Chapter One

LINEAR FUNCTIONS AND CHANGE

Contents

1.1 FUNCTIONS AND FUNCTION NOTATION

In everyday language, the word *function* expresses the notion of dependence. For example, a person might say that election results are a function of the economy, meaning that the winner of an election is determined by how the economy is doing. Someone else might claim that car sales are a function of the weather, meaning that the number of cars sold on a given day is affected by the weather.

In mathematics, the meaning of the word *function* is more precise, but the basic idea is the same. A function is a relationship between two quantities. If the value of the first quantity determines exactly one value of the second quantity, we say the second quantity is a function of the first. We make the following definition:

> A **function** is a rule that takes certain numbers as inputs and assigns to each input number exactly one output number. The output is a function of the input.

The inputs and outputs are also called *variables*.

Representing Functions: Words, Tables, Graphs, and Formulas

A function can be described using words, data in a table, points on a graph, or a formula.

Example 1 It is a surprising biological fact that most crickets chirp at a rate that increases as the temperature increases. For the snowy tree cricket (*Oecanthus fultoni*), the relationship between temperature and chirp rate is so reliable that this type of cricket is called the thermometer cricket. We can estimate the temperature (in degrees Fahrenheit) by counting the number of times a snowy tree cricket chirps in 15 seconds and adding 40. For instance, if we count 20 chirps in 15 seconds, then a good estimate of the temperature is $20 + 40 = 60°$F.

The rule used to find the temperature T (in °F) from the chirp rate R (in chirps per minute) is an example of a function. The input is chirp rate and the output is temperature. Describe this function using words, a table, a graph, and a formula.

Solution

- **Words**: To estimate the temperature, we count the number of chirps in fifteen seconds and add forty. Alternatively, we can count R chirps per minute, divide R by four and add forty. This is because there are one-fourth as many chirps in fifteen seconds as there are in sixty seconds. For instance, 80 chirps per minute works out to $\frac{1}{4} \cdot 80 = 20$ chirps every 15 seconds, giving an estimated temperature of $20 + 40 = 60°$F.
- **Table**: Table 1.1 gives the estimated temperature, T, as a function of R, the number of chirps per minute. Notice the pattern in Table 1.1: each time the chirp rate, R, goes up by 20 chirps per minute, the temperature, T, goes up by $5°$F.
- **Graph**: The data from Table 1.1 are plotted in Figure 1.1. For instance, the pair of values $R = 80$, $T = 60$ is plotted as the point P, which is 80 units along the horizontal axis and 60 units up the vertical axis. Data represented in this way are said to be plotted on the *Cartesian plane*. The precise position of P is shown by its coordinates, written $P = (80, 60)$.

Table 1.1 *Chirp rate and temperature*

R, chirp rate (chirps/minute)	T, predicted temperature (°F)
20	45
40	50
60	55
80	60
100	65
120	70
140	75
160	80

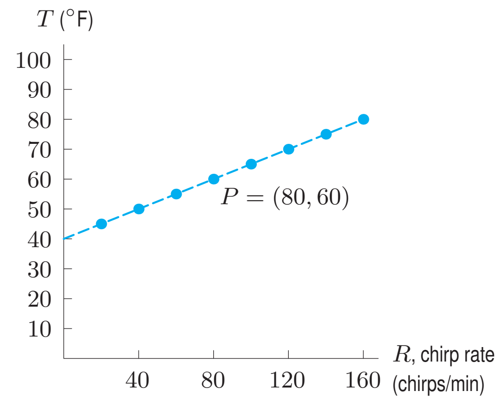


Figure 1.1: Chirp rate and temperature

- **Formula**: A formula is an equation giving T in terms of R. Dividing the chirp rate by four and adding forty gives the estimated temperature, so:

$$\underbrace{\text{Estimated temperature (in °F)}}_{T} = \frac{1}{4} \cdot \underbrace{\text{Chirp rate (in chirps/min)}}_{R} + 40.$$

Rewriting this using the variables T and R gives the formula:

$$T = \frac{1}{4}R + 40.$$

Let's check the formula. Substituting $R = 80$, we have

$$T = \frac{1}{4} \cdot 80 + 40 = 60,$$

which agrees with point $P = (80, 60)$ in Figure 1.1. The formula $T = \frac{1}{4}R + 40$ also tells us that if $R = 0$, then $T = 40$. Thus, the dashed line in Figure 1.1 crosses (or intersects) the T-axis at $T = 40$; we say the T-*intercept* is 40.

All the descriptions given in Example 1 provide the same information, but each description has a different emphasis. A relationship between variables is often given in words, as at the beginning of Example 1. Table 1.1 is useful because it shows the predicted temperature for various chirp rates. Figure 1.1 is more suggestive of a trend than the table, although it is harder to read exact values of the function. For example, you might have noticed that every point in Figure 1.1 falls on a straight line that slopes up from left to right. In general, a graph can reveal a pattern that might otherwise go unnoticed. Finally, the formula has the advantage of being both compact and precise. However, this compactness can also be a disadvantage since it may be harder to gain as much insight from a formula as from a table or a graph.

Mathematical Models

When we use a function to describe an actual situation, the function is referred to as a **mathematical model**. The formula $T = \frac{1}{4}R + 40$ is a mathematical model of the relationship between the temperature and the cricket's chirp rate. Such models can be powerful tools for understanding phenomena and making predictions. For example, this model predicts that when the chirp rate is 80 chirps per minute, the temperature is 60°F. In addition, since $T = 40$ when $R = 0$, the model predicts that the chirp rate is 0 at 40°F. Whether the model's predictions are accurate for chirp rates down to 0 and temperatures as low as 40°F is a question that mathematics alone cannot answer; an understanding of the biology of crickets is needed. However, we can safely say that the model does not apply for temperatures below 40°F, because the chirp rate would then be negative. For the range of chirp rates and temperatures in Table 1.1, the model is remarkably accurate.

In everyday language, saying that T is a function of R suggests that making the cricket chirp faster would somehow make the temperature change. Clearly, the cricket's chirping does not cause the temperature to be what it is. In mathematics, saying that the temperature "depends" on the chirp rate means only that knowing the chirp rate is sufficient to tell us the temperature.

Function Notation

To indicate that a quantity Q is a function of a quantity t, we abbreviate

$$Q \text{ is a function of } t \quad \text{to} \quad Q \text{ equals "} f \text{ of } t\text{"}$$

and, using function notation, to

$$Q = f(t).$$

Thus, applying the rule f to the input value, t, gives the output value, $f(t)$. In other words, $f(t)$ represents a value of Q. Here Q is called the *dependent variable* and t is called the *independent variable*. Symbolically,

$$\text{Output} = f(\text{Input})$$

or

$$\text{Dependent} = f(\text{Independent}).$$

We could have used any letter, not just f, to represent the rule.

Example 2 The number of gallons of paint needed to paint a house depends on the size of the house. A gallon of paint typically covers 250 square feet. Thus, the number of gallons of paint, n, is a function of the area to be painted, A ft^2. We write $n = f(A)$.

(a) Find a formula for f.
(b) Explain in words what the statement $f(10{,}000) = 40$ tells us about painting houses.

Solution

(a) If $A = 5000$ ft^2, then $n = 5000/250 = 20$ gallons of paint. In general, n and A are related by the formula

$$n = \frac{A}{250}.$$

(b) The input of the function $n = f(A)$ is an area and the output is an amount of paint. The statement $f(10{,}000) = 40$ tells us that an area of $A = 10{,}000$ ft^2 requires $n = 40$ gallons of paint.

The expressions "Q depends on t" or "Q is a function of t" do *not* imply a cause-and-effect relationship, as the snowy tree cricket example illustrates.

Example 3 Example 1 gives the following formula for estimating air temperature based on the chirp rate of the snowy tree cricket:

$$T = \frac{1}{4}R + 40.$$

In this formula, T depends on R. Writing $T = f(R)$ indicates that the relationship is a function.

Functions Don't Have to Be Defined by Formulas

People sometimes think that functions are always represented by formulas. However, the next example shows a function that is not given by a formula.

Example 4 The average monthly rainfall, R, at Chicago's O'Hare airport is given in Table 1.2, where time, t, is in months and $t = 1$ is January, $t = 2$ is February, and so on. The rainfall is a function of the month, so we write $R = f(t)$. However, there is no equation that gives R when t is known. Evaluate $f(1)$ and $f(11)$. Explain what your answers mean.

Table 1.2 *Average monthly rainfall at Chicago's O'Hare airport*

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Rainfall, R (inches)	1.8	1.8	2.7	3.1	3.5	3.7	3.5	3.4	3.2	2.5	2.4	2.1

Solution The value of $f(1)$ is the average rainfall in inches at Chicago's O'Hare airport in a typical January. From the table, $f(1) = 1.8$ inches. Similarly, $f(11) = 2.4$ means that in a typical November, there are 2.4 inches of rain at O'Hare.

When Is a Relationship Not a Function?

It is possible for two quantities to be related and yet for neither quantity to be a function of the other.

Example 5 A national park contains foxes that prey on rabbits. Table 1.3 gives the two populations, F and R, over a 12-month period, where $t = 0$ means January 1, $t = 1$ means February 1, and so on.

Table 1.3 *Number of foxes and rabbits in a national park, by month*

t, month	0	1	2	3	4	5	6	7	8	9	10	11
R, rabbits	1000	750	567	500	567	750	1000	1250	1433	1500	1433	1250
F, foxes	150	143	125	100	75	57	50	57	75	100	125	143

(a) Is F a function of t? Is R a function of t?
(b) Is F a function of R? Is R a function of F?

Solution

(a) Both F and R are functions of t. For each value of t, there is exactly one value of F and exactly one value of R. For example, Table 1.3 shows that if $t = 5$, then $R = 750$ and $F = 57$. This means that on June 1 there are 750 rabbits and 57 foxes in the park. If we write $R = f(t)$ and $F = g(t)$, then $f(5) = 750$ and $g(5) = 57$.

(b) No, F is not a function of R. For example, suppose $R = 750$, meaning there are 750 rabbits. This happens both at $t = 1$ (February 1) and at $t = 5$ (June 1). In the first instance, there are 143 foxes; in the second instance, there are 57 foxes. Since there are R-values which correspond to more than one F-value, F is not a function of R.

Similarly, R is not a function of F. At time $t = 5$, we have $R = 750$ when $F = 57$, while at time $t = 7$, we have $R = 1250$ when $F = 57$ again. Thus, the value of F does not uniquely determine the value of R.

How to Tell if a Graph Represents a Function: Vertical Line Test

What does it mean graphically for y to be a function of x? Look at the graph of y against x. For a function, each x-value corresponds to exactly one y-value. This means that the graph intersects any vertical line at most once. If a vertical line cuts the graph twice, the graph would contain two points with different y-values but the same x-value; this would violate the definition of a function. Thus, we have the following criterion:

Vertical Line Test: If there is a vertical line that intersects a graph in more than one point, then the graph does not represent a function.

Example 6 In which of the graphs in Figures 1.2 and 1.3 could y be a function of x?

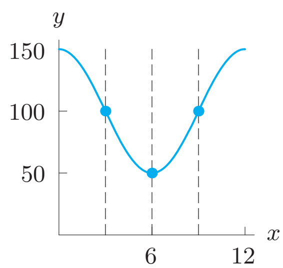


Figure 1.2: Since no vertical line intersects this curve at more than one point, y could be a function of x

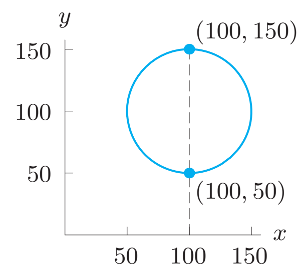


Figure 1.3: Since one vertical line intersects this curve at more than one point, y is not a function of x

Solution The graph in Figure 1.2 could represent y as a function of x because no vertical line intersects this curve in more than one point. The graph in Figure 1.3 does not represent a function because the vertical line shown intersects the curve at two points.

A graph fails the vertical line test if at least one vertical line cuts the graph more than once, as in Figure 1.3. However, if a graph represents a function, then *every* vertical line must intersect the graph at no more than one point.

Exercises and Problems for Section 1.1

Skill Refresher

In Exercises S1–S4, simplify each expression.

S1. $c + \frac{1}{2}c$

S2. $P + 0.07P + 0.02P$

S3. $2\pi r^2 + 2\pi r \cdot 2r$

S4. $\dfrac{12\pi - 2\pi}{6\pi}$

In Exercises S5–S8, find the value of the expressions for the given value of x and y.

S5. $x - 5y$ for $x = \frac{1}{2}$, $y = -5$.

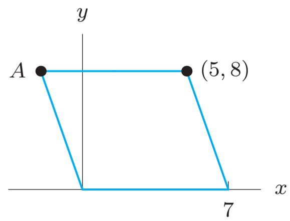
S6. $1 - 12x + x^2$ for $x = 3$.

S7. $\dfrac{3}{2 - x^3}$ for $x = -1$.

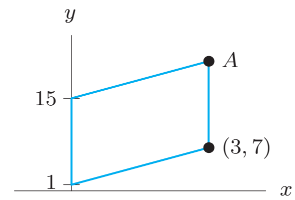
S8. $\dfrac{4}{1 + 1/x}$ for $x = -\frac{3}{4}$.

The figures in Exercises S9–S10 are parallelograms. Find the coordinates of the labeled point(s).

S9.



S10.



Exercises

1. Figure 1.4 gives the depth of the water at Montauk Point, New York, for a day in November.

(a) How many high tides took place on this day?
(b) How many low tides took place on this day?
(c) How much time elapsed in between high tides?

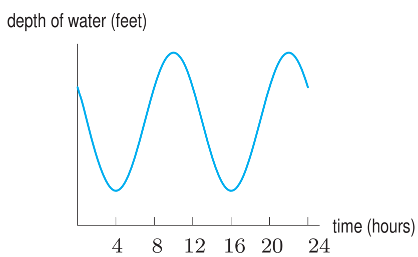


Figure 1.4

In Exercises 2–3, write the relationship using function notation (that is, y is a function of x is written $y = f(x)$).

2. Number of molecules, m, in a gas, is a function of the volume of the gas, v.

3. Weight, w, is a function of caloric intake, c.

In Exercises 4–7, label the axes for a sketch to illustrate the given statement.

4. "Over the past century we have seen changes in the population, P (in millions), of the city. . ."

5. "Sketch a graph of the cost of manufacturing q items. . ."

6. "Graph the pressure, p, of a gas as a function of its volume, v, where p is in pounds per square inch and v is in cubic inches."

7. "Graph D in terms of y. . ."

8. Using Table 1.4, graph $n = f(A)$, the number of gallons of paint needed to cover a house of area A. Identify the independent and dependent variables.

Table 1.4

A	0	250	500	750	1000	1250	1500
n	0	1	2	3	4	5	6

9. Use Table 1.5 to fill in the missing values. (There may be more than one answer.)

(a) $f(0) = ?$ **(b)** $f(?) = 0$
(c) $f(1) = ?$ **(d)** $f(?) = 1$

Table 1.5

x	0	1	2	3	4
$f(x)$	4	2	1	0	1

10. Use Figure 1.5 to fill in the missing values:

(a) $f(0) = ?$ **(b)** $f(?) = 0$

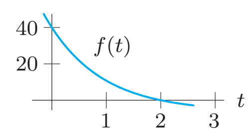


Figure 1.5

Exercises 11–14 use Figure 1.6.

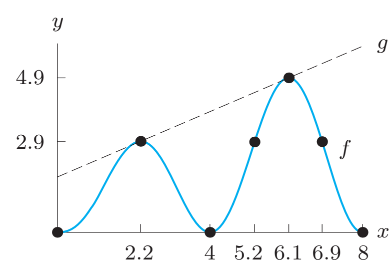


Figure 1.6

11. Find $f(6.9)$.

12. Give the coordinates of two points on the graph of g.

13. Solve $f(x) = 0$ for x.

14. Solve $f(x) = g(x)$ for x.

15. **(a)** You are going to graph $p = f(w)$. Which variable goes on the horizontal axis?
(b) If $10 = f(-4)$, give the coordinates of a point on the graph of f.
(c) If 6 is a solution of the equation $f(w) = 1$, give a point on the graph of f.

16. **(a)** Suppose x and y are the coordinates of a point on the circle $x^2 + y^2 = 1$. Is y a function of x? Why or why not?
(b) Suppose x and y are the coordinates of a point on the part of the circle $x^2 + y^2 = 1$ that is above the x-axis. Is y a function of x? Why or why not?

17. **(a)** Is the area, A, of a square a function of the length of one of its sides, s?
(b) Is the area, A, of a rectangle a function of the length of one of its sides, s?

18. Which of the following graphs represent functions?

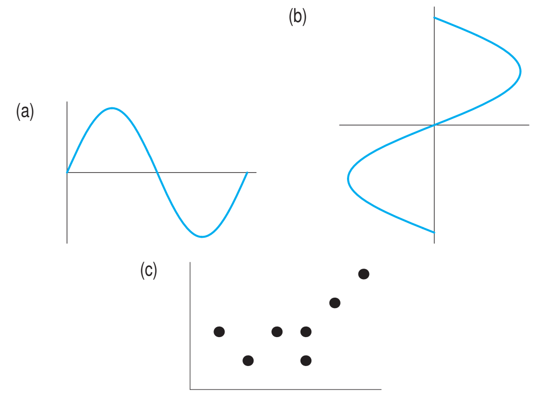


Figure 1.7

Problems

19. A buzzard is circling high overhead when it spies some road kill. It swoops down, lands, and eats. Later it takes off sluggishly, and resumes circling overhead, but at a lower altitude. Sketch a possible graph of the height of the buzzard as a function of time.

20. A person's blood sugar level at a particular time of the day is partially determined by the time of the most recent meal. After a meal, blood sugar level increases rapidly, then slowly comes back down to a normal level. Sketch a person's blood sugar level as a function of time over the course of a day. Label the axes to indicate normal blood sugar level and the time of each meal.

21. Let $f(t)$ be the number of people, in millions, who own cell phones t years after 1990. Explain the meaning of the following statements.

(a) $f(10) = 100.3$ **(b)** $f(a) = 20$
(c) $f(20) = b$ **(d)** $n = f(t)$

22. At the end of a semester, students' math grades are listed in a table which gives each student's ID number in the left column and the student's grade in the right column. Let N represent the ID number and the G represent the grade. Which quantity, N or G, must necessarily be a function of the other?

23. Table 1.6 gives the ranking r for three different names—Hannah, Alexis, and Madison. Of the three names, which was most popular and which was least popular in

(a) 1995? **(b)** 2004?

Table 1.6 *Ranking of names—Hannah (r_h), Alexis (r_a), and Madison (r_m)—for girls born between 1995 ($t = 0$) and 2004 ($t = 9$)*[1]

t	0	1	2	3	4	5	6	7	8	9
r_h	7	7	5	2	2	2	3	3	4	5
r_a	14	8	8	6	3	6	5	5	7	11
r_m	29	15	10	9	7	3	2	2	3	3

24. Table 1.6 gives information about the popularity of the names Hannah, Madison, and Alexis. Describe in words what your answers to parts (a)–(c) tell you about these names.

(a) Evaluate $r_m(0) - r_h(0)$.
(b) Evaluate $r_m(9) - r_h(9)$.
(c) Solve $r_m(t) < r_a(t)$.

