1, -2), (4, 1, 6), (1, 0, -3)$

24. $(5, -1, 4), (1, -1, 2), (2, 1, -3)$

In Exercises 25–28, find an equation of the plane.

25. The plane passes through the point $(2, 5, 3)$ and is parallel to the xz-plane.

26. The plane passes through the point $(2, 5, 3)$ and is parallel to the xy-plane.

27. The plane passes through the points $(4, 0, 0)$ and $(0, 2, 0)$ and is perpendicular to the plane $x + 2y + 2z = 4$.

28. The plane passes through the points $(2, 2, 1)$ and $(-1, 1, -1)$ and is perpendicular to the plane $2x - 3y + z = 3$.

In Exercises 29–32, determine whether the planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.

29. $3x + y - 4z = 3$, $-9x - 3y + 12z = 4$

30. $3x + 2y - z = 7$, $x - 4y + 2z = 0$

31. $2x - z = 1$
$4x + y + 8z = 10$

32. $x - 5y - z = 1$
$5x - 25y - 5z = -3$

In Exercises 33–36, mark the intercepts and sketch a graph of the plane.

33. $x + 2y + 3z = 6$
34. $2x - y + 4z = 4$
35. $x + z = 3$
36. $y + 2z = 4$

In Exercises 37–40, use a graphing utility to graph the plane.

37. $3x + 2y - z = 6$
38. $x - 3z = 6$
39. $x + 2y - 6z = 8$
40. $3x - 4y - z = -12$

In Exercises 41–44, find the distance between the point and the plane.

41. $(0, 0, 0)$
$3x + 2y + z = 12$

42. $(1, 2, 3)$
$2x - y + z = 4$

43. $(4, -2, -2)$
$2x - y + z = 4$

44. $(-1, 2, 5)$
$2x + 3y + z = 12$

In Exercises 45–48, (a) find the angle between the two planes and (b) find the parametric equations for their line of intersection.

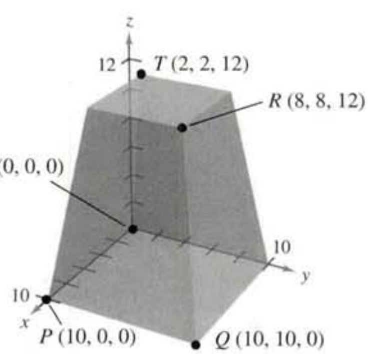
45. $3x - 4y + 5z = 6$
$x + y - z = 2$

46. $x - 3y + z = -2$
$2x + 5z + 3 = 0$

47. $x + y - z = 0$
$2x - 5y - z = 1$

48. $2x + 4y - 2z = 1$
$-3x - 6y + 3z = 10$

49. ***Machine Design*** A tractor fuel tank has the shape and dimensions shown in the figure. In fabricating the tank, it is necessary to know the angle between two adjacent sides. Find the angle.



50. ***Mechanical Design*** A chute at the top of a grain elevator of a combine funnels the grain into a bin as shown in the figure. Find the angle between two adjacent sides.

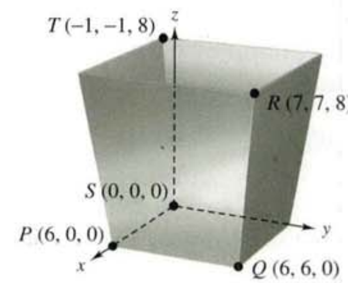


FIGURE FOR 50

Synthesis

True or False? **In Exercises 51–53, determine whether the statement is true or false. Justify your answer.**

51. Every two lines in space are either intersecting or parallel.
52. Two nonparallel lines in space will always intersect.
53. Two nonparallel planes in space will always intersect.

54. The direction numbers of two distinct lines in space are $10, -18, 20$, and $-15, 27, -30$. What is the relationship between the lines? Explain.

55. ***Exploration***
(a) Describe and find an equation for the surface generated by all points (x, y, z) that are two units from the point $(4, -1, 1)$.
(b) Describe and find an equation for the surface generated by all points (x, y, z) that are two units from the plane $4x - 3y + z = 10$.

Review

In Exercises 56–59, convert the polar equation to rectangular form.

56. $r = 10$
57. $\theta = \dfrac{3\pi}{4}$
58. $r = 3 \cos \theta$
59. $r = \dfrac{1}{2 - \cos \theta}$

In Exercises 60–65, convert the rectangular equation to polar form.

60. $x^2 + y^2 = 49$
61. $x^2 + y^2 - 4x = 0$
62. $y = 5$
63. $x = 3$
64. $2x - y + 1 = 0$
65. $5x - 6y + 4 = 0$

11 Chapter Summary

What did you learn?