25. Figure 1.8 shows the fuel consumption (in miles per gallon, mpg) of a car traveling at various speeds.

(a) How much gas is used on a 300-mile trip at 40 mph?
(b) How much gas is saved by traveling 60 mph instead of 70 mph on a 200-mile trip?
(c) According to this graph, what is the most fuel-efficient speed to travel? Explain.

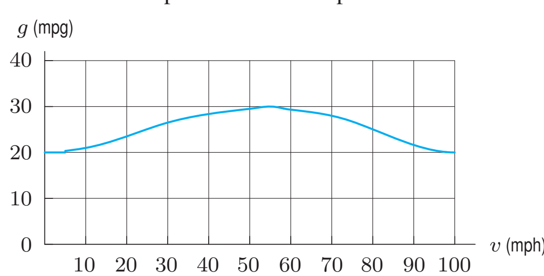


Figure 1.8

[1]Data from the SSA website at www.ssa.gov, accessed January 12, 2006.

26. **(a)** Ten inches of snow is equivalent to about one inch of rain.[2] Write an equation for the amount of precipitation, measured in inches of rain, $r = f(s)$, as a function of the number of inches of snow, s.

(b) Evaluate and interpret $f(5)$.

(c) Find s such that $f(s) = 5$ and interpret your result.

27. An 8-foot-tall cylindrical water tank has a base of diameter 6 feet.

(a) How much water can the tank hold?

(b) How much water is in the tank if the water is 5 feet deep?

(c) Write a formula for the volume of water as a function of its depth in the tank.

28. Match each story about a bike ride to one of the graphs (i)–(v), where d represents distance from home and t is time in hours since the start of the ride. (A graph may be used more than once.)

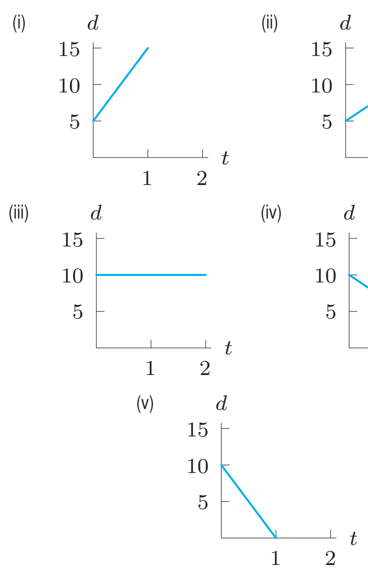
(a) Starts 5 miles from home and rides 5 miles per hour away from home.

(b) Starts 5 miles from home and rides 10 miles per hour away from home.

(c) Starts 10 miles from home and arrives home one hour later.

(d) Starts 10 miles from home and is halfway home after one hour.

(e) Starts 5 miles from home and is 10 miles from home after one hour.



29. Table 1.7 shows the daily low temperature for a one-week period in New York City during July.

(a) What was the low temperature on July 19?

(b) When was the low temperature 73°F?

(c) Is the daily low temperature a function of the date?

(d) Is the date a function of the daily low temperature?

Table 1.7

Date	17	18	19	20	21	22	23
Low temp (°F)	73	77	69	73	75	75	70

30. Use the data from Table 1.3 on page 5.

(a) Plot R on the vertical axis and t on the horizontal axis. Use this graph to explain why you believe that R is a function of t.

(b) Plot F on the vertical axis and t on the horizontal axis. Use this graph to explain why you believe that F is a function of t.

(c) Plot F on the vertical axis and R on the horizontal axis. From this graph show that F is not a function of R.

(d) Plot R on the vertical axis and F on the horizontal axis. From this graph show that R is not a function of F.

31. Since Roger Bannister broke the 4-minute mile on May 6, 1954, the record has been lowered by over sixteen seconds. Table 1.8 shows the year and times (as min:sec) of new world records for the one-mile run.[3] The last time the record was broken was in 1999.

(a) Is the time a function of the year? Explain.

(b) Is the year a function of the time? Explain.

(c) Let $y(r)$ be the year in which the world record, r, was set. Explain what is meant by the statement $y(3{:}47.33) = 1981$.

(d) Evaluate and interpret $y(3{:}51.1)$.

Table 1.8

Year	Time	Year	Time	Year	Time
1954	3:59.4	1966	3:51.3	1981	3:48.53
1954	3:58.0	1967	3:51.1	1981	3:48.40
1957	3:57.2	1975	3:51.0	1981	3:47.33
1958	3:54.5	1975	3:49.4	1985	3:46.32
1962	3:54.4	1979	3:49.0	1993	3:44.39
1964	3:54.1	1980	3:48.8	1999	3:43.13
1965	3:53.6				

[2] http://mo.water.usgs.gov/outreach/rain, accessed May 7, 2006.
[3] www.infoplease.com/ipsa/A0112924.html, accessed January 15, 2006.

32. Table 1.9 gives $A = f(d)$, the amount of money in bills of denomination d circulating in US currency in 2008.[4] For example, there were \$64.7 billion worth of \$50 bills in circulation.

(a) Find $f(100)$. What does this tell you about money?
(b) Are there more \$1 bills or \$5 bills in circulation?

Table 1.9

Denomination (\$)	1	2	5	10	20	50	100
Circulation (\$bn)	9.5	1.7	11	16.3	125.1	64.7	625

33. There are x male job-applicants at a certain company and y female applicants. Suppose that 15% of the men are accepted and 18% of the women are accepted. Write an expression in terms of x and y representing each of the following quantities:

(a) The total number of applicants to the company.
(b) The total number of applicants accepted.
(c) The percentage of all applicants accepted.

34. The sales tax on an item is 6%. Express the total cost, C, in terms of the price of the item, P.

35. Write a formula for the area of a circle as a function of its radius and determine the percent increase in the area if the radius is increased by 10%.

36. A price increases 5% due to inflation and is then reduced 10% for a sale. Express the final price as a function of the original price, P.

37. A chemical company spends \$2 million to buy machinery before it starts producing chemicals. Then it spends \$0.5 million on raw materials for each million liters of chemical produced.

(a) The number of liters produced ranges from 0 to 5 million. Make a table showing the relationship between the number of million liters produced, l, and the total cost, C, in millions of dollars, to produce that number of million liters.
(b) Find a formula that expresses C as a function of l.

38. A person leaves home and walks due west for a time and then walks due north.

(a) The person walks 10 miles in total. If w represents the (variable) distance west she walks, and D represents her (variable) distance from home at the end of her walk, is D a function of w? Why or why not?
(b) Suppose now that x is the distance that she walks in total. Is D a function of x? Why or why not?

1.2 RATE OF CHANGE

Sales of digital video disc (DVD) players have been increasing since they were introduced in early 1998. To measure how fast sales were increasing, we calculate a *rate of change* of the form

$$\frac{\text{Change in sales}}{\text{Change in time}}.$$

At the same time, sales of video cassette recorders (VCRs) have been decreasing. See Table 1.10.

Let us calculate the rate of change of DVD player and VCR sales between 1998 and 2003. Table 1.10 gives

$$\begin{array}{c}\text{Average rate of change of DVD}\\ \text{player sales from 1998 to 2003}\end{array} = \frac{\text{Change in DVD player sales}}{\text{Change in time}} = \frac{3050-421}{2003-1998} \approx 525.8 \begin{array}{l}\text{mn \$/}\\ \text{year.}\end{array}$$

Thus, DVD player sales increased on average by \$525.8 million per year between 1998 and 2003. See Figure 1.9. Similarly, Table 1.10 gives

$$\begin{array}{c}\text{Average rate of change of VCR sales}\\ \text{from 1998 to 2003}\end{array} = \frac{\text{Change in VCR sales}}{\text{Change in time}} = \frac{407-2409}{2003-1998} \approx -400.4 \begin{array}{l}\text{mn \$/}\\ \text{year.}\end{array}$$

[4] www.visualeconomics.com/the-value-of-united-states-currency-in-circulation, The Value of United States Currency in Circulation, 2008, accessed November 16, 2009.

Thus, VCR sales decreased on average by $\$400.4$ million per year between 1998 and 2003. See Figure 1.10.

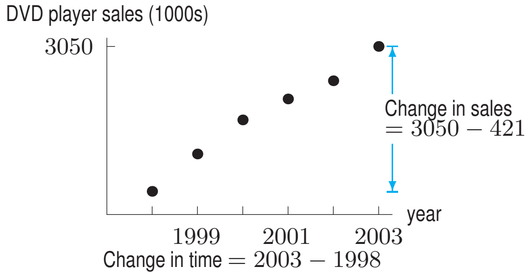


Figure 1.9: DVD player sales

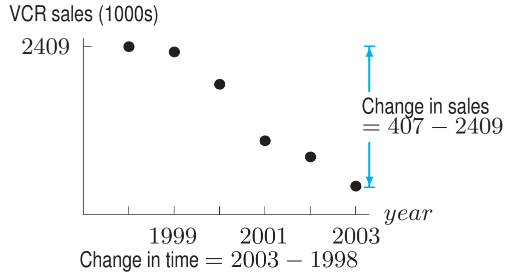


Figure 1.10: VCR sales

Table 1.10 *Annual sales of VCRs and DVD players in millions of dollars*[5]

Year	1998	1999	2000	2001	2002	2003
VCR sales (million \$)	2409	2333	1869	1058	826	407
DVD player sales (million \$)	421	1099	1717	2097	2427	3050

Rate of Change of a Function

The rate of change of sales is an example of the rate of change of a function. In general, if $Q = f(t)$, we write ΔQ for a change in Q and Δt for a change in t. We define:[6]

> The **average rate of change**, or **rate of change**, of Q with respect to t over an interval is
>
> $$\text{Average rate of change over an interval} = \frac{\text{Change in } Q}{\text{Change in } t} = \frac{\Delta Q}{\Delta t}.$$

The average rate of change of the function $Q = f(t)$ over an interval tells us how much Q changes, on average, for each unit change in t within that interval. On some parts of the interval, Q may be changing rapidly, while on other parts Q may be changing slowly. The average rate of change evens out these variations.

Increasing and Decreasing Functions

In the previous example, the average rate of change of DVD player sales is positive on the interval from 1998 to 2003 since sales of DVD players increased over this interval. Similarly, the average rate of change of VCR sales is negative on the same interval since sales of VCRs decreased over this interval. The annual sales of DVD players is an example of an *increasing function* and the annual sales of VCRs is an example of a *decreasing function*. In general we say the following:

> If $Q = f(t)$ for t in the interval $a \leq t \leq b$,
> - f is an **increasing function** if the values of f increase as t increases in this interval.
> - f is a **decreasing function** if the values of f decrease as t increases in this interval.

[5] www.census.gov/prod/2005pubs/06statab/manufact.pdf, accessed January 16, 2006.

[6] The Greek letter Δ, delta, is often used in mathematics to represent change. In this book, we use rate of change to mean average rate of change across an interval. In calculus, rate of change means something called instantaneous rate of change.

Looking at DVD player sales, we see that an increasing function has a positive rate of change. From the VCR sales, we see that a decreasing function has a negative rate of change. In general:

> If $Q = f(t)$,
> - If f is an increasing function, then the average rate of change of Q with respect to t is positive on every interval.
> - If f is a decreasing function, then the average rate of change of Q with respect to t is negative on every interval.

Example 1 The function $A = q(r) = \pi r^2$ gives the area, A, of a circle as a function of its radius, r. Graph q. Explain how the fact that q is an increasing function can be seen on the graph.

Solution The area increases as the radius increases, so $A = q(r)$ is an increasing function. We can see this in Figure 1.11 because the graph climbs as we move from left to right and the average rate of change, $\Delta A/\Delta r$, is positive on every interval.

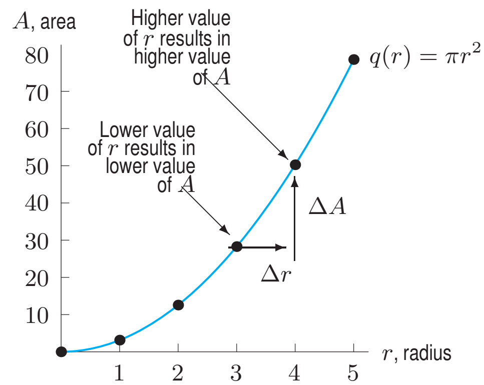


Figure 1.11: The graph of an increasing function, $A = q(r)$, rises when read from left to right

Example 2 Carbon-14 is a radioactive element that exists naturally in the atmosphere and is absorbed by living organisms. When an organism dies, the carbon-14 present at death begins to decay. Let $L = g(t)$ represent the quantity of carbon-14 (in micrograms, μg) in a tree t years after its death. See Table 1.11. Explain why we expect g to be a decreasing function of t. How is this represented on a graph?

Table 1.11 *Quantity of carbon-14 as a function of time*

t, time (years)	0	1000	2000	3000	4000	5000
L, quantity of carbon-14 (μg)	200	177	157	139	123	109

Solution Since the amount of carbon-14 is decaying over time, g is a decreasing function. In Figure 1.12, the graph falls as we move from left to right and the average rate of change in the level of carbon-14 with respect to time, $\Delta L/\Delta t$, is negative on every interval.

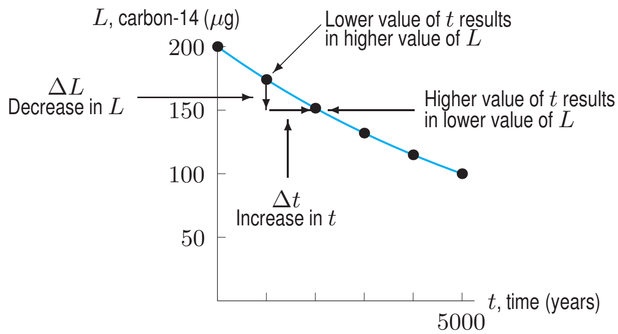


Figure 1.12: The graph of a decreasing function, $L = g(t)$, falls when read from left to right

In general, we can identify an increasing or decreasing function from its graph as follows:

- The graph of an increasing function rises when read from left to right.
- The graph of a decreasing function falls when read from left to right.

Many functions have some intervals on which they are increasing and other intervals on which they are decreasing. These intervals can often be identified from the graph.

Example 3 On what intervals is the function graphed in Figure 1.13 increasing? Decreasing?

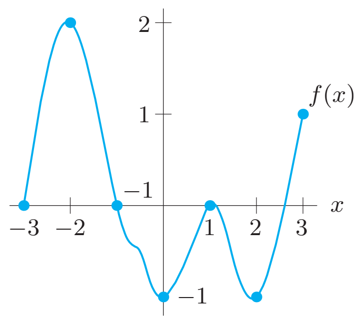


Figure 1.13: Graph of a function that is increasing on some intervals and decreasing on others

Solution The function appears to be increasing for values of x between -3 and -2, for x between 0 and 1, and for x between 2 and 3. The function appears to be decreasing for x between -2 and 0 and for x between 1 and 2. Using inequalities, we say that f is increasing for $-3 < x < -2$, for $0 < x < 1$, and for $2 < x < 3$. Similarly, f is decreasing for $-2 < x < 0$ and $1 < x < 2$.

Function Notation for the Average Rate of Change

Suppose we want to find the average rate of change of a function $Q = f(t)$ over the interval $a \leq t \leq b$. On this interval, the change in t is given by

$$\Delta t = b - a.$$

At $t = a$, the value of Q is $f(a)$, and at $t = b$, the value of Q is $f(b)$. Therefore, the change in Q is given by

$$\Delta Q = f(b) - f(a).$$

Using function notation, we express the average rate of change as follows:

$$\begin{array}{c}\text{Average rate of change of } Q = f(t) \\ \text{over the interval } a \le t \le b\end{array} = \frac{\text{Change in } Q}{\text{Change in } t} = \frac{\Delta Q}{\Delta t} = \frac{f(b) - f(a)}{b - a}.$$

In Figure 1.14, notice that the average rate of change is given by the ratio of the rise, $f(b)-f(a)$, to the run, $b-a$. This ratio is also called the *slope* of the dashed line segment.[7]

In the future, we may drop the word "average" and talk about the rate of change over an interval.

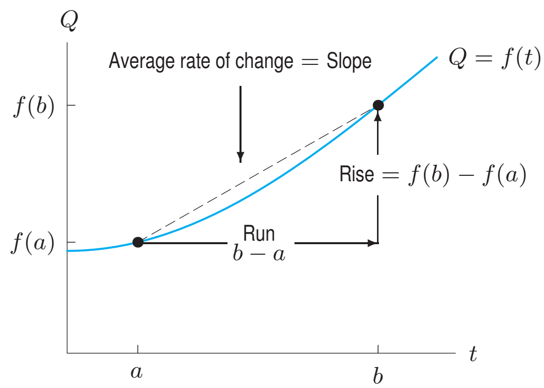


Figure 1.14: The average rate of change is the ratio Rise/Run

In previous examples we calculated the average rate of change from data. We now calculate average rates of change for functions given by formulas.

Example 4 Calculate the average rates of change of the function $f(x) = x^2$ between $x = 1$ and $x = 3$ and between $x = -2$ and $x = 1$. Show your results on a graph.

Solution Between $x = 1$ and $x = 3$, we have

$$\begin{array}{c}\text{Average rate of change of } f(x) \\ \text{over the interval } 1 \le x \le 3\end{array} = \frac{\text{Change in } f(x)}{\text{Change in } x} = \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{3^2 - 1^2}{3 - 1} = \frac{9 - 1}{2} = 4.$$

Between $x = -2$ and $x = 1$, we have

$$\begin{array}{c}\text{Average rate of change of } f(x) \\ \text{over the interval } -2 \le x \le 1\end{array} = \frac{\text{Change in } f(x)}{\text{Change in } x} = \frac{f(1) - f(-2)}{1 - (-2)}$$

$$= \frac{1^2 - (-2)^2}{1 - (-2)} = \frac{1 - 4}{3} = -1.$$

The average rate of change between $x = 1$ and $x = 3$ is positive because $f(x)$ is increasing on this interval. See Figure 1.15. However, on the interval from $x = -2$ and $x = 1$, the function is partly decreasing and partly increasing. The average rate of change on this interval is negative because the decrease on the interval is larger than the increase.

[7] See Section 1.3 for further discussion of slope.

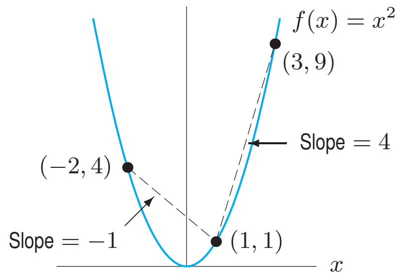


Figure 1.15: Average rate of change of $f(x)$ on an interval is the slope of the dashed line on that interval

Exercises and Problems for Section 1.2

Skill Refresher

In Exercises S1–S10, simplify each expression.

S1. $\dfrac{4-6}{3-2}$

S2. $\dfrac{1-3}{2^2-(-3)^2}$

S3. $\dfrac{-3-(-9)}{-1-2}$

S4. $\dfrac{(1-3^2)-(1-4^2)}{3-4}$

S5. $\dfrac{(\frac{1}{2}-(-4)^2-(\frac{1}{2}-(5^2))}{-4-5}$

S6. $2(x+a)-3(x-b)$

S7. $x^2-(2x+a)^2$

S8. $4x^2-(x-b)^2$

S9. $\dfrac{x^2-\frac{3}{4}-\left(y^2-\frac{3}{4}\right)}{x-y}$

S10. $\dfrac{2(x+h)^2-2x^2}{(x+h)-x}$

Exercises

1. In 2005, you have 40 CDs in your collection. In 2008, you have 120 CDs. In 2012, you have 40. What is the average rate of change in your CD collection's size between

(a) 2005 and 2008? **(b)** 2008 and 2012?
(c) 2005 and 2012?

2. Table 1.10 on page 11 gives the annual sales (in millions) of VCRs and DVD players. What was the average rate of change of annual sales of each of them between

(a) 1998 and 2000? **(b)** 2000 and 2003?
(c) Interpret these results in terms of sales.

3. Table 1.10 on page 11 shows that VCR sales are a function of DVD player sales. Is it an increasing or decreasing function?

4. Table 1.12 shows data for two populations (in hundreds) for five different years. Find the average rate of change of each population over the following intervals.

(a) 1990 to 2000 **(b)** 1995 to 2007
(c) 1990 to 2007

Table 1.12

Year	1990	1992	1995	2000	2007
P_1	53	63	73	83	93
P_2	85	80	75	70	65

Exercises 5–9 use Figure 1.16.

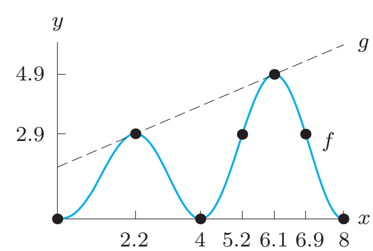


Figure 1.16

5. Find the average rate of change of f for $2.2 \le x \le 6.1$.

6. Give two different intervals on which $\Delta f(x)/\Delta x = 0$.

7. What is the average rate of change of g between $x = 2.2$ and $x = 6.1$?

8. What is the relation between the average rate of change of f and the average rate of change of g between $x = 2.2$ and $x = 6.1$?

9. Is the rate of change of f positive or negative on the following intervals?

(a) $2.2 \le x \le 4$ **(b)** $5 \le x \le 6$

10. If G is an increasing function, what can you say about $G(3) - G(-1)$?

11. If F is a decreasing function, what can you say about $F(-2)$ compared to $F(2)$?

12. Figure 1.17 shows distance traveled as a function of time.

(a) Find ΔD and Δt between:

(i) $t = 2$ and $t = 5$ (ii) $t = 0.5$ and $t = 2.5$
(iii) $t = 1.5$ and $t = 3$

(b) Compute the rate of change, $\Delta D/\Delta t$, over each of the intervals in part (a), and interpret its meaning.

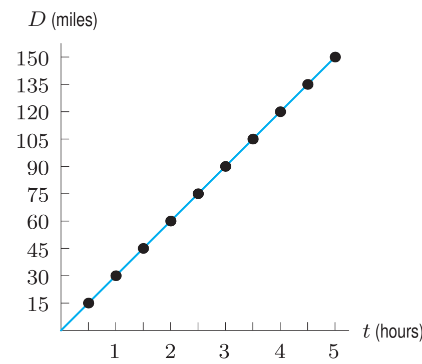


Figure 1.17

Problems

13. Figure 1.18 shows the percent of the side of the moon toward the earth illuminated by the sun at different times during the year 2008. Use the figure to answer the following questions.

(a) Give the coordinates of the points A, B, C, D, E.
(b) Plot the point $F = (15, 60)$ and $G = (60, 15)$. Which point is on the graph?
(c) During which time intervals is the function increasing?
(d) During which time intervals is the function decreasing?

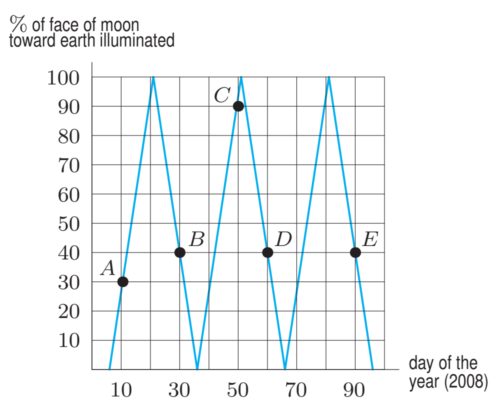


Figure 1.18: Moon phases

14. Imagine you constructed a list of the world record times for a particular event—such as the mile footrace, or the 100-meter freestyle swimming race—in terms of when they were established. Is the world record time a function of the date when it was established? If so, is this function increasing or decreasing? Explain. Could a world record be established twice in the same year? Is the world record time a function of the year it was established?

15. **(a)** What is the average rate of change of $g(x) = 2x - 3$ between the points $(-2, -7)$ and $(3, 3)$?
(b) Based on your answer to part (a), is g increasing or decreasing on the given interval? Explain.
(c) Graph the function and determine over what intervals g is increasing and over what intervals g is decreasing.

16. **(a)** Let $f(x) = 16 - x^2$. Compute each of the following expressions, and interpret each as an average rate of change.

(i) $\dfrac{f(2) - f(0)}{2 - 0}$ (ii) $\dfrac{f(4) - f(2)}{4 - 2}$
(iii) $\dfrac{f(4) - f(0)}{4 - 0}$

(b) Graph $f(x)$. Illustrate each ratio in part (a) by sketching the line segment with the given slope. Over which interval is the average rate of decrease the greatest?

17. Figure 1.19 gives the population of two different towns over a 50-year period of time.

(a) Which town starts (in year $t = 0$) with the most people?
(b) Which town is growing faster over these 50 years?

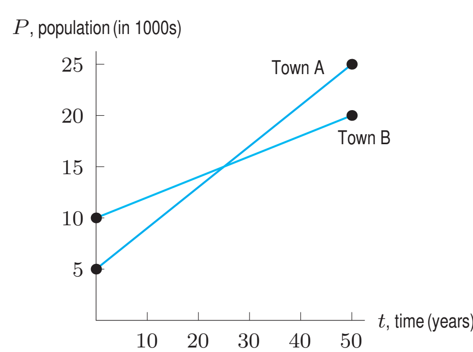


Figure 1.19

18. You have zero dollars now and the average rate of change in your net worth is \$5000 per year. How much money will you have in forty years?

19. The most freakish change in temperature ever recorded was from $-4°$F to $45°$F between 7:30 am and 7:32 am on January 22, 1943 at Spearfish, South Dakota.[8] What was the average rate of change of the temperature for this time period?

20. The surface of the sun has dark areas known as sunspots, that are cooler than the rest of the sun's surface. The number of sunspots fluctuates with time, as shown in Figure 1.20.[9]

(a) Explain how you know the number of sunspots, s, in year t is a function of t.
(b) Approximate the time intervals on which s is an increasing function of t.

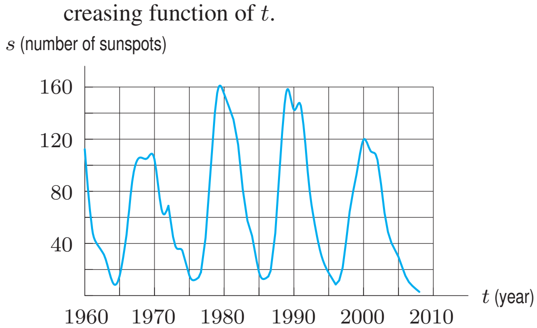


Figure 1.20

21. Table 1.13 shows the number of calories used per minute as a function of body weight for three sports.[10]

(a) Determine the number of calories that a 200-lb person uses in one half-hour of walking.
(b) Who uses more calories, a 120-lb person swimming for one hour or a 220-lb person bicycling for a half-hour?
(c) Does the number of calories used by a person walking increase or decrease as weight increases?

Table 1.13

Activity	100 lb	120 lb	150 lb	170 lb	200 lb	220 lb
Walking	2.7	3.2	4.0	4.6	5.4	5.9
Bicycling	5.4	6.5	8.1	9.2	10.8	11.9
Swimming	5.8	6.9	8.7	9.8	11.6	12.7

22. Because scientists know how much carbon-14 a living organism should have in its tissues, they can measure the amount of carbon-14 present in the tissue of a fossil and then calculate how long it took for the original amount to decay to the current level, thus determining the time of the organism's death. A tree fossil is found to contain 130 μg of carbon-14, and scientists determine from the size of the tree that it would have contained 200 μg of carbon-14 at the time of its death. Using Table 1.11 on page 12, approximately how long ago did the tree die?

23. Find the average rate of change of $f(x) = 3x^2 + 1$ between the points

(a) $(1, 4)$ and $(2, 13)$ **(b)** (j, k) and (m, n)
(c) $(x, f(x))$ and $(x+h, f(x+h))$

24. Figure 1.21 shows the graph of the function $g(x)$.

(a) Estimate $\dfrac{g(4) - g(0)}{4 - 0}$.
(b) The ratio in part (a) is the slope of a line segment joining two points on the graph. Sketch this line segment on the graph.
(c) Estimate $\dfrac{g(b) - g(a)}{b - a}$ for $a = -9$ and $b = -1$.
(d) On the graph, sketch the line segment whose slope is given by the ratio in part (c).

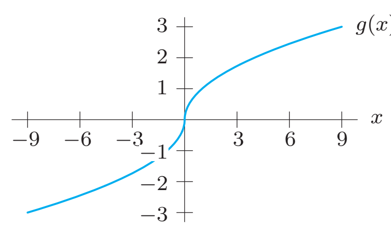


Figure 1.21

25. Table 1.14 gives the amount of garbage, G, in millions of tons, produced[11] in the US in year t.

(a) What is the value of Δt for consecutive entries in this table?
(b) Calculate the value of ΔG for each pair of consecutive entries in this table.
(c) Are all the values of ΔG you found in part (b) the same? What does this tell you?
(d) The function G changed from increasing to decreasing between 2007 and 2008. To what might this be attributed?

Table 1.14

t	1960	1970	1980	1990	2000	2007	2008
G	88.1	121.1	151.6	205.2	239.1	254.6	249.6

[8] *The Guinness Book of Records.* 1995.
[9] ftp://ftp.ngdc.noaa.gov/STP/SOLAR DATA/SUNSPOT NUMBERS/YEARLY.PLT, accessed November 30, 2009.
[10] From *1993 World Almanac.*
[11] http://www.epa.gov/osw/nonhaz/municipal/pubs/msw2008rpt.pdf, accessed November 23, 2009.

26. Table 1.15 shows the times, t, in sec, achieved every 10 meters by Carl Lewis in the 100-meter final of the World Championship in Rome in 1987.[12] Distance, d, is in meters.

(a) For each successive time interval, calculate the average rate of change of distance. What is a common name for the average rate of change of distance?

(b) Where did Carl Lewis attain his maximum speed during this race? Some runners are running their fastest as they cross the finish line. Does that seem to be true in this case?

Table 1.15

t	0.00	1.94	2.96	3.91	4.78	5.64
d	0	10	20	30	40	50
t	6.50	7.36	8.22	9.07	9.93	
d	60	70	80	90	100	

1.3 LINEAR FUNCTIONS

Constant Rate of Change

In the previous section, we introduced the average rate of change of a function on an interval. For many functions, the average rate of change is different on different intervals. For the remainder of this chapter, we consider functions that have the same average rate of change on every interval. Such a function has a graph that is a line and is called *linear*.

Population Growth

Mathematical models of population growth are used by city planners to project the growth of towns and states. Biologists model the growth of animal populations and physicians model the spread of an infection in the bloodstream. One possible model, a linear model, assumes that the population changes at the same average rate on every time interval.

Example 1 A town of 30,000 people grows by 2000 people every year. Since the population, P, is growing at the constant rate of 2000 people per year, P is a linear function of time, t, in years.

(a) What is the average rate of change of P over every time interval?
(b) Make a table that gives the town's population every five years over a 20-year period. Graph the population.
(c) Find a formula for P as a function of t.

Solution

(a) The average rate of change of population with respect to time is 2000 people per year.
(b) The initial population in year $t = 0$ is $P = 30{,}000$ people. Since the town grows by 2000 people every year, after five years it has grown by

$$\frac{2000 \text{ people}}{\text{year}} \cdot 5 \text{ years} = 10{,}000 \text{ people.}$$

Thus, in year $t = 5$ the population is given by

$$P = \text{Initial population} + \text{New population} = 30{,}000 + 10{,}000 = 40{,}000.$$

In year $t = 10$ the population is given by

$$P = 30{,}000 + \underbrace{2000 \text{ people/year} \cdot 10 \text{ years}}_{20{,}000 \text{ new people}} = 50{,}000.$$

[12] W. G. Pritchard, "Mathematical Models of Running", *SIAM Review.* 35, 1993, pp. 359–379.

Similar calculations for year $t = 15$ and year $t = 20$ give the values in Table 1.16. See Figure 1.22; the dashed line shows the trend in the data.

Table 1.16 *Population over 20 years*

t, years	P, population
0	30,000
5	40,000
10	50,000
15	60,000
20	70,000

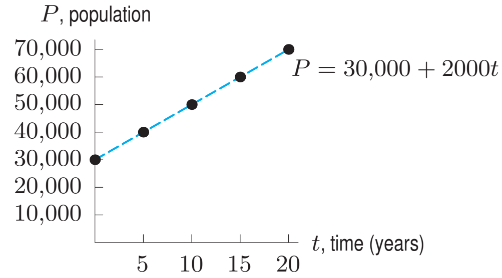


Figure 1.22: Town's population over 20 years

(c) From part (b), we see that the size of the population is given by

$$\begin{aligned} P &= \text{Initial population} + \text{Number of new people} \\ &= 30{,}000 + 2000 \text{ people/year} \cdot \text{Number of years}, \end{aligned}$$

so a formula for P in terms of t is

$$P = 30{,}000 + 2000t.$$

The graph of the population data in Figure 1.22 is a straight line. The average rate of change of the population over every interval is the same, namely 2000 people per year. Any linear function has the same average rate of change over every interval. Thus, we talk about *the* rate of change of a linear function. In general:

- A **linear function** has a constant rate of change.
- The graph of any linear function is a straight line.

Financial Models

Economists and accountants use linear functions for *straight-line depreciation*. For tax purposes, the value of certain equipment is considered to decrease, or depreciate, over time. For example, computer equipment may be state-of-the-art today, but after several years it is outdated. Straight-line depreciation assumes that the rate of change of value with respect to time is constant.

Example 2 A small business spends $\$20{,}000$ on new computer equipment and, for tax purposes, chooses to depreciate it to $\$0$ at a constant rate over a five-year period.

(a) Make a table and a graph showing the value of the equipment over the five-year period.
(b) Give a formula for value as a function of time.

Solution (a) After five years, the equipment is valued at $\$0$. If V is the value in dollars and t is the number of years, we see that

$$\begin{array}{c}\text{Rate of change of value} \\ \text{from } t = 0 \text{ to } t = 5\end{array} = \frac{\text{Change in value}}{\text{Change in time}} = \frac{\Delta V}{\Delta t} = \frac{-\$20{,}000}{5 \text{ years}} = -\$4000 \text{ per year.}$$

Thus, the value drops at the constant rate of $\$4000$ per year. (Notice that ΔV is negative because the value of the equipment decreases.) See Table 1.17 and Figure 1.23. Since V changes at a constant rate, $V = f(t)$ is a linear function and its graph is a straight line. The rate of change, $-\$4000$ per year, is negative because the function is decreasing and the graph slopes down.

Table 1.17 *Value of equipment depreciated over a 5-year period*

t, year	V, value (\$)
0	20,000
1	16,000
2	12,000
3	8,000
4	4,000
5	0

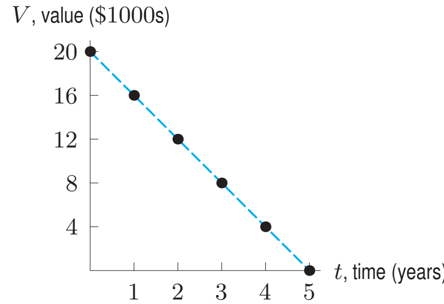


Figure 1.23: Value of equipment depreciated over a 5-year period

(b) After t years have elapsed,

$$\text{Decrease in value of equipment} = \$4000 \cdot \text{ Number of years } = \$4000t.$$

The initial value of the equipment is \$20,000, so at time t,

$$V = 20{,}000 - 4000t.$$

The total cost of production is another application of linear functions in economics.

A General Formula for the Family of Linear Functions

Example 1 involved a town whose population is growing at a constant rate with formula

$$\text{Current population} = \underbrace{\text{Initial population}}_{30{,}000 \text{ people}} + \underbrace{\text{Growth rate}}_{2000 \text{ people per year}} \times \underbrace{\text{Number of years}}_{t}$$

so

$$P = 30{,}000 + 2000t.$$

In Example 2, the value, V, as a function of t is given by

$$\text{Total cost} = \underbrace{\text{Initial value}}_{\$20{,}000} + \underbrace{\text{Change per year}}_{-\$4000 \text{ per year}} \times \underbrace{\text{Number of years}}_{t}$$

so

$$V = 20{,}000 + (-4000)t.$$

Using the symbols x, y, b, m, we see formulas for both of these linear functions follow the same pattern:

$$\underbrace{\text{Output}}_{y} = \underbrace{\text{Initial value}}_{b} + \underbrace{\text{Rate of change}}_{m} \times \underbrace{\text{Input}}_{x}.$$

Summarizing, we get the following results:

If $y = f(x)$ is a linear function, then for some constants b and m:

$$y = b + mx.$$

- m is called the **slope**, and gives the rate of change of y with respect to x. Thus,

$$m = \frac{\Delta y}{\Delta x}.$$

If (x_0, y_0) and (x_1, y_1) are any two distinct points on the graph of f, then

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}.$$

- b is called the **vertical intercept**, or ***y*-intercept**, and gives the value of y for $x = 0$. In mathematical models, b typically represents an initial, or starting, value of the output.

Every linear function can be written in the form $y = b + mx$. Different linear functions have different values for m and b. These constants are known as *parameters*.

Example 3 In Example 1, the population function, $P = 30{,}000 + 2000t$, has slope $m = 2000$ and vertical intercept $b = 30{,}000$. In Example 2, the value of the computer equipment, $V = 20{,}000 - 4000t$, has slope $m = -4000$ and vertical intercept $b = 20{,}000$.

Tables for Linear Functions

A table of values could represent a linear function if the rate of change is constant, for all pairs of points in the table; that is,

$$\text{Rate of change of linear function} = \frac{\text{Change in output}}{\text{Change in input}} = \text{Constant}.$$

Thus, if the value of x goes up by equal steps in a table for a linear function, then the value of y goes up (or down) by equal steps as well. We say that changes in the value of y are *proportional* to changes in the value of x.

Example 4 Table 1.18 gives values of two functions, p and q. Could either of these functions be linear?

Table 1.18 *Values of two functions p and q*

x	50	55	60	65	70
$p(x)$	0.10	0.11	0.12	0.13	0.14
$q(x)$	0.01	0.03	0.06	0.14	0.15

Solution The value of x goes up by equal steps of $\Delta x = 5$. The value of $p(x)$ also goes up by equal steps of $\Delta p = 0.01$, so $\Delta p/\Delta x$ is a constant. See Table 1.19. Thus, p could be a linear function.

Table 1.19 *Values of $\Delta p/\Delta x$*

x	$p(x)$	Δp	$\Delta p/\Delta x$
50	0.10		
		0.01	0.002
55	0.11		
		0.01	0.002
60	0.12		
		0.01	0.002
65	0.13		
		0.01	0.002
70	0.14		

Table 1.20 *Values of $\Delta q/\Delta x$*

x	$q(x)$	Δq	$\Delta q/\Delta x$
50	0.01		
		0.02	0.004
55	0.03		
		0.03	0.006
60	0.06		
		0.08	0.016
65	0.14		
		0.01	0.002
70	0.15		

In contrast, the value of $q(x)$ does not go up by equal steps. The value climbs by 0.02, then by 0.03, and so on. See Table 1.20. This means that $\Delta q/\Delta x$ is not constant. Thus, q could not be a linear function.