Section 11.1	Review Exercises
☐ How to plot points in the three-dimensional coordinate system	1–4
☐ How to find distances between points in space	5–8
☐ How to find midpoints of line segments joining points in space	9–12
☐ How to write equations of spheres in standard form	13–18
☐ How to find traces of surfaces in space	19, 20
Section 11.2	
☐ How to find the component form, the unit vector in the same direction, and magnitude of vectors in space	21–24
☐ How to find dot products of and angles between vectors in space	25–30
☐ How to determine whether vectors in space are parallel or orthogonal	31–34
☐ How to use vectors in space to solve real-life problems	35, 36
Section 11.3	
☐ How to find cross products of vectors in space	37, 38
☐ How to use geometric properties of cross products of vectors in space	39–42
☐ How to use triple scalar products to find volumes of parallelepipeds	43
Section 11.4	
☐ How to find parametric and symmetric equations of lines in space	44–47
☐ How to find equations of planes in space	48–51
☐ How to sketch planes in space	52–55
☐ How to find distances between points and planes in space	56–59

11 Review Exercises

11.1 In Exercises 1 and 2, plot the points in the same three-dimensional coordinate system.

1. (a) $(4, -1, 2)$
(b) $(-3, 3, 0)$

2. (a) $(2, 4, -3)$
(b) $(0, -4, 1)$

3. Find the coordinates of the point in the xy-plane four units to the right of the xz-plane and five units behind the yz-plane.

4. Find the coordinates of the point located on the y-axis and seven units to the left of the xz-plane.

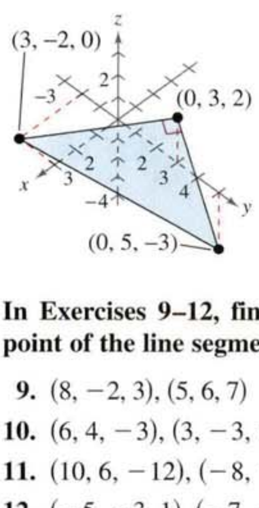
In Exercises 5 and 6, find the distance between the indicated points.

5. $(4, 0, 7)$ and $(5, 2, 1)$

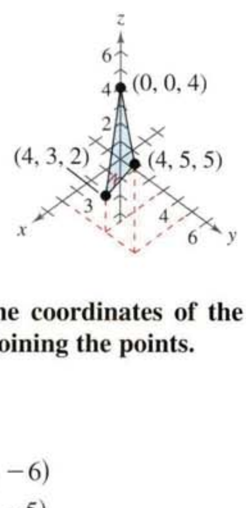
6. $(2, 3, -4)$ and $(-1, -3, 0)$

In Exercises 7 and 8, find the lengths of the sides of the right triangle. Show that these lengths satisfy the Pythagorean Theorem.

7.



8.



In Exercises 9–12, find the coordinates of the midpoint of the line segment joining the points.

9. $(8, -2, 3), (5, 6, 7)$

10. $(6, 4, -3), (3, -3, 10)$

11. $(10, 6, -12), (-8, -2, -6)$

12. $(-5, -3, 1), (-7, -9, -5)$

In Exercises 13–16, find the standard form of the equation of the sphere.

13. Center: $(2, 3, 5)$; Radius: 1

14. Center: $(3, -2, 4)$; Radius: 4

15. Center: $(1, 5, 2)$; Diameter: 12

16. Center: $(3, -2, 6)$; Diameter: 15

In Exercises 17 and 18, find the center and radius of the sphere and sketch its graph.

17. $x^2 + y^2 + z^2 - 4x - 6y + 4 = 0$

18. $x^2 + y^2 + z^2 - 10x + 6y - 4z + 34 = 0$

In Exercises 19 and 20, sketch the graph of the equation and sketch the specified trace.

19. $x^2 + (y - 3)^2 + z^2 = 16$
(a) xz-trace (b) yz-trace

20. $(x + 2)^2 + (y - 1)^2 + z^2 = 9$
(a) xy-trace (b) yz-trace

11.2 In Exercises 21–24, find the component form and the magnitude of the vector with initial and terminal points P and Q, respectively.

21. $P(2, -1, 4)$
$Q(3, 3, 0)$

22. $P(2, -1, 2)$
$Q(-3, 2, 3)$

23. $P(7, -4, 3)$
$Q(-3, 2, 10)$

24. $P(0, 3, -1)$
$Q(5, -8, 6)$

In Exercises 25–28, find the dot product of u and v.

25. $\mathbf{u} = \langle 2, -3, 4 \rangle$
$\mathbf{v} = \langle 0, 6, 5 \rangle$

26. $\mathbf{u} = \langle 8, -4, 2 \rangle$
$\mathbf{v} = \langle 2, 5, 2 \rangle$

27. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$
$\mathbf{v} = \mathbf{i} - \mathbf{k}$

28. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
$\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

In Exercises 29 and 30, find the angle θ between the vectors u and v.

29. $\mathbf{u} = \langle 2\sqrt{2}, -4, 4 \rangle$
$\mathbf{v} = \langle -\sqrt{2}, 1, 2 \rangle$

30. $\mathbf{u} = \langle 3, 1, -1 \rangle$
$\mathbf{v} = \langle 4, 5, 2 \rangle$

In Exercises 31 and 32, use a graphing utility to determine whether u and v are orthogonal, parallel, or neither.