It is possible to have data from a linear function in which neither the x-values nor the y-values go up by equal steps. However the rate of change must be constant, as in the following example.

Example 5 The former Republic of Yugoslavia exported cars called Yugos to the US between 1985 and 1989. The car is now a collector's item.[13] Table 1.21 gives the quantity of Yugos sold, Q, and the price, p, for each year from 1985 to 1988.

(a) Using Table 1.21, explain why Q could be a linear function of p.
(b) What does the rate of change of this function tell you about Yugos?

Table 1.21 *Price and sales of Yugos in the US*

Year	Price in \$, p	Number sold, Q
1985	3990	49,000
1986	4110	43,000
1987	4200	38,500
1988	4330	32,000

Solution (a) We are interested in Q as a function of p, so we plot Q on the vertical axis and p on the horizontal axis. The data points in Figure 1.24 appear to lie on a straight line, suggesting a linear function.

[13] www.inet.hr/~pauric/epov.htm, accessed January 16, 2006.

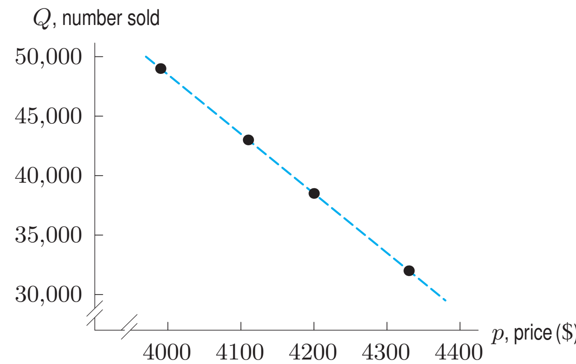


Figure 1.24: Since the data from Table 1.21 falls on a straight line, the table could represent a linear function

To provide further evidence that Q is a linear function, we check that the rate of change of Q with respect to p is constant for the points given. When the price of a Yugo rose from $3990 to $4110, sales fell from 49,000 to 43,000. Thus,

$$\Delta p = 4110 - 3990 = 120,$$

$$\Delta Q = 43{,}000 - 49{,}000 = -6000.$$

Since the number of Yugos sold decreased, ΔQ is negative. Thus, as the price increased from $3990 to $4110,

$$\text{Rate of change of quantity as price increases} = \frac{\Delta Q}{\Delta p} = \frac{-6000}{120} = -50 \text{ cars per dollar.}$$

Next, we calculate the rate of change as the price increased from $4110 to $4200 to see if the rate remains constant:

$$\text{Rate of change} = \frac{\Delta Q}{\Delta p} = \frac{38{,}500 - 43{,}000}{4200 - 4110} = \frac{-4500}{90} = -50 \text{ cars per dollar,}$$

and as the price increased from $4200 to $4330:

$$\text{Rate of change} = \frac{\Delta Q}{\Delta p} = \frac{32{,}000 - 38{,}500}{4330 - 4200} = \frac{-6500}{130} = -50 \text{ cars per dollar.}$$

Since the rate of change, -50, is constant, Q could be a linear function of p. Given additional data, $\Delta Q/\Delta p$ might not remain constant. However, based on the table, it appears that the function is linear.

(b) Since ΔQ is the change in the number of cars sold and Δp is the change in price, the rate of change is -50 cars per dollar. Thus the number of Yugos sold decreased by 50 each time the price increased by $1.

Warning: Not All Graphs That Look Like Lines Represent Linear Functions

The graph of any linear function is a line. However, a function's graph can look like a line without actually being one. Consider the following example.

Example 6 The function $P = 100(1.02)^t$ approximates the population of Mexico in the early 2000s. Here P is the population (in millions) and t is the number of years since 2000. Table 1.22 and Figure 1.25 show values of P over a 5-year period. Is P a linear function of t?

Table 1.22 *Population of Mexico t years after 2000*

t (years)	P (millions)
0	100
1	102
2	104.04
3	106.12
4	108.24
5	110.41

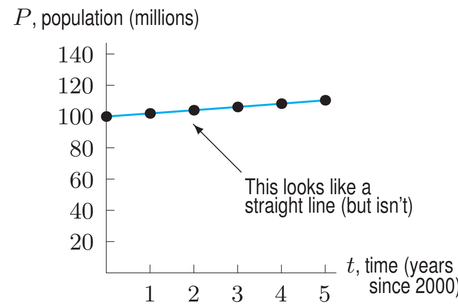


Figure 1.25: Graph of $P = 100(1.02)^t$ over 5-year period: Looks linear (but is not)

Solution The formula $P = 100(1.02)^t$ is not of the form $P = b + mt$, so P is not a linear function of t. However, the graph of P in Figure 1.25 appears to be a straight line. We check P's rate of change in Table 1.22. When $t = 0$, $P = 100$ and when $t = 1$, $P = 102$. Thus, between 2000 and 2001,

$$\text{Rate of change of population} = \frac{\Delta P}{\Delta t} = \frac{102 - 100}{1 - 0} = 2.$$

For the interval from 2001 to 2002, we have

$$\text{Rate of change} = \frac{\Delta P}{\Delta t} = \frac{104.04 - 102}{2 - 1} = 2.04,$$

and for the interval from 2004 to 2005, we have

$$\text{Rate of change} = \frac{\Delta P}{\Delta t} = \frac{110.41 - 108.24}{5 - 4} = 2.17.$$

Thus, P's rate of change is not constant. In fact, P appears to be increasing at a faster and faster rate. Table 1.23 and Figure 1.26 show values of P over a longer (60-year) period. On this scale, these points do not appear to fall on a straight line. However, the graph of P curves upward so gradually at first that over the short interval shown in Figure 1.25, it barely curves at all. The graphs of many nonlinear functions, when viewed on a small scale, appear to be linear.

Table 1.23 *Population over 60 years*

t (years since 2000)	P (millions)
0	100
10	121.90
20	148.59
30	181.14
40	220.80
50	269.16
60	328.10

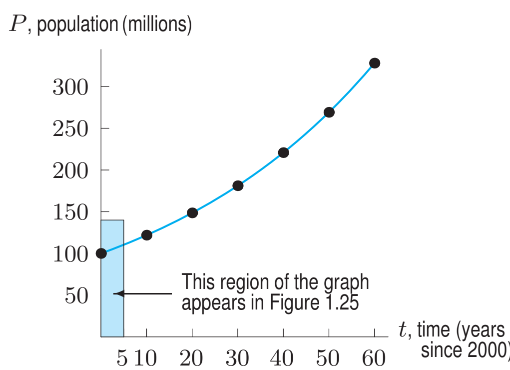


Figure 1.26: Graph of $P = 100(1.02)^t$ over 60 years: Not linear

Exercises and Problems for Section 1.3

Skill Refresher

In Exercises S1–S2, find $f(0)$ and $f(3)$.

S1. $f(x) = \frac{2}{3}x + 5$

S2. $f(t) = 17 - 4t$

In Exercises S3–S4, find $f(2) - f(0)$.

S3.

x	0	1	2	3
$f(x)$	-2	0	3	4

S4.

t	-1	0	1	2
$f(t)$	0	2	7	-1

In Exercises S5–S6, find the coordinates of the x and y intercepts.

S5. $y = -4x + 3$

S6. $5x - 2y = 4$

For each of the linear expressions in x in Exercises S7–S10, give the constant term and the coefficient of x.

S7. $3 - 2x + \frac{1}{2}$

S8. $4 - 3(x + 2) + 6(2x - 1)$

S9. $ax - ab - 3x + a + 3$

S10. $5(x - 1) + 3$

Exercises

Which of the tables in Exercises 1–6 could represent a linear function?

1.

x	0	100	300	600
$g(x)$	50	100	150	200

2.

x	0	10	20	30
$h(x)$	20	40	50	55

3.

t	1	2	3	4	5
$g(t)$	5	4	5	4	5

4.

x	0	5	10	15
$f(x)$	10	20	30	40

5.

γ	9	8	7	6	5
$p(\gamma)$	42	52	62	72	82

6.

x	-3	-1	0	3
$j(x)$	5	1	-1	-7

In Exercises 7–8, which line has the greater

(a) Slope? **(b)** y-intercept?

7. $y = 7 + 2x, \quad y = 8 - 15x$

8. $y = 5 - 2x, \quad y = 7 - 3x$

In Exercises 9–12, identify the vertical intercept and the slope, and explain their meanings in practical terms.

9. The population of a town can be represented by the formula $P(t) = 54.25 - \dfrac{2}{7}t$, where $P(t)$ represents the population, in thousands, and t represents the time, in years, since 1970.

10. A stalactite grows according to the formula $L(t) = 17.75 + \dfrac{1}{250}t$, where $L(t)$ represents the length of the stalactite, in inches, and t represents the time, in years, since the stalactite was first measured.

11. The profit, in dollars, of selling n items is given by $P(n) = 0.98n - 3000$.

12. A phone company charges according to the formula $C(n) = 29.99 + 0.05n$, where n is the number of minutes, and $C(n)$ is the monthly phone charge, in dollars.

Problems

13. Table 1.24 gives the proposed fine $r = f(v)$ to be imposed on a motorist for speeding, where v is the motorist's speed and 55 mph is the speed limit.

(a) Decide whether f appears to be linear.
(b) What would the rate of change represent in practical terms for the motorist?
(c) Plot the data points.

Table 1.24

v (mph)	60	65	70	75	80	85
r (dollars)	75	100	125	150	175	200

14. In 2006, the population of a town was 18,310 and growing by 58 people per year. Find a formula for P, the town's population, in terms of t, the number of years since 2006.

15. A new Toyota RAV4 costs \$21,500. The car's value depreciates linearly to \$11,900 in three years time. Write a formula which expresses its value, V, in terms of its age, t, in years.

16. In 2003, the number, N, of cases of SARS (Severe Acute Respiratory Syndrome) reported in Hong Kong[14] was initially approximated by $N = 78.9 + 30.1t$, where t is the number of days since March 17. Interpret the constants 78.9 and 30.1.

17. Table 1.25 shows the cost C, in dollars, of selling x cups of coffee per day from a cart.

(a) Using the table, show that the relationship appears to be linear.
(b) Plot the data in the table.
(c) Find the slope of the line. Explain what this means in the context of the given situation.
(d) Why should it cost \$50 to serve zero cups of coffee?

Table 1.25

x	0	5	10	50	100	200
C	50.00	51.25	52.50	62.50	75.00	100.00

18. In each case, graph a linear function with the given rate of change. Label and put scales on the axes.

(a) Increasing at 2.1 inches/day
(b) Decreasing at 1.3 gallons/mile

19. A flight costs \$10,000 to operate, regardless of the number of passengers. Each ticket costs \$127. Express profit, π, as a linear function of the number of passengers, n, on the flight.

20. A small café sells drip coffee for \$0.95 per cup. On average, it costs the café \$0.25 to make a cup of coffee (for grounds, hot water, filters). The café also has a fixed daily cost of \$200 (for rent, wages, utilities).

(a) Let R, C, and P be the café's daily revenue, costs, and profit, respectively, for selling x cups of coffee in a day. Find formulas for R, C, and P as functions of x. [Hint: The revenue, R, is the total amount of money that the café brings in. The cost, C, includes the fixed daily cost as well as the cost for all x cups of coffee sold. P is the café's profit after costs have been accounted for.]
(b) Plot P against x. For what x-values is the graph of P below the x-axis? Above the x-axis? Interpret your results.
(c) Interpret the slope and both intercepts of your graph in practical terms.

21. Owners of an inactive quarry in Australia have decided to resume production. They estimate that it will cost them \$1000 per month to maintain and insure their equipment and that monthly salaries will be \$3000. It costs \$80 to mine a ton of rocks. Write a formula that expresses the total cost each month, c, as a function of r, the number of tons of rock mined per month.

22. Table 1.26 gives the area and perimeter of a square as a function of the length of its side.

(a) From the table, decide if either area or perimeter could be a linear function of side length.
(b) From the data make two graphs, one showing area as a function of side length, the other showing perimeter as a function of side length. Connect the points.
(c) If you find a linear relationship, give its corresponding rate of change and interpret its significance.

Table 1.26

Length of side	0	1	2	3	4	5	6
Area of square	0	1	4	9	16	25	36
Perimeter of square	0	4	8	12	16	20	24

23. Make two tables, one comparing the radius of a circle to its area, the other comparing the radius of a circle to its circumference. Repeat parts (a), (b), and (c) from Problem 22, this time comparing radius with circumference, and radius with area.

24. Sri Lanka is an island that experienced approximately linear population growth from 1950 to 2000. On the other hand, Afghanistan was torn by warfare in the 1980s and did not experience linear nor near-linear growth.[15]

(a) Table 1.27 gives the population of these two countries, in millions. Which of these two countries is A and which is B? Explain.
(b) What is the approximate rate of change of the linear function? What does the rate of change represent in practical terms?
(c) Estimate the population of Sri Lanka in 1988.

Table 1.27

Year	1950	1960	1970	1980	1990	2000
Population of country A	8.2	9.8	12.4	15.1	14.7	23.9
Population of country B	7.5	9.9	12.5	14.9	17.2	19.2

[14] World Health Organization, www.who.int/csr/sars/country/en.
[15] www.census.gov/ipc/www/idbsusum.html, accessed January 12, 2006.

25. Table 1.44 on page 54 gives the temperature-depth profile, $T = f(d)$, in a borehole in Belleterre, Quebec, where T is the average temperature at a depth d.

(a) Could f be linear?
(b) Graph f. What do you notice about the graph for $d \geq 150$?
(c) What can you say about the average rate of change of f for $d \geq 150$?

26. The summit of Africa's largest peak, Mt. Kilimanjaro, consists of the northern and southern ice fields and the Furtwanger glacier. An article in the Proceedings of the National Academy of Sciences[16] indicates that in 2000 ($t = 0$) the area of the ice cover at the peak of Mt. Kilimanjaro was approximately 1951 m^2. By 2007, the area had shrunk to approximately 1555 m^2.

(a) If this decline is modeled by a linear function, find $A = f(t)$, the equation of the ice-cover area as a function of time. Explain what the slope and A-intercept mean in terms of the ice cover.
(b) Evaluate $f(11)$.
(c) If this model is correct, when would you expect the ice cover to disappear?

27. Tuition cost T (in dollars) for part-time students at Stonewall College is given by $T = 300 + 200C$, where C represents the number of credits taken.

(a) Find the tuition cost for eight credits.
(b) How many credits were taken if the tuition was \$1700?
(c) Make a table showing costs for taking from one to twelve credits. For each value of C, give both the tuition cost, T, and the cost per credit, T/C. Round to the nearest dollar.
(d) Which of these values of C has the smallest cost per credit?
(e) What does the 300 represent in the formula for T?
(f) What does the 200 represent in the formula for T?

28. A company finds that there is a linear relationship between the amount of money that it spends on advertising and the number of units it sells. If it spends no money on advertising, it sells 300 units. For each additional \$5000 spent, an additional 20 units are sold.

(a) If x is the amount of money that the company spends on advertising, find a formula for y, the number of units sold as a function of x.
(b) How many units does the firm sell if it spends \$25,000 on advertising? \$50,000?
(c) How much advertising money must be spent to sell 700 units?
(d) What is the slope of the line you found in part (a)? Give an interpretation of the slope that relates units sold and advertising costs.

29. When each of the following equations are written in the form $y = b + mx$, the result is $y = 5 + 4$x. Find the constants r, s, k, j in these equations.

(a) $y = 2r + x\sqrt{s}$
(b) $y = \dfrac{1}{k} - (j-1)x$.

30. Graph the following function in the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Is this graph a line? Explain.

$$y = -x\left(\frac{x - 1000}{900}\right)$$

31. Graph $y = 2x + 400$ using the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Describe what happens, and how you can fix it by using a better window.

32. Graph $y = 200x + 4$ using the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Describe what happens and how you can fix it by using a better window.

33. Figure 1.27 shows the graph of $y = x^2/1000 + 5$ in the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Discuss whether this is a linear function.

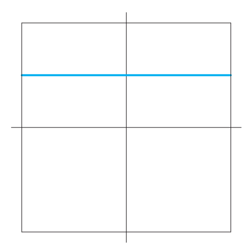


Figure 1.27

34. The cost of a cab ride is given by the function $C = 2.50 + 2d$, where d is the number of miles traveled and C is in dollars. Choose an appropriate window and graph the cost of a ride for a cab that travels no farther than a 10-mile radius from the center of the city.

[16]http://www.pnas.org/content/early/2009/10/30/0906029106.full.pdf+html, accessed November 27, 2009.

1.4 FORMULAS FOR LINEAR FUNCTIONS

To find a formula for a linear function we find values for the slope, m, and the vertical intercept, b, in the formula $y = b + mx$.

Finding a Formula for a Linear Function from a Table of Data

If a table of data represents a linear function, we first calculate m and then determine b.

Example 1 A grapefruit is thrown into the air. Its velocity, v, is a linear function of t, the time since it was thrown. A positive velocity indicates the grapefruit is rising and a negative velocity indicates it is falling. Check that the data in Table 1.28 corresponds to a linear function. Find a formula for v in terms of t.

Table 1.28 *Velocity of a grapefruit t seconds after being thrown into the air*

t, time (sec)	1	2	3	4
v, velocity (ft/sec)	48	16	-16	-48

Solution Figure 1.28 shows the data in Table 1.28. The points appear to fall on a line. To check that the velocity function is linear, calculate the rates of change of v and see that they are constant. From time $t = 1$ to $t = 2$, we have

$$\text{Rate of change of velocity with time} = \frac{\Delta v}{\Delta t} = \frac{16 - 48}{2 - 1} = -32.$$

For the next second, from $t = 2$ to $t = 3$, we have

$$\text{Rate of change} = \frac{\Delta v}{\Delta t} = \frac{-16 - 16}{3 - 2} = -32.$$

You can check that the rate of change from $t = 3$ to $t = 4$ is also -32.

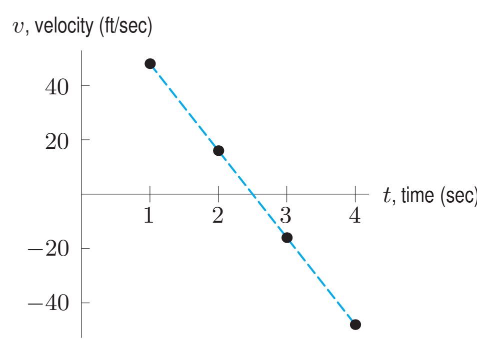


Figure 1.28: Velocity of a grapefruit is a linear function of time

A formula for v is of the form $v = b + mt$. Since m is the rate of change, we have $m = -32$ so $v = b - 32t$. The initial velocity (at $t = 0$) is represented by b. We are not given the value of v when $t = 0$, but we can use any data point to calculate b. For example, $v = 48$ when $t = 1$, so

$$48 = b - 32 \cdot 1,$$

which gives

$$b = 80.$$

Thus, a formula for the velocity is $v = 80 - 32t$.

What does the rate of change, m, in Example 1 tell us about the grapefruit? Think about the units:

$$m = \frac{\Delta v}{\Delta t} = \frac{\text{Change in velocity}}{\text{Change in time}} = \frac{-32 \text{ ft/sec}}{1 \text{ sec}} = -32 \text{ ft/sec per second.}$$

The value of m, -32 ft/sec per second, tells us that the grapefruit's velocity is decreasing by 32 ft/sec for every second that goes by. We say the grapefruit is accelerating at -32 ft/sec per second. (The units ft/sec per second are often written ft/sec^2. Negative acceleration is also called deceleration.)[17]

Finding a Formula for a Linear Function from a Graph

We can calculate the slope, m, of a linear function using two points on its graph. Having found m, we can use either of the points to calculate b, the vertical intercept.

Example 2 Figure 1.29 shows oxygen consumption as a function of heart rate for two people.

(a) Assuming linearity, find formulas for these two functions.
(b) Interpret the slope of each graph in terms of oxygen consumption.

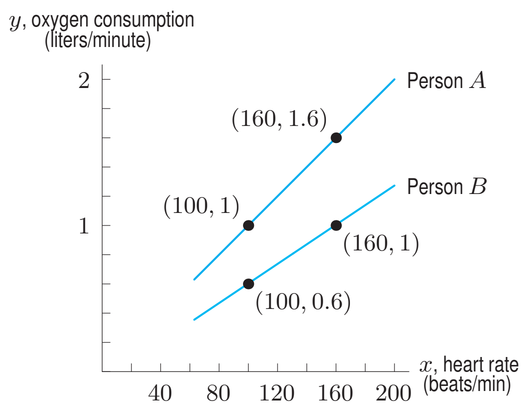


Figure 1.29: Oxygen consumption of two people running on treadmills

Solution (a) Let x be heart rate and let y be oxygen consumption. Since we are assuming linearity, $y = b + mx$. The two points on person A's line, $(100, 1)$ and $(160, 1.6)$, give

$$\text{Slope of A's line } = m = \frac{\Delta y}{\Delta x} = \frac{1.6 - 1}{160 - 100} = 0.01.$$

Thus $y = b + 0.01x$. To find b, use the fact that $y = 1$ when $x = 100$:

$$1 = b + 0.01(100)$$
$$1 = b + 1$$
$$b = 0.$$

[17]The notation ft/sec^2 is shorthand for ft/sec per second; it does not mean a "square second" in the same way that areas are measured square feet or square meters.

Alternatively, b can be found using the fact that $x = 160$ if $y = 1.6$. Either way leads to the formula $y = 0.01x$.

For person B, we again begin with the formula $y = b + mx$. In Figure 1.29, two points on B's line are $(100, 0.6)$ and $(160, 1)$, so

$$\text{Slope of B's line } = m = \frac{\Delta y}{\Delta x} = \frac{1 - 0.6}{160 - 100} = \frac{0.4}{60} \approx 0.0067.$$

To find b, use the fact that $y = 1$ when $x = 160$:

$$\begin{aligned} 1 &= b + (0.4/60) \cdot 160 \\ 1 &= b + 1.067 \\ b &= -0.067. \end{aligned}$$

Thus, for person B, we have $y = -0.067 + 0.0067x$.

(b) The slope for person A is $m = 0.01$, so

$$m = \frac{\text{Change in oxygen consumption}}{\text{Change in heart rate}} = \frac{\text{Change in liters/min}}{\text{Change in beats/min}} = 0.01 \frac{\text{liters}}{\text{heartbeat}}.$$

Every additional heartbeat (per minute) for person A translates to an additional 0.01 liters (per minute) of oxygen consumed.

The slope for person B is $m = 0.0067$. Thus, for every additional beat (per minute), person B consumes an additional 0.0067 liter of oxygen (per minute). Since the slope for person B is smaller than for person A, person B consumes less additional oxygen than person A for the same increase in pulse.

What do the y-intercepts of the functions in Example 2 say about oxygen consumption? Often the y-intercept of a function is a starting value. In this case, the y-intercept would be the oxygen consumption of a person whose pulse is zero (i.e. $x = 0$). Since a person running on a treadmill must have a pulse, in this case it makes no sense to interpret the y-intercept this way. The formula for oxygen consumption is useful only for realistic values of the pulse.

Finding a Formula for a Linear Function from a Verbal Description

Sometimes the verbal description of a linear function is less straightforward than those we saw in Section 1.3. Consider the following example.

Example 3 We have \$24 to spend on soda and chips for a party. A six-pack of soda costs \$3 and a bag of chips costs \$2. The number of six-packs we can afford, y, is a function of the number of bags of chips we decide to buy, x.

(a) Find an equation relating x and y.
(b) Graph the equation. Interpret the intercepts and the slope in the context of the party.

Solution (a) If we spend all \$24 on soda and chips, then we have the following equation:

$$\text{Amount spent on chips } + \text{ Amount spent on soda } = \$24.$$

If we buy x bags of chips at \$2 per bag, then the amount spent on chips is $\$2x$. Similarly, if we buy y six-packs of soda at \$3 per six-pack, then the amount spent on soda is $\$3y$. Thus,

$$2x + 3y = 24.$$

We can solve for y, giving

$$3y = 24 - 2x$$
$$y = 8 - \frac{2}{3}x.$$

This is a linear function with slope $m = -2/3$ and y-intercept $b = 8$.

(b) The graph of this function is a discrete set of points, since the number of bags of chips and the number of six-packs of soda must be (nonnegative) integers.

To find the y-intercept, we set $x = 0$, giving

$$2 \cdot 0 + 3y = 24.$$

So $3y = 24$, giving $y = 8$.

Substituting $y = 0$ gives the x-intercept,

$$2x + 3 \cdot 0 = 24.$$

So $2x = 24$, giving $x = 12$. Thus the points $(0, 8)$ and $(12, 0)$ are on the graph.

The point $(0, 8)$ indicates that we can buy 8 six-packs of soda if we buy no chips. The point $(12, 0)$ indicates that we can buy 12 bags of chips if we buy no soda. The other points on the line describe affordable options between these two extremes. For example, the point $(6, 4)$ is on the line, because

$$2 \cdot 6 + 3 \cdot 4 = 24.$$

This means that if we buy 6 bags of chips, we can afford 4 six-packs of soda.

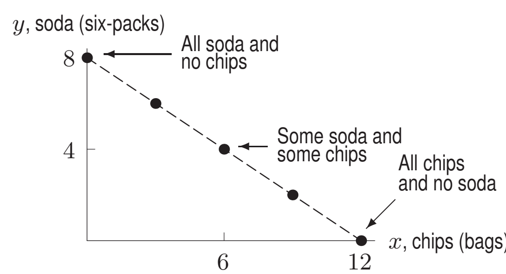


Figure 1.30: Relation between the number of six-packs, y, and the number of bags of chips, x

The points marked in Figure 1.30 represent affordable options. All affordable options lie on or below the line $2x + 3y = 24$. Not all points on the line are affordable options. For example, suppose we purchase one six-pack of soda for \$3.00. That leaves \$21.00 to spend on chips, meaning we would have to buy 10.5 bags of chips, which is not possible. Therefore, the point $(10.5, 1)$ is not an option, although it is a point on the line $2x + 3y = 24$.

To interpret the slope, notice that

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in number of six-packs}}{\text{Change in number of bags of chips}},$$

so the units of m are six-packs of soda per bags of chips. The fact that $m = -2/3$ means that for each additional 3 bags of chips purchased, we can purchase 2 fewer six-packs of soda. This occurs because 2 six-packs cost \$6, the same as 3 bags of chips. Thus, $m = -2/3$ is the rate at which the amount of soda we can buy decreases as we buy more chips.

Alternative Forms for the Equation of a Line

In Example 3, the equation $2x + 3y = 24$ represents a linear relationship between x and y even though the equation is not in the form $y = b + mx$. The following equations represent lines.

- The *slope-intercept form* is
  $y = b + mx$ where m is the slope and b is the y-intercept.
- The *point-slope form* is
  $y - y_0 = m(x - x_0)$ where m is the slope and (x_0, y_0) is a point on the line.
- The *standard form* is
  $Ax + By + C = 0$ where A, B, and C are constants.

If we know the slope of a line and the coordinates of a point on the line, it is often convenient to use the point-slope form of the equation.

Example 4 Use the point-slope form to find the equation of the line for the oxygen consumption of person A in Example 2.

Solution In Example 2, we found the slope of person A's line to be $m = 0.01$. Since the point $(100, 1)$ lies on the line, the point-slope form gives the equation

$$y - 1 = 0.01(x - 100).$$

To check that this gives the same equation we got in Example 2, we multiply out and simplify:

$$\begin{aligned} y - 1 &= 0.01x - 1 \\ y &= 0.01x. \end{aligned}$$

Alternatively, we could have used the point $(160, 1.6)$ instead of $(100, 1)$, giving

$$y - 1.6 = 0.01(x - 160).$$

Multiplying out again gives $y = 0.01x$.

Exercises and Problems for Section 1.4

Skill Refresher

Solve the equations in Exercises S1–S5.

S1. $y - 5 = 21$

S2. $2x - 5 = 13$

S3. $2x - 5 = 4x - 9$

S4. $17 - 28y = 13y + 24$

S5. $\frac{5}{3}(y + 2) = \frac{1}{2} - y$

In Exercises S6–S10, solve for the indicated variable.

S6. $I = Prt,$ for P.

S7. $C = \frac{5}{9}(F - 32),$ for F.

S8. $C = 2\pi r,$ for r.

S9. $ab + ax = c - ax$, for x.

S10. $by - d = ay + c$, for y.

Exercises

If possible, rewrite the equations in Exercises 1–9 in slope-intercept form, $y = b + mx$.

1. $5(x + y) = 4$

2. $3x + 5y = 20$

3. $0.1y + x = 18$

4. $5x - 3y + 2 = 0$

5. $y - 0.7 = 5(x - 0.2)$

6. $y = 5$

7. $3x + 2y + 40 = x - y$

8. $x = 4$

9. $\dfrac{x + y}{7} = 3$

Is each function in Exercises 10–15 linear? If so, rewrite it the form $y = b + mx$.

10. $g(w) = -\dfrac{1 - 12w}{3}$

11. $F(P) = 13 - \dfrac{2^{-1}}{4}P$

12. $j(s) = 3s^{-1} + 7$

13. $C(r) = 2\pi r$

14. $h(x) = 3^x + 12$

15. $f(x) = m^2x + n^2$

Find formulas for the linear functions in Exercises 16–23.

16. Slope -4 and x-intercept 7

17. Slope 3 and y-intercept 8

18. Passes through the points $(-1, 5)$ and $(2, -1)$

19. Slope $2/3$ and passes through the point $(5, 7)$

20. Has x-intercept 3 and y-intercept -5

21. Slope 0.1, passes through $(-0.1, 0.02)$

22. Function f has $f(0.3) = 0.8$ and $f(0.8) = -0.4$

23. Function f has $f(-2) = 7$ and $f(3) = -3$

Exercises 24–30 give data from a linear function. Find a formula for the function.

24.

Year, t	0	1	2
Value of computer, $\$V = f(t)$	2000	1500	1000

25.

Price per bottle, p ($)	0.50	0.75	1.00
Number of bottles sold, $q = f(p)$	1500	1000	500

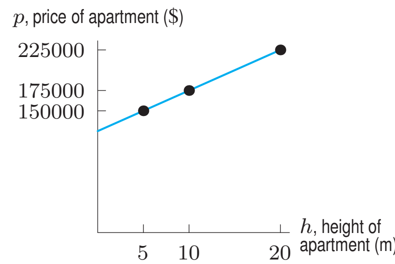
26.

Temperature, $y = f(x)$ (°C)	0	5	20
Temperature, x (°F)	32	41	68

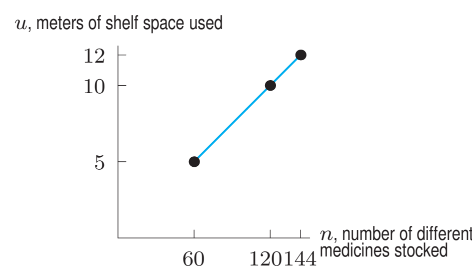
27.

Temperature, $y = f(x)$, (°R)	459.7	469.7	489.7
Temperature, x (°F)	0	10	30

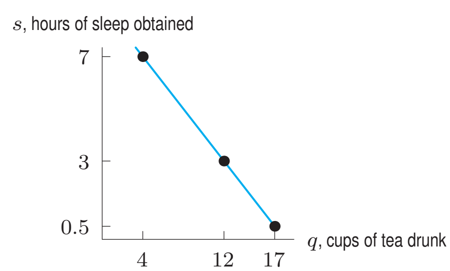
28.



29.



30.



Problems

Find formulas for the linear functions in Problems 31–34.

31. The graph of f contains $(-3, -8)$ and $(5, -20)$.

32. $g(100) = 2000$ and $g(400) = 3800$

33. $P = h(t)$ gives the size of a population that begins with 12,000 members and grows by 225 members each year.

34. The graph of h intersects the graph of $y = x^2$ at $x = -2$ and $x = 3$.

Table 1.29 gives the cost, $C(n)$, of producing a certain good as a linear function of n, the number of units produced. Use the table to answer Problems 35–37.

Table 1.29

n (units)	100	125	150	175
$C(n)$ (dollars)	11000	11125	11250	11375

35. Evaluate the following expressions. Give economic interpretations for each.

(a) $C(175)$ **(b)** $C(175) - C(150)$

(c) $\dfrac{C(175) - C(150)}{175 - 150}$

36. Estimate $C(0)$. What is the economic significance of this value?

37. The *fixed cost* of production is the cost incurred before any goods are produced. The *unit cost* is the cost of producing an additional unit. Find a formula for $C(n)$ in terms of n, given that

$$\text{Total cost} = \text{Fixed cost} + \text{Unit cost} \cdot \text{Number of units}$$

38. In a college meal plan you pay a membership fee; then all your meals are at a fixed price per meal.

(a) If 90 meals cost \$1005 and 140 meals cost \$1205, write a linear function that describes the cost of a meal plan, C, in terms of the number of meals, n.
(b) What is the cost per meal and what is the membership fee?
(c) Find the cost for 120 meals.
(d) Find n in terms of C.
(e) Use part (d) to determine the maximum number of meals you can buy on a budget of \$1285.

39. An empty champagne bottle is tossed from a hot-air balloon. Its upward velocity is measured every second and recorded in Table 1.30.

(a) Describe the motion of the bottle in words. What do negative values of v represent?
(b) Find a formula for v in terms of t.
(c) Explain the physical significance of the slope of your formula.
(d) Explain the physical significance of the t-axis and v-axis intercepts.

Table 1.30

t (sec)	0	1	2	3	4	5
v (ft/sec)	40	8	−24	−56	−88	−120

40. John wants to buy a dozen rolls. The local bakery sells sesame and poppy-seed rolls for the same price.

(a) Make a table of all the possible combinations of rolls if he buys a dozen, where s is the number of sesame seed rolls and p is the number of poppy-seed rolls.
(b) Find a formula for p as a function of s.
(c) Graph this function.

41. The demand for gasoline can be modeled as a linear function of price. If the price of gasoline is $p = \$3.10$ per gallon, the quantity demanded in a fixed period is $q = 65$ gallons. If the price rises to \$3.50 per gallon, the quantity demanded falls to 45 gallons in that period.

(a) Find a formula for q in terms of p.
(b) Explain the economic significance of the slope of your formula.
(c) Explain the economic significance of the q-axis and p-axis intercepts.

42. The solid waste generated each year in the cities of the US is increasing.[18] The solid waste generated, in millions of tons, was 88.1 in 1960 and 239.1 in 2000. The trend appears linear during this time.

(a) Construct a formula for the amount of municipal solid waste generated in the US by finding the equation of the line through these two points.
(b) Use this formula to predict the amount of municipal solid waste generated in the US, in millions of tons, in the year 2020.

43. Find the equation of the line l, shown in Figure 1.31, if its slope is $m = 4$.

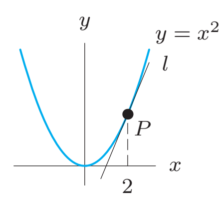


Figure 1.31

44. Find a formula for the line intersecting the graph of $f(x)$ at $x = 1$ and $x = 3$, where

$$f(x) = \frac{10}{x^2 + 1}.$$

[18] http://www.epa.gov/osw/nonhaz/municipal/pubs/msw2008rpt.pdf, accessed November 23, 2009.

45. Find the equation of line l in Figure 1.32.

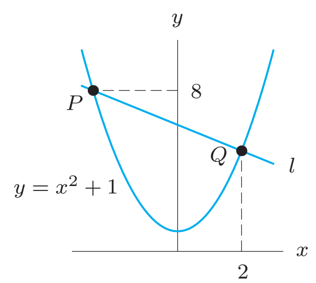


Figure 1.32

46. Find an equation for the line l in Figure 1.33 in terms of the constant A and values of the function f.

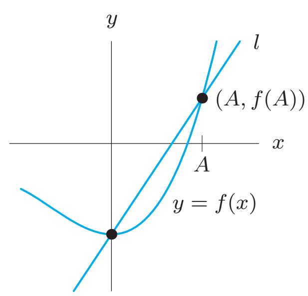


Figure 1.33

47. You can type four pages in 50 minutes and nine pages in an hour and forty minutes.

- **(a)** Find a linear function for the number of pages typed, p, as a function of time, t. If time is measured in minutes, what values of t make sense in this example?
- **(b)** How many pages can be typed in two hours?
- **(c)** Interpret the slope of the function in practical terms.
- **(d)** Use the result in part (a) to solve for time as a function of the number of pages typed.
- **(e)** How long does it take to type a 15-page paper?
- **(f)** Write a short paragraph explaining why it is useful to know both of the formulas obtained in part (a) and part (d).

48. Wire is sold by gauge size, where the diameter of the wire is a decreasing linear function of gauge. Gauge 2 wire has a diameter of 0.2656 inches and gauge 8 wire has a diameter of 0.1719 inches. Find the diameter for wires of gauge 12.5 and gauge 0. What values of the gauge do not make sense in this model?

49. A dose-response function can be used to describe the increase in risk associated with the increase in exposure to various hazards. For example, the risk of contracting lung cancer depends, among other things, on the number of cigarettes a person smokes per day. This risk can be described by a linear dose-response function. For example, it is known that smoking 10 cigarettes per day increases a person's probability of contracting lung cancer by a factor of 25, while smoking 20 cigarettes a day increases the probability by a factor of 50.

- **(a)** Find a formula for $i(x)$, the increase in the probability of contracting lung cancer for a person who smokes x cigarettes per day as compared to a nonsmoker.
- **(b)** Evaluate $i(0)$.
- **(c)** Interpret the slope of the function i.

In Problems 50–51, write the functions in slope-intercept form. Identify the values of b and m.

50. $v(s) = \pi x^2 - 3xr - 4rs - s\sqrt{x}$

51. $w(r) = \pi x^2 - 3xr - 4rs - s\sqrt{x}$

52. The development time, t, of an organism is the number of days required for the organism to mature, and the development rate is defined as $r = 1/t$. In cold-blooded organisms such as insects, the development rate depends on temperature: the colder it is, the longer the organism takes to develop. For such organisms, the degree-day model[19] assumes that the development rate r is a linear function of temperature H (in °C):

$$r = b + kH.$$

- **(a)** According to the degree-day model, there is a minimum temperature H_{min} below which an organism never matures. Find a formula for H_{min} in terms of the constants b and k.
- **(b)** Define S as $S = (H - H_{\text{min}})t$, where S is the number of degree-days. That is, S is the number of days t times the number of degrees between H and H_{min}. Use the formula for r to show that S is a constant. In other words, find a formula for S that does not involve H. Your formula will involve k.
- **(c)** A certain organism requires $t = 25$ days to develop at a constant temperature of $H = 20°$C and has $H_{\text{min}} = 15°$C. Using the fact that S is a constant, how many days does it take for this organism to develop at a temperature of 25°C?