31. $\mathbf{u} = \langle 39, -12, 21 \rangle$
$\mathbf{v} = \langle -26, 8, -14 \rangle$

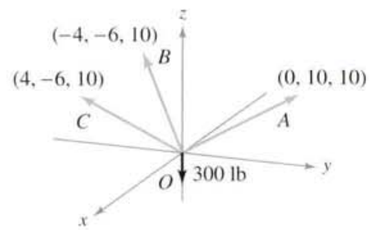
32. $\mathbf{u} = \langle 8, 5, -8 \rangle$
$\mathbf{v} = \left\langle -2, 4, \frac{1}{2} \right\rangle$

In Exercises 33 and 34, use vectors to show that the points form the vertices of a parallelogram.

33. $(5, 2, 0), (2, 6, 1), (2, 4, 7), (5, 0, 6)$

34. $(1, 1, 1), (2, 3, 4), (6, 5, 2), (7, 7, 5)$

35. ***Load-Supporting Cables*** A load of 300 pounds is supported by three cables, as shown in the figure. Find the tension in each of the support cables.



36. ***Load-Supporting Cables*** Determine the tension in each of the support cables in Exercise 35 if the load is 200 pounds.

11.3 In Exercises 37 and 38, find u × v. Use a graphing utility to verify your answer.

37. $\mathbf{u} = \langle -2, 8, 2\rangle$, $\mathbf{v} = \langle 1, 1, -1\rangle$

38. $\mathbf{u} = \langle 10, 15, 5\rangle$, $\mathbf{v} = \langle 5, -3, 0\rangle$

In Exercises 39 and 40, find a unit vector orthogonal to u and v.

39. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = 10\mathbf{i} - 15\mathbf{j} + 2\mathbf{k}$

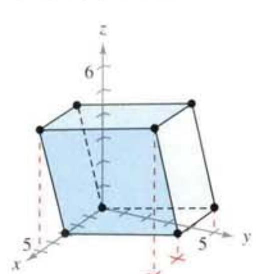
40. $\mathbf{u} = 4\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 12\mathbf{k}$

In Exercises 41 and 42, verify that the points are the vertices of a parallelogram and find its area.

41. $(2, -1, 1), (5, 1, 4), (0, 1, 1), (3, 3, 4)$

42. $(0, 4, 0), (1, 4, 1), (0, 6, 0), (1, 6, 1)$

43. ***Volume*** Use the triple scalar product to find the volume of the parallelepiped with vertices $(0, 0, 0)$, $(3, 0, 0)$, $(0, 5, 1)$, $(3, 5, 1)$, $(2, 0, 5)$, $(5, 0, 5)$, $(2, 5, 6)$, $(5, 5, 6)$.



11.4 In Exercises 44–47, find a set of (a) parametric equations and (b) symmetric equations for the specified line.

44. The line passes through the points $(-1, 3, 5)$ and $(3, 6, -1)$.

45. The line passes through the points $(0, -10, 3)$ and $(5, 10, 0)$.

46. The line passes through the point $(3, 1, 2)$ and is parallel to the line given by $x = y = z$.

47. The line passes through the point $(3, 2, 1)$ and is parallel to the line given by $x = y = z$.

In Exercises 48–51, find an equation of the plane.

48. The plane passes through the points $(0, 0, 0)$, $(5, 0, 2)$, and $(2, 3, 8)$.

49. The plane passes through the points $(-1, 3, 4)$, $(4, -2, 2)$, and $(2, 8, 6)$.

50. The plane passes through the point $(5, 3, 2)$ and is parallel to the xy-plane.

51. The plane passes through the point $(3, 1, 2)$ and is orthogonal to the line given by $x = y = z$.

In Exercises 52–55, mark the intercepts and sketch a graph of the plane.

52. $3x - 2y + 3z = 6$

53. $5x - y - 5z = 5$

54. $2x - 3z = 6$

55. $4y - 3z = 12$

In Exercises 56–59, find the distance from the point to the plane.

56. $(2, 3, 10)$, $2x - 20y + 6z = 6$

57. $(1, 2, 3)$, $2x - y + z = 4$

58. $(0, 0, 0)$, $x - 10y + 3z = 2$

59. $(0, 0, 0)$, $2x + 3y + z = 12$

Synthesis

True or False? **In Exercises 60 and 61, determine whether the statement is true or false. Justify your answer.**

60. The cross product is commutative.

61. The triple scalar product of three vectors in space is a scalar.

In Exercises 62–65, let $\mathbf{u} = \langle 3, -2, 1\rangle$, $\mathbf{v} = \langle 2, -4, -3\rangle$, and $\mathbf{w} = \langle -1, 2, 2\rangle$.

62. Show that $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.

63. Show that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

64. Show that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

65. Show that $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.