[19]Information drawn from a web site created by Dr. Alexei A. Sharov at the Virginia Polytechnic Institute, http://www.ento.vt.edu/ sharov/PopEcol/popecol.html.

(d) In part (c) we assumed that the temperature H is constant throughout development. If the temperature varies from day to day, the number of degree-days can be accumulated until they total S, at which point the organism completes development. For instance, suppose on the first day the temperature is $H = 20°$C and that on the next day it is $H = 22°$C. Then for these first two days

Total number of degree days

$$= (20 - 15) \cdot 1 + (22 - 15) \cdot 1 = 12.$$

Based on Table 1.31, on what day does the organism reach maturity?

Table 1.31

Day	1	2	3	4	5	6	7	8	9	10	11	12
H (°C)	20	22	27	28	27	31	29	30	28	25	24	26

53. (Continuation of Problem 52.) Table 1.32 gives the development time t (in days) for an insect as a function of temperature H (in °C).

(a) Find a linear formula for r, the development rate, in terms of H.
(b) Find the value of S, the number of degree-days required for the organism to mature.

Table 1.32

H, °C	20	22	24	26	28	30
t, days	14.3	12.5	11.1	10.0	9.1	8.3

1.5 GEOMETRIC PROPERTIES OF LINEAR FUNCTIONS

Interpreting the Parameters of a Linear Function

The slope-intercept form for a linear function is $y = b + mx$, where b is the y-intercept and m is the slope. The parameters b and m can be used to compare linear functions.

Example 1 With time, t, in years, the populations of four towns, P_A, P_B, P_C and P_D, are given by the following formulas:

$$P_A = 20{,}000 + 1600t, \quad P_B = 50{,}000 - 300t, \quad P_C = 650t + 45{,}000, \quad P_D = 15{,}000(1.07)^t.$$

(a) Which populations are represented by linear functions?
(b) Describe in words what each linear model tells you about that town's population. Which town starts out with the most people? Which town is growing fastest?

Solution

(a) The populations of towns A, B, and C are represented by linear functions because they are written in the form $P = b + mt$. Town D's population does not grow linearly since its formula, $P_D = 15{,}000(1.07)^t$, cannot be expressed in the form $P_D = b + mt$.

(b) For town A, we have

$$P_A = \underbrace{20{,}000}_{b} + \underbrace{1600}_{m} \cdot t,$$

so $b = 20{,}000$ and $m = 1600$. This means that in year $t = 0$, town A has 20,000 people. It grows by 1600 people per year.

For town B, we have

$$P_B = \underbrace{50{,}000}_{b} + \underbrace{(-300)}_{m} \cdot t,$$

so $b = 50{,}000$ and $m = -300$. This means that town B starts with 50,000 people. The negative slope indicates that the population is decreasing at the rate of 300 people per year.

For town C, we have

$$P_C = \underbrace{45{,}000}_{b} + \underbrace{650}_{m} \cdot t,$$

so $b = 45{,}000$ and $m = 650$. This means that town C begins with 45,000 people and grows by 650 people per year.

Town B starts out with the most people, 50,000, but town A, with a rate of change of 1600 people per year, grows the fastest of the three towns that grow linearly.

The Effect of the Parameters on the Graph of a Linear Function

The graph of a linear function is a line. Changing the values of b and m gives different members of the family of linear functions. In summary:

> Let $y = b + mx$. Then the graph of y against x is a line.
> - The y-intercept, b, tells us where the line crosses the y-axis.
> - If the slope, m, is positive, the line climbs from left to right. If the slope, m, is negative, the line falls from left to right.
> - The slope, m, tells us how fast the line is climbing or falling.
> - The larger the magnitude of m (either positive or negative), the steeper the graph of f.

Example 2 (a) Graph the three linear functions P_A, P_B, P_C from Example 1 and show how to identify the values of b and m from the graph.

(b) Graph P_D from Example 1 and explain how the graph shows P_D is not a linear function.

Solution (a) Figure 1.34 gives graphs of the three functions:

$$P_A = 20{,}000 + 1600t, \qquad P_B = 50{,}000 - 300t, \qquad \text{and} \qquad P_C = 45{,}000 + 650t.$$

The values of b identified in Example 1 tell us the vertical intercepts. Figure 1.34 shows that the graph of P_A crosses the P-axis at $P = 20{,}000$, the graph of P_B crosses at $P = 50{,}000$, and the graph of P_C crosses at $P = 45{,}000$.

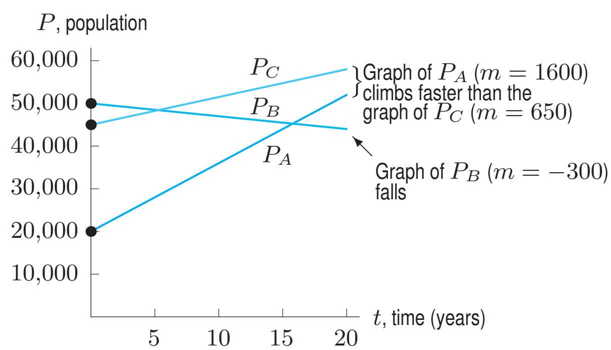


Figure 1.34: Graphs of three linear functions, P_A, P_B, and P_C, showing starting values and rates of climb

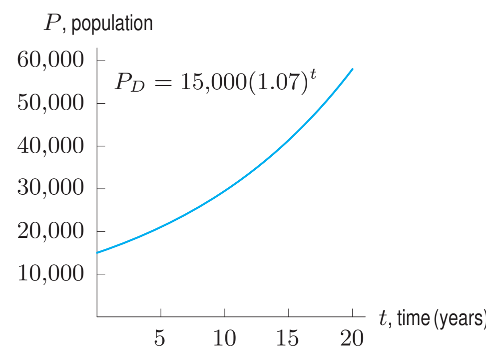


Figure 1.35: Graph of $P_D = 15{,}000(1.07)^t$ is not a line

Notice that the graphs of P_A and P_C are both climbing and that P_A climbs faster than P_C. This corresponds to the fact that the slopes of these two functions are positive ($m = 1600$ for P_A and $m = 650$ for P_C) and the slope of P_A is larger than the slope of P_C.

The graph of P_B falls when read from left to right, indicating that population decreases over time. This corresponds to the fact that the slope of P_C is negative ($m = -300$).

(b) Figure 1.35 gives a graph of P_D. Since it is not a line, P_D is not a linear function.

Intersection of Two Lines

To find the point at which two lines intersect, notice that the (x, y)-coordinates of such a point must satisfy the equations for both lines. Thus, in order to find the point of intersection algebraically, solve the equations simultaneously.[20]

If linear functions are modeling real quantities, their points of intersection often have practical significance. Consider the next example.

Example 3 The cost in dollars of renting a car for a day from three different rental agencies and driving it d miles is given by the following functions:

$$C_1 = 50 + 0.10d, \qquad C_2 = 30 + 0.20d, \qquad C_3 = 0.50d.$$

(a) Describe in words the daily rental arrangements made by each of these three agencies.
(b) Which agency is cheapest?

Solution

(a) Agency 1 charges \$50 plus \$0.10 per mile driven. Agency 2 charges \$30 plus \$0.20 per mile. Agency 3 charges \$0.50 per mile driven.

(b) The answer depends on how far we want to drive. If we are not driving far, agency 3 may be cheapest because it only charges for miles driven and has no other fees. If we want to drive a long way, agency 1 may be cheapest (even though it charges \$50 up front) because it has the lowest per-mile rate.

The three functions are graphed in Figure 1.36. The graph shows that for d up to 100 miles, the value of C_3 is less than C_1 and C_2 because its graph is below the other two. For d between 100 and 200 miles, the value of C_2 is less than C_1 and C_3. For d more than 200 miles, the value of C_1 is less than C_2 and C_3.

By graphing these three functions on a calculator, we can estimate the coordinates of the points of intersection by tracing. To find the exact coordinates, we solve simultaneous equations. Starting with the intersection of lines C_1 and C_2, we set the costs equal, $C_1 = C_2$, and solve for d:

$$\begin{aligned} 50 + 0.10d &= 30 + 0.20d \\ 20 &= 0.10d \\ d &= 200. \end{aligned}$$

Thus, the cost of driving 200 miles is the same for agencies 1 and 2. Solving $C_2 = C_3$ gives

$$\begin{aligned} 30 + 0.20d &= 0.50d \\ 0.30d &= 30 \\ d &= 100, \end{aligned}$$

[20] If you have questions about the algebra in this section, see the Skills Refresher on page 61.

which means the cost of driving 100 miles is the same for agencies 2 and 3.

Thus, agency 3 is cheapest up to 100 miles. Agency 1 is cheapest for more than 200 miles. Agency 2 is cheapest between 100 and 200 miles. See Figure 1.36. Notice that the point of intersection of C_1 and C_3, $(125, 62.5)$, does not influence our decision as to which agency is the cheapest.

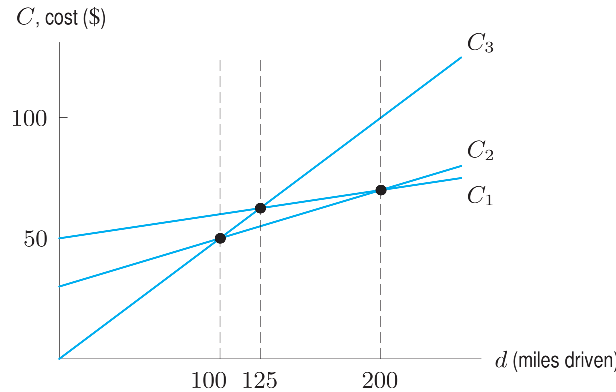


Figure 1.36: Cost of driving a car d miles when renting from three different agencies. Cheapest agency corresponds to the lowest graph for a given d value

Equations of Horizontal and Vertical Lines

An increasing linear function has positive slope and a decreasing linear function has negative slope. What about a line with slope $m = 0$? If the rate of change of a quantity is zero, then the quantity does not change. Thus, if the slope of a line is zero, the value of y must be constant. Such a line is horizontal.

Example 4 Explain why the equation $y = 4$ represents a horizontal line and the equation $x = 4$ represents a vertical line.

Solution The equation $y = 4$ represents a linear function with slope $m = 0$. To see this, notice that this equation can be rewritten as $y = 4 + 0 \cdot x$. Thus, the value of y is 4 no matter what the value of x is. See Figure 1.37. Similarly, the equation $x = 4$ means that x is 4 no matter what the value of y is. Every point on the line in Figure 1.38 has x equal to 4, so this line is the graph of $x = 4$.

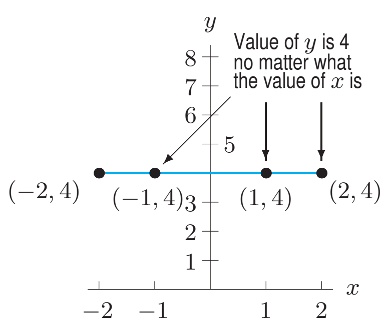


Figure 1.37: The horizontal line $y = 4$ has slope 0

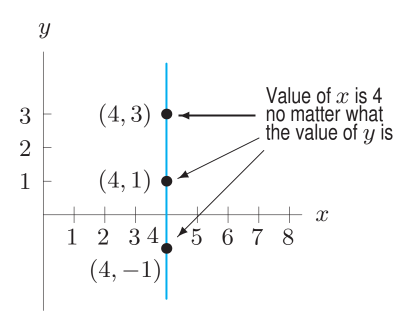


Figure 1.38: The vertical line $x = 4$ has an undefined slope

What is the slope of a vertical line? Figure 1.38 shows three points, $(4,-1)$, $(4,1)$, and $(4,3)$ on a vertical line. Calculating the slope, gives

$$m = \frac{\Delta y}{\Delta x} = \frac{3-1}{4-4} = \frac{2}{0}.$$

The slope is undefined because the denominator, Δx, is 0. The slope of every vertical line is undefined for the same reason. All the x-values on such a line are equal, so Δx is 0, and the denominator of the expression for the slope is 0. A vertical line is not the graph of a function, since it fails the vertical line test. It does not have an equation of the form $y = b + mx$.

In summary,

> For any constant k:
> - The graph of the equation $y = k$ is a horizontal line and its slope is zero.
> - The graph of the equation $x = k$ is a vertical line and its slope is undefined.

Slopes of Parallel and Perpendicular Lines

Figure 1.39 shows two parallel lines. These lines are parallel because they have equal slopes.

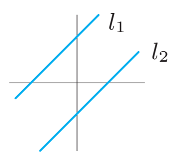


Figure 1.39: Parallel lines: l_1 and l_2 have equal slopes

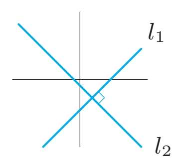


Figure 1.40: Perpendicular lines: l_1 has a positive slope and l_2 has a negative slope

What about perpendicular lines? Two perpendicular lines are graphed in Figure 1.40. We can see that if one line has a positive slope, then any perpendicular line must have a negative slope. Perpendicular lines have slopes with opposite signs.

We show that if l_1 and l_2 are two perpendicular lines with slopes, m_1 and m_2, then m_1 is the negative reciprocal of m_2. If m_1 and m_2 are not zero, we have the following result:

> Let l_1 and l_2 be two lines having slopes m_1 and m_2, respectively. Then:
> - These lines are parallel if and only if $m_1 = m_2$.
> - These lines are perpendicular if and only if $m_1 = -\dfrac{1}{m_2}$.

In addition, any two horizontal lines are parallel and $m_1 = m_2 = 0$. Any two vertical lines are parallel and m_1 and m_2 are undefined. A horizontal line is perpendicular to a vertical line. See Figures 1.41–1.43.

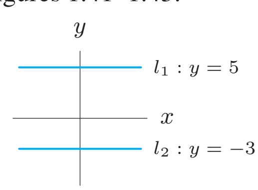


Figure 1.41: Any two horizontal lines are parallel

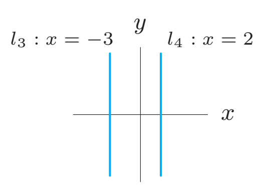


Figure 1.42: Any two vertical lines are parallel

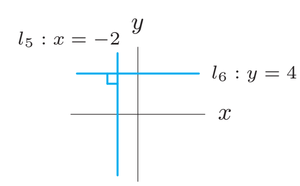


Figure 1.43: A horizontal line and a vertical line are perpendicular

Justification of Formula for Slopes of Perpendicular Lines

Figure 1.44 shows l_1 and l_2, two perpendicular lines with slopes m_1 and m_2. Neither line is horizontal or vertical, so m_1 and m_2 are both defined and nonzero. We will show that

$$m_2 = -\frac{1}{m_1}.$$

Using right triangle ΔPQR with side lengths a and b we see that

$$m_1 = \frac{b}{a}.$$

Rotating ΔPQR by $90°$ about the point P produces triangle ΔPST. Using ΔPST we see that

$$m_2 = -\frac{a}{b} = -\frac{1}{b/a} = -\frac{1}{m_1}.$$

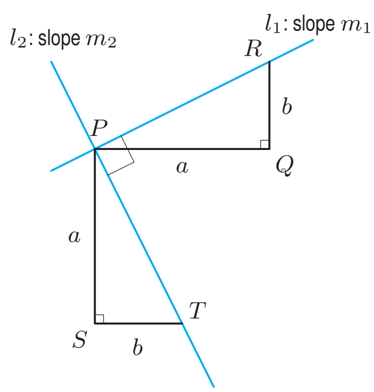


Figure 1.44

Exercises and Problems for Section 1.5

Skill Refresher

Solve the systems of equations in Exercises S1–S6, if possible.

S1. $\begin{cases} x + y = 3 \\ y = 5 \end{cases}$

S2. $\begin{cases} x + y = 3 \\ x - y = 5 \end{cases}$

S3. $\begin{cases} x + y = 2 \\ 2x + 2y = 7 \end{cases}$

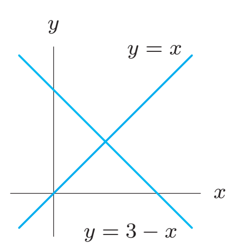
S4. $\begin{cases} y = x - 3 \\ 2y - 2x = -6 \end{cases}$

S5. $\begin{cases} 2x - y = 10 \\ x + 2y = 15 \end{cases}$

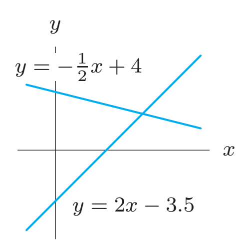
S6. $\begin{cases} 2(x + y) = 3 \\ x = y + 3(x - 5) \end{cases}$

Determine the points of intersection for Exercises S7–S8.

S7.



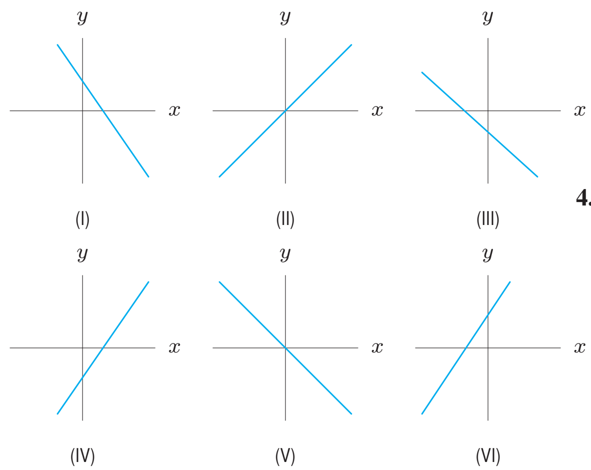
S8.



Exercises

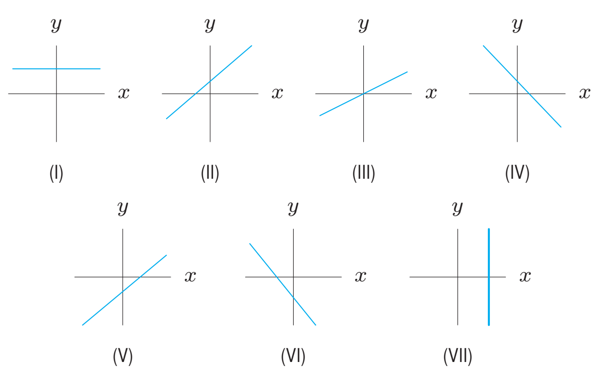
1. Without using a calculator, match the equations (a) – (f) to the graphs (I) – (VI).

(a) $y = -2.72x$ **(b)** $y = 0.01 + 0.001x$
(c) $y = 27.9 - 0.1x$ **(d)** $y = 0.1x - 27.9$
(e) $y = -5.7 - 200x$ **(f)** $y = x/3.14$



2. Without a calculator, match the equations (a)–(g) to the graphs (I)–(VII).

(a) $y = x - 5$ **(b)** $-3x + 4 = y$
(c) $5 = y$ **(d)** $y = -4x - 5$
(e) $y = x + 6$ **(f)** $y = x/2$
(g) $5 = x$



3. Figure 1.45 gives lines, A, B, C, D, and E. Without a calculator, match each line to f, g, h, u or v:

$f(x) = 20 + 2x$
$g(x) = 20 + 4x$
$h(x) = 2x - 30$
$u(x) = 60 - x$
$v(x) = 60 - 2x$

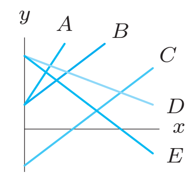


Figure 1.45

4. Without a calculator, match the following functions to the lines in Figure 1.46:

$f(x) = 5 + 2x$
$g(x) = -5 + 2x$
$h(x) = 5 + 3x$
$j(x) = 5 - 2x$
$k(x) = 5 - 3x$

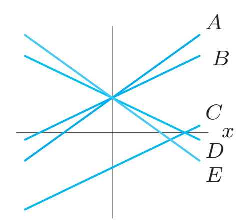


Figure 1.46

5. **(a)** By hand, graph $y = 3$ and $x = 3$.
(b) Can the equations in part (a) be written in slope-intercept form?

Are the lines in Exercises 6–11 perpendicular? Parallel? Neither?

6. $y = 5x - 7$; $y = 5x + 8$

7. $y = 4x + 3$; $y = 13 - \frac{1}{4}x$

8. $y = 2x + 3$; $y = 2x - 7$

9. $y = 4x + 7$; $y = \frac{1}{4}x - 2$

10. $f(q) = 12q + 7$; $g(q) = \frac{1}{12}q + 96$

11. $2y = 16 - x$; $4y = -8 - 2x$

Problems

12. Sketch a family of functions $y = -2 - ax$ for five different values of a with $a < 0$.

13. Find the equation of the line parallel to $3x + 5y = 6$ and passing through the point $(0, 6)$.

14. Find the equation of the line passing through the point $(2, 1)$ and perpendicular to the line $y = 5x - 3$.

15. Find the equations of the lines parallel to and perpendicular to the line $y + 4x = 7$, and through the point $(1, 5)$.

16. Estimate the slope of the line in Figure 1.47 and find an approximate equation for the line.

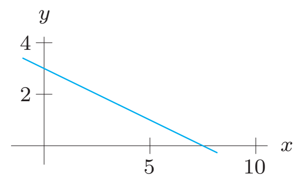


Figure 1.47

17. Line l in Figure 1.48 is parallel to the line $y = 2x + 1$. Find the coordinates of the point P.

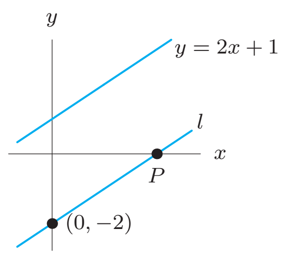


Figure 1.48

18. Find the equation of the line l_2 in Figure 1.49.

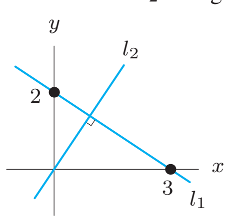


Figure 1.49

19. The cost of a Frigbox refrigerator is \$950, and it depreciates \$50 each year. The cost of an Arctic Air refrigerator is \$1200, and it depreciates \$100 per year.

(a) If a Frigbox and an Arctic Air are bought at the same time, when do the two refrigerators have equal value?

(b) If both refrigerators continue to depreciate at the same rates, what happens to the values of the refrigerators in 20 years' time? What does this mean?

20. You need to rent a car and compare the charges of three different companies. Company A charges 20 cents per mile plus \$20 per day. Company B charges 10 cents per mile plus \$35 per day. Company C charges \$70 per day with no mileage charge.

(a) Find formulas for the cost of driving cars rented from companies A, B, and C, in terms of x, the distance driven in miles in one day.

(b) Graph the costs for each company for $0 \le x \le 500$. Put all three graphs on the same set of axes.

(c) What do the slope and the vertical intercept tell you in this situation?

(d) Use the graph in part (b) to find under what circumstances company A is the cheapest. What about Company B? Company C? Explain why your results make sense.

21. Line l is given by $y = 3 - \frac{2}{3}x$ and point P has coordinates $(6, 5)$.

(a) Find the equation of the line containing P and parallel to l.

(b) Find the equation of the line containing P and perpendicular to l.

(c) Graph the equations in parts (a) and (b).

22. Assume A, B, C are constants with $A \neq 0$, $B \neq 0$. Consider the equation

$$Ax + By = C.$$

(a) Show that $y = f(x)$ is linear. State the slope and the x- and y-intercepts of $f(x)$.

(b) Graph $y = f(x)$, labeling the x- and y-intercepts in terms of A, B, and C, assuming

(i) $A > 0, B > 0, C > 0$

(ii) $A > 0, B > 0, C < 0$

(iii) $A > 0, B < 0, C > 0$

23. Fill in the missing coordinates for the points in the following figures.

(a) The triangle in Figure 1.50.

(b) The parallelogram in Figure 1.51.

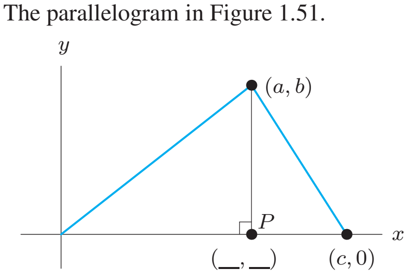


Figure 1.50

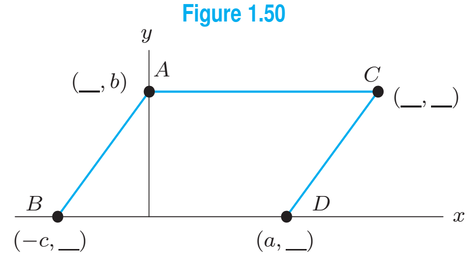


Figure 1.51

24. Using the window $-10 \leq x \leq 10, -10 \leq y \leq 10$, graph $y = x$, $y = 10x$, $y = 100x$, and $y = 1000x$.

(a) Explain what happens to the graphs of the lines as the slopes become large.
(b) Write an equation of a line that passes through the origin and is horizontal.

25. Graph $y = x + 1$, $y = x + 10$, and $y = x + 100$ in the window $-10 \leq x \leq 10, -10 \leq y \leq 10$.

(a) Explain what happens to the graph of a line, $y = b + mx$, as b becomes large.
(b) Write a linear equation whose graph cannot be seen in the window $-10 \leq x \leq 10, -10 \leq y \leq 10$ because all its y-values are less than the y-values shown.

26. The graphical interpretation of the slope is that it shows steepness. Using a calculator or a computer, graph the function $y = 2x - 3$ in the following windows:

(a) $-10 \leq x \leq 10$ by $-10 \leq y \leq 10$
(b) $-10 \leq x \leq 10$ by $-100 \leq y \leq 100$
(c) $-10 \leq x \leq 10$ by $-1000 \leq y \leq 1000$
(d) Write a sentence about how steepness is related to the window being used.

In Problems 27–28, what is true about the constant β in the following linear equation if its graph has the given property?

$$y = \frac{x}{\beta - 3} + \frac{1}{6 - \beta}.$$

27. Positive slope, positive y-intercept.

28. Perpendicular to the line $y = (\beta - 7)x - 3$.

29. A circle of radius 2 is centered at the origin and goes through the point $(-1, \sqrt{3})$.

(a) Find an equation for the line through the origin and the point $(-1, \sqrt{3})$.
(b) Find an equation for the tangent line to the circle at $(-1, \sqrt{3})$. [Hint: A tangent line is perpendicular to the radius at the point of tangency.]

30. Find an equation for the altitude through point A of the triangle ABC, where A is $(-4, 5)$, B is $(-3, 2)$, and C is $(9, 8)$. [Hint: The altitude of a triangle is perpendicular to the base.]

31. Fill in the missing coordinates in Figure 1.52. Write an equation for the line connecting the two points. Check your answer by solving the system of two equations.

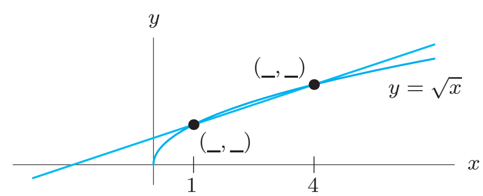


Figure 1.52

32. Two lines are given by $y = b_1 + m_1 x$ and $y = b_2 + m_2 x$, where b_1, b_2, m_1, and m_2 are constants.

(a) What conditions are imposed on b_1, b_2, m_1, and m_2 if the two lines have no points in common?
(b) What conditions are imposed on b_1, b_2, m_1, and m_2 if the two lines have all points in common?
(c) What conditions are imposed on b_1, b_2, m_1, and m_2 if the two lines have exactly one point in common?
(d) What conditions are imposed on b_1, b_2, m_1, and m_2 if the two lines have exactly two points in common?

1.6 FITTING LINEAR FUNCTIONS TO DATA

When real data are collected in the laboratory or the field, they are often subject to experimental error. Even if there is an underlying linear relationship between two quantities, real data may not fit this relationship perfectly. However, even if a data set does not perfectly conform to a linear function, we may still be able to use a linear function to help us analyze the data.

Laboratory Data: The Viscosity of Motor Oil

The viscosity of a liquid, or its resistance to flow, depends on the liquid's temperature. Pancake syrup is a familiar example: straight from the refrigerator, it pours very slowly. When warmed on the stove, its viscosity decreases and it becomes quite runny.

The viscosity of motor oil is a measure of its effectiveness as a lubricant in the engine of a car. Thus, the effect of engine temperature is an important determinant of motor-oil performance. Table 1.33 gives the viscosity, v, of motor oil as measured in the lab at different temperatures, T.

Table 1.33 *The measured viscosity, v, of motor oil as a function of the temperature, T*

T, temperature (°F)	v, viscosity (lbs·sec/in^2)
160	28
170	26
180	24
190	21
200	16
210	13
220	11
230	9

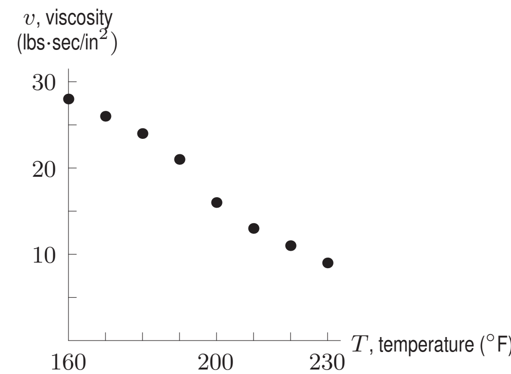


Figure 1.53: The viscosity data from Table 1.33

The *scatter plot* of the data in Figure 1.53 shows that the viscosity of motor oil decreases, approximately linearly, as its temperature rises. To find a formula relating viscosity and temperature, we fit a line to these data points.

Fitting the best line to a set of data is called *linear regression*. One way to fit a line is to draw a line "by eye." Alternatively, many computer programs and calculators compute regression lines. Figure 1.54 shows the data from Table 1.33 together with the computed regression line,

$$v = 75.6 - 0.293T.$$

Notice that none of the data points lie exactly on the regression line, although it fits the data well.

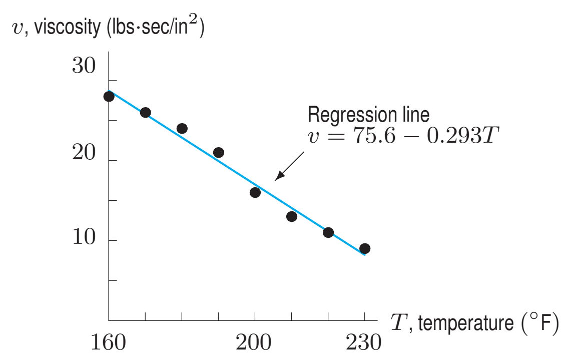


Figure 1.54: A graph of the viscosity data from Table 1.33, together with a regression line (provided by a calculator)

The Assumptions Involved In Finding a Regression Line

When we find a regression line for the data in Table 1.33, we are assuming that the value of v is related to the value of T. However, there may be experimental errors in our measurements. For example, if we measure viscosity twice at the same temperature, we may get two slightly different values. Alternatively, something besides engine temperature could be affecting the oil's viscosity (the oil pressure, for example). Thus, even if we assume that the temperature readings are exact, the viscosity readings include some degree of uncertainty.

Interpolation and Extrapolation

The formula for viscosity can be used to make predictions. Suppose we want to know the viscosity of motor oil at $T = 196°$F. The formula gives

$$v = 75.6 - 0.293 \cdot 196 \approx 18.2 \text{ lb} \cdot \text{sec/in}^2.$$

To see that this is a reasonable estimate, compare it to the entries in Table 1.33. At 190°F, the measured viscosity was 21, and at 200°F, it was 16; the predicted viscosity of 18.2 is between 16 and 21. See Figure 1.55. Of course, if we measured the viscosity at $T = 196°$F in the lab, we might not get exactly 18.2.

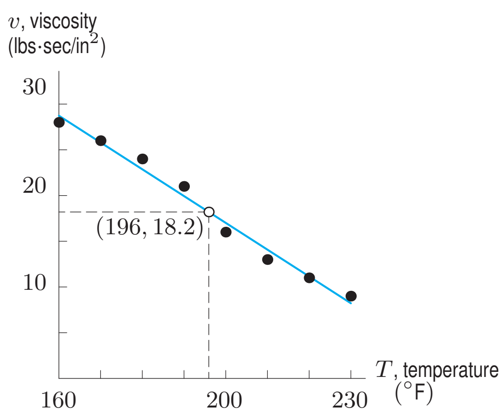


Figure 1.55: Regression line used to predict the viscosity at 196°F

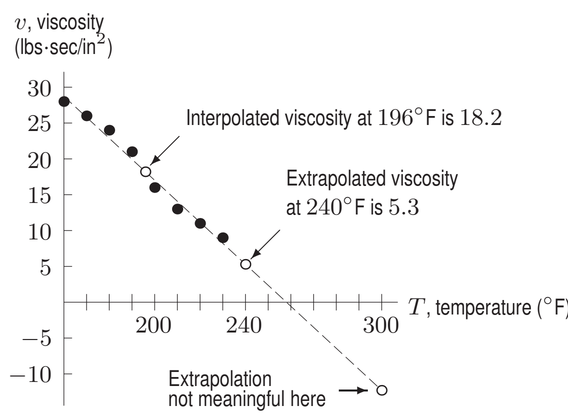


Figure 1.56: The data from Table 1.33 together with the predicted viscosity at $T = 196°$F, $T = 240°$F, and $T = 300°$F

Since the temperature $T = 196°$F is between two temperatures for which v is known (190°F and 200°F), the estimate of 18.2 is said to be an *interpolation*. If instead we estimate the value of v at a temperature outside the values for T in Table 1.33, our estimate is called an *extrapolation*.

Example 1 Predict the viscosity of motor oil at 240°F and at 300°F.

Solution At $T = 240°$F, the formula for the regression line predicts that the viscosity of motor oil is

$$v = 75.6 - 0.293 \cdot 240 = 5.3 \text{ lb} \cdot \text{sec/in}^2.$$

This is reasonable. Figure 1.56 shows that the predicted point—represented by an open circle on the graph—is consistent with the trend in the data points from Table 1.33.

On the other hand, at $T = 300°$F the regression-line formula gives

$$v = 75.6 - 0.293 \cdot 300 = -12.3 \text{ lb} \cdot \text{sec/in}^2.$$

This is unreasonable because viscosity cannot be negative. To understand what went wrong, notice that in Figure 1.56, the open circle representing the point $(300, -12.3)$ is far from the plotted data points. By making a prediction at 300°F, we have assumed—incorrectly—that the trend observed in laboratory data extended as far as 300°F.

In general, interpolation tends to be more reliable than extrapolation because we are making a prediction on an interval we already know something about instead of making a prediction beyond the limits of our knowledge.

How Regression Works

How does a calculator or computer decide which line fits the data best? We assume that the value of y is related to the value of x, although other factors could influence y as well. Thus, we assume that we can pick the value of x exactly but that the value of y may be only partially determined by this x-value.

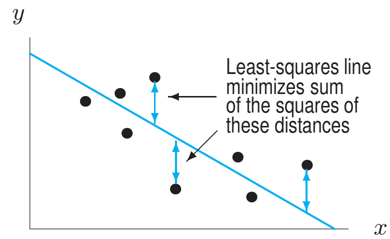


Figure 1.57: A given set of data and the corresponding least-squares regression line

One way to fit a line to the data is shown in Figure 1.57. The line shown was chosen to minimize the sum of the squares of the vertical distances between the data points and the line. Such a line is called a *least-squares line*. There are formulas which a calculator or computer uses to calculate the slope, m, and the y-intercept, b, of the least-squares line.

Correlation

When a computer or calculator calculates a regression line, it also gives a *correlation coefficient*, r. This number lies between -1 and $+1$ and measures how well a particular regression line fits the data. If $r = 1$, the data lie exactly on a line of positive slope. If $r = -1$, the data lie exactly on a line of negative slope. If r is close to 0, the data may be completely scattered, or there may be a nonlinear relationship between the variables. (See Figure 1.58.)

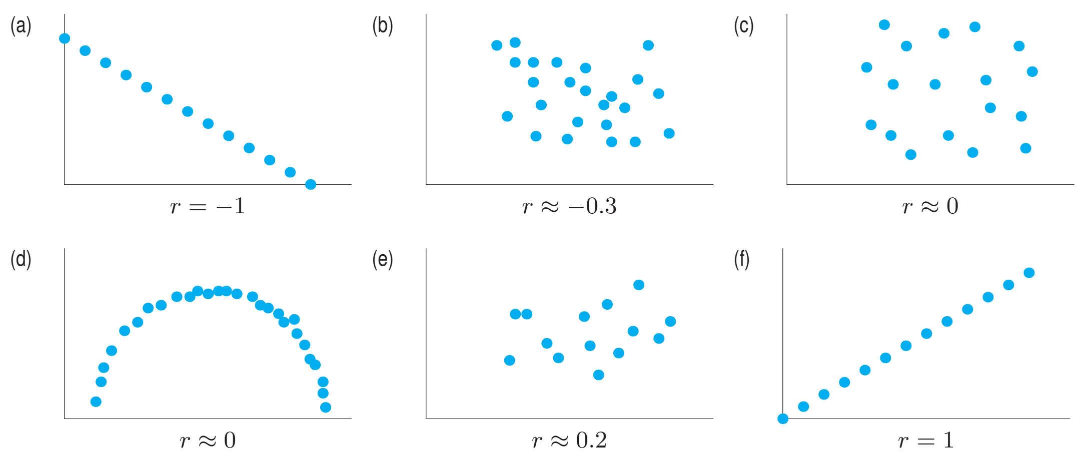


Figure 1.58: Various data sets and correlation coefficients

Example 2 The correlation coefficient for the viscosity data in Table 1.33 on page 45 is $r \approx -0.99$. The fact that r is negative tells us that the regression line has negative slope. The fact that r is close to -1 tells us that the regression line fits the data well.

The Difference Between Relation, Correlation, and Causation

It is important to understand that a high correlation (either positive or negative) between two quantities does *not* imply causation. For example, there is a high correlation between children's reading level and shoe size.[21] However, large feet do not cause a child to read better (or vice versa). Larger feet and improved reading ability are both a consequence of growing older.

Notice also that a correlation of 0 does not imply that there is no relationship between x and y. For example, in Figure 1.58(d) there is a relationship between x and y-values, while Figure 1.58(c) exhibits no apparent relationship. Both data sets have a correlation coefficient of $r \approx 0$. Thus a correlation of $r = 0$ usually implies there is no linear relationship between x and y, but this does not mean there is no relationship at all.

Exercises and Problems for Section 1.6

For data in Problems 1–6 is the given value of r reasonable? Give an explanation for your answer.

1. $r = 0.93$

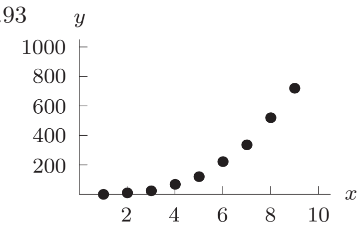


Figure 1.59

2. $r = -0.9$

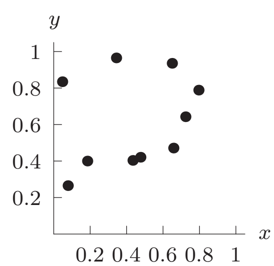


Figure 1.60

3. $r = 1$

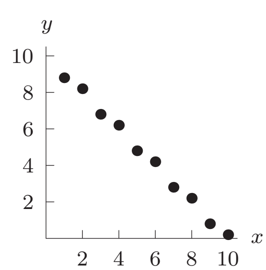


Figure 1.61

4. $r = 0.92$

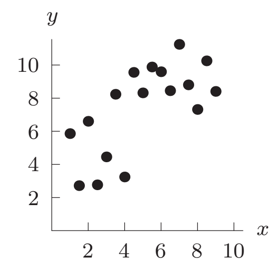


Figure 1.62

5. $r = 1$

Table 1.34

x	1	2	3	4	5
y	3.8	3.2	1.8	1.2	-0.2

6. $r = 0.343$

Table 1.35

x	1	2	3	4	5
y	3.477	5.531	14.88	5.924	8.049

[21] From *Statistics*, 2nd edition, by David Freedman. Robert Pisani, Roger Purves, Ani Adhikari, p. 142 (New York: W.W. Norton, 1991).

7. Match the r values with scatter plots in Figure 1.63.

$$r = -0.98, \quad r = -0.5, \quad r = -0.25,$$

$$r = 0, \quad r = 0.7, \quad r = 1.$$

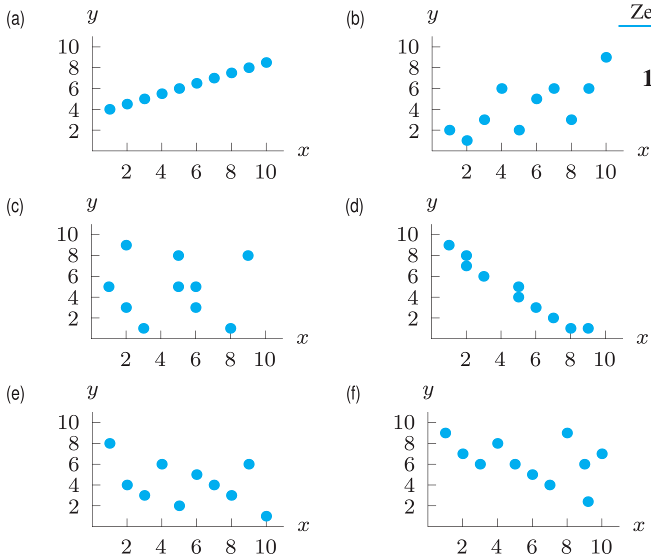


Figure 1.63

8. Table 1.36 shows the number of calories burned per minute by a person walking at 3 mph.

(a) Make a scatter plot of this data.
(b) Draw a regression line by eye.
(c) Roughly estimate the correlation coefficient by eye.

Table 1.36

Body weight (lb)	100	120	150	170	200	220
Calories	2.7	3.2	4.0	4.6	5.4	5.9

9. An ecologist tracked 290 zebra that were born in 2000. The number of zebra, z, living each subsequent year is recorded in Table 1.37.

(a) Make a scatter plot of this data. Let $t = 0$ represent 2000.
(b) Draw by eye a line of good fit and estimate its equation. (Round the coefficients to integers.)
(c) Use a calculator or computer to find the equation of the least squares line. (Round the coefficients to integers.)
(d) Interpret the slope and each intercept of the line.
(e) Interpret the correlation between the year and the number of zebra born in 2000 that are still alive.

Table 1.37

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
Zebra	290	288	268	206	140	90	64	44	8

10. The rate of oxygen consumption for Colorado beetles increases with temperature. See Table 1.38.

(a) Make a scatter plot of this data.
(b) Draw an estimated regression line by eye.
(c) Use a calculator or computer to find the equation of the regression line. (Alternatively, find the equation of your line in part (b).) Round constants in the equation to the nearest integer.
(d) Interpret the slope and each intercept of the regression equation.
(e) Interpret the correlation between temperature and oxygen rate.

Table 1.38

°C	10	15	20	25	30
Oxygen consumption rate	90	125	200	300	375

11. Table 1.39 gives the data on hand strength collected from college freshman using a grip meter.

(a) Make a scatter plot of these data treating the strength of the preferred hand as the independent variable.
(b) Draw a line on your scatter plot that is a good fit for these data and use it to find an approximate equation for the regression line.
(c) Using a graphing calculator or computer, find the equation of the least squares line.
(d) What would the predicted grip strength in the nonpreferred hand be for a student with a preferred hand strength of 37?
(e) Discuss interpolation and extrapolation using specific examples in relation to this regression line.
(f) Discuss why r, the correlation coefficient, is both positive and close to 1.
(g) Why do the points tend to cluster into two groups on your scatter plot?

Table 1.39 *Hand strength for 20 students in kilograms*

Preferred	28	27	45	20	40	47	28	54	52	21
Nonpreferred	24	26	43	22	40	45	26	46	46	22
Preferred	53	52	49	45	39	26	25	32	30	32
Nonpreferred	47	47	41	44	33	20	27	30	29	29

12. Table 1.40 shows men's and women's world records for swimming distances from 50 meters to 1500 meters.[22]

(a) What values would you add to Table 1.40 to represent the time taken by both men and women to swim 0 meters?

(b) Plot men's time against distance, with time t in seconds on the vertical axis and distance d in meters on the horizontal axis. It is claimed that a straight line models this behavior well. What is the equation for that line? What does its slope represent? On the same graph, plot women's time against distance and find the equation of the straight line that models this behavior well. Is this line steeper or flatter than the men's line? What does that mean in terms of swimming? What are the values of the vertical intercepts? Do these values have a practical interpretation?

(c) On another graph plot the women's times against the men's times, with women's times, w, on the vertical axis and men's times, m, on the horizontal axis. It should look linear. How could you have predicted this linearity from the equations you found in part (b)? What is the slope of this line and how can it be interpreted? A newspaper reporter claims that the women's records are about 8% slower than the men's. Do the facts support this statement? What is the value of the vertical intercept? Does this value have a practical interpretation?

Table 1.40 *Men's and women's world swimming records*

Distance (m)	50	100	200	400	800	1500
Men (sec)	21.64	47.84	104.06	220.08	458.65	874.56
Women (sec)	24.13	53.62	116.64	243.85	496.22	952.10

CHAPTER SUMMARY

- **Functions**
  Definition: a rule which takes certain numbers as inputs and assigns to each input exactly one output number.
  Function notation, $y = f(x)$.
  Use of vertical line test.
- **Average Rate of Change**
  Average rate of change of $Q = f(t)$ on $[a, b]$ is
  $$\frac{\Delta Q}{\Delta t} = \frac{f(b) - f(a)}{b - a}.$$
  Increasing, decreasing functions; identifying from average rate of change.
- **Linear Functions**
  Value of y changes at constant rate.
  Tables for linear functions.
- **Formulas for Linear Functions**
  Slope-intercept form: $y = b + mx$.
  Point-slope form: $y - y_0 = m(x - x_0)$.
  Standard form: $Ax + By + C = 0$.
- **Properties of Linear Functions**
  Interpretation of slope, vertical and horizontal intercepts.
  Intersection of lines: Solution of equations.
  Horizontal and vertical lines.
  Parallel lines: $m_1 = m_2$.
  Perpendicular lines: $m_1 = -\dfrac{1}{m_2}$.
- **Fitting Lines to Data**
  Linear regression; correlation. Interpolation, extrapolation; dangers of extrapolation.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER ONE

Exercises

In Exercises 1–5 a relationship is given between two quantities. Are both quantities functions of the other one, or is one or neither a function of the other? Explain.

1. $7w^2 + 5 = z^2$ **2.** $y = x^4 - 1$ **3.** $m = \sqrt{t}$

4. The number of gallons of gas, g, at \$2 per gallon and the number of pounds of coffee, c, at \$10 per pound that can be bought for a total of \$100.

5.

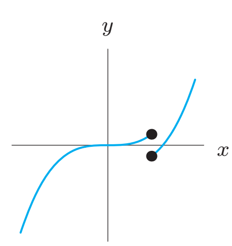


Figure 1.64

[22]Data from *The World Almanac and Book of Facts: 2006*, World Almanac Education Group, Inc., New York, 2006.

6. (a) Which of the graphs in Figure 1.65 represent y as a function of x? (Note that an open circle indicates a point that is not included in the graph; a solid dot indicates a point that is included in the graph.)

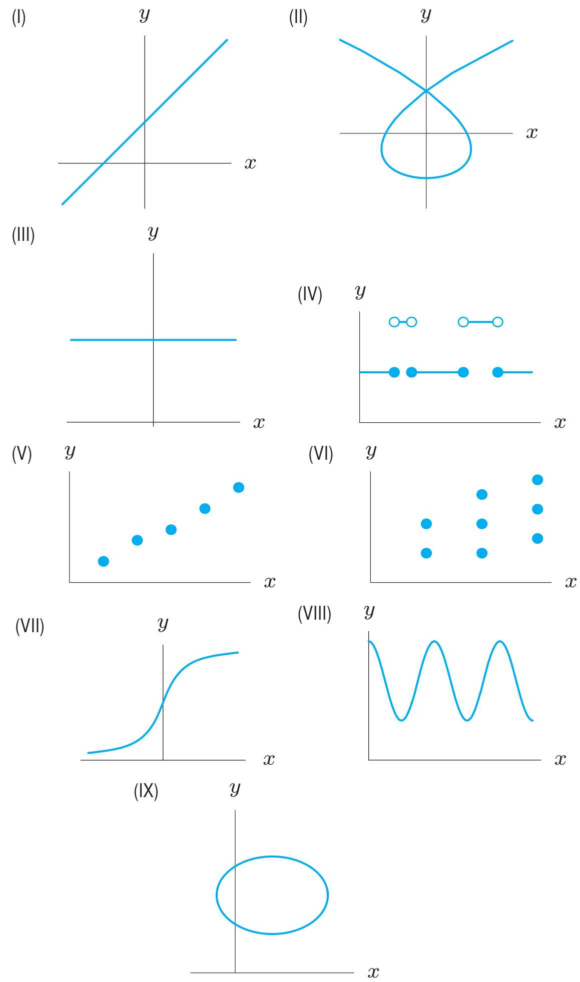


Figure 1.65

(b) Which of the graphs in Figure 1.65 could represent the following situations? Give reasons.

(i) SAT Math score versus SAT Verbal score for a small number of students.

(ii) Total number of daylight hours as a function of the day of the year, shown over a period of several years.

(c) Among graphs (I)–(IX) in Figure 1.65, find two which could give the cost of train fare as a function of the time of day. Explain the relationship between cost and time for both choices.

7. (a) Make a table of values for $f(x) = 10/(1 + x^2)$ for $x = 0, 1, 2, 3$.

(b) What x-value gives the largest $f(x)$ value in your table? How could you have predicted this before doing any calculations?

8. Table 1.41 gives the populations of two cities (in thousands) over a 17-year period.

(a) Find the average rate of change of each population on the following intervals:

(i) 1990 to 2000 (ii) 1990 to 2007 (iii) 1995 to 2007

(b) What do you notice about the average rate of change of each population? Explain what the average rate of change tells you about each population.

Table 1.41

Year	1990	1992	1995	2000	2007
P_1	42	46	52	62	76
P_2	82	80	77	72	65

9. The following tables represent the relationship between the button number, N, that you push, and the snack, S, delivered by three different vending machines.[23]

(a) One of these vending machines is not a good one to use, because S is not a function of N. Which one?

(b) For which vending machine(s) is S a function of N?

(c) For which of the vending machines is N not a function of S?

Vending Machine #1

N	S
1	M&Ms
2	pretzels
3	dried fruit
4	Hersheys
5	fat-free cookies
6	Snickers

Vending Machine #2

N	S
1	M&Ms or dried fruit
2	pretzels or Hersheys
3	Snickers or fat-free cookies

Vending Machine #3

N	S
1	M&Ms
2	M&Ms
3	pretzels
4	dried fruit
5	Hersheys
6	Hersheys
7	fat-free cookies
8	Snickers
9	Snickers

[23]For each N, vending machine #2 dispenses one or the other product at random.

10. Figure 1.66 shows the average monthly temperature in Albany, New York, over a twelve-month period. (January is month 1.)

(a) Make a table showing average temperature as a function of the month of the year.
(b) What is the warmest month in Albany?
(c) Over what interval of months is the temperature increasing? Decreasing?

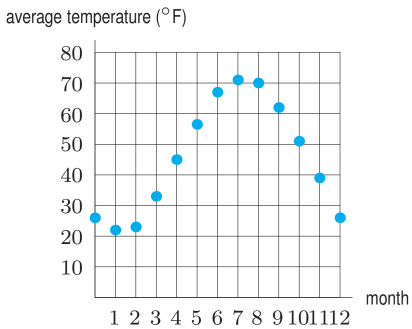


Figure 1.66

11. In 1947, Jesse Owens, the US gold medal track star of the 1930s and 1940s, ran a 100-yard race against a horse. The race, "staged" in Havana, Cuba, is filled with controversy; some say Owens received a head start, others claim the horse was drugged. Owens himself revealed some years later that the starting gun was placed next to the horse's ear, causing the animal to rear and remain at the gate for a few seconds. Figure 1.67 depicts speeds measured against time for the race.

(a) How fast were Owens and the horse going at the end of the race?
(b) When were the participants both traveling at the same speed?

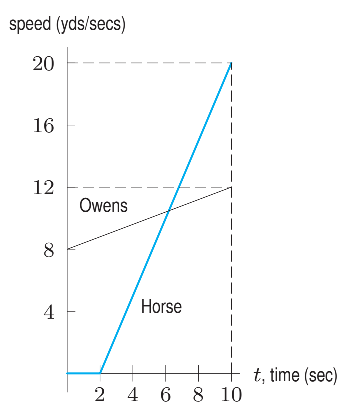


Figure 1.67

In Exercises 12–13, could the table represent a linear function?

12.

λ	1	2	3	4	5
$q(\lambda)$	2	4	8	16	32

13.

t	3	6	9	12	15
$a(t)$	2	4	6	8	10

Problems 14–16 give data from a linear function. Find a formula for the function.

14.

x	200	230	300	320	400
$g(x)$	70	68.5	65	64	60

15.

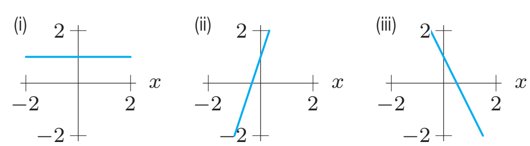
t	1.2	1.3	1.4	1.5
$f(t)$	0.736	0.614	0.492	0.37

16.

t	5.2	5.3	5.4	5.5
$f(t)$	73.6	61.4	49.2	37

17. Without a calculator, match the functions (a)–(c) to the graphs (i)–(iii).

(a) $f(x) = 3x + 1$ **(b)** $g(x) = -2x + 1$
(c) $h(x) = 1$



In Exercises 18–20, which line has the greater

(a) Slope? **(b)** y-intercept?

18. $y = -1 + 2x;\quad y = -2 + 3x$

19. $y = 3 + 4x;\quad y = 5 - 2x$

20. $y = \frac{1}{4}x;\quad y = 1 - 6x$

Are the lines in Exercises 21–24 perpendicular? Parallel? Neither?

21. $y = 5x + 2;\quad y = 2x + 5$

22. $y = 14x - 2;\quad y = -\frac{1}{14}x + 2$

23. $y = 3x + 3;\quad y = -\frac{1}{3}x + 3$

24. $7y = 8 + 21x;\ 9y = 77 - 3x$

Problems

In Problems 25–27, use Table 1.42, which gives values of $v = r(s)$, the eyewall wind profile of a typical hurricane.[24] The eyewall of a hurricane is the band of clouds that surrounds the eye of the storm. The eyewall wind speed v (in mph) is a function of the height above the ground s (in meters).

Table 1.42

s	0	100	200	300	400	500
v	90	110	116	120	121	122
s	600	700	800	900	1000	1100
v	121	119	118	117	116	115

25. Evaluate and interpret $r(300)$.

26. At what altitudes does the eyewall wind speed appear to equal or exceed 116 mph?

27. At what height is the eyewall wind speed greatest?

28. You are looking at the graph of y, a function of x.

(a) What is the maximum number of times that the graph can intersect the y-axis? Explain.
(b) Can the graph intersect the x-axis an infinite number of times? Explain.

29. A bug starts out ten feet from a light, flies closer to the light, then farther away, then closer than before, then farther away. Finally the bug hits the bulb and flies off. Sketch the distance of the bug from the light as a function of time.

30. Although there were 17 women in the Senate in 2009, the first woman elected to the Senate was Hattie Wyatt Caraway of Arkansas. She was appointed to fill the vacancy caused by the death of her husband, then won election in 1932, was reelected in 1938, and served until 1945. Table 1.43 shows the number of female senators at the beginning of the first session of each Congress.[25]

(a) Is the number of female senators a function of the Congress's number, c? Explain.
(b) Is the Congress's number a function of the number of female senators? Explain.
(c) Let $S(c)$ represent the number of female senators serving in the c^{th} Congress. What does the statement $S(104) = 8$ mean?
(d) Evaluate and interpret $S(110)$.

Table 1.43 *Female senators, S, in Congress c*

c	96	98	100	102	104	106	108	110	111
S	1	2	2	2	8	9	14	16	17

31. A light is turned off for several hours. It is then turned on. After a few hours it is turned off again. Sketch the light bulb's temperature as a function of time.

32. According to Charles Osgood, CBS news commentator, it takes about one minute to read 15 double-spaced typewritten lines on the air.[26]

(a) Construct a table showing the time Charles Osgood is reading on the air in seconds as a function of the number of double-spaced lines read for $0, 1, 2, \ldots, 10$ lines. From your table, how long does it take Charles Osgood to read 9 lines?
(b) Plot this data on a graph with the number of lines on the horizontal axis.
(c) From your graph, estimate how long it takes Charles Osgood to read 9 lines. Estimate how many lines Charles Osgood can read in 30 seconds.
(d) Construct a formula which relates the time T to n, the number of lines read.

33. The distance between Cambridge and Wellesley is 10 miles. A person walks part of the way at 5 miles per hour, then jogs the rest of the way at 8 mph. Find a formula that expresses the total amount of time for the trip, $T(d)$, as a function of d, the distance walked.

34. A cylindrical can is closed at both ends and its height is twice its radius. Express its surface area, S, as a function of its radius, r. [Hint: The surface of a can consists of a rectangle plus two circular disks.]

35. A lawyer does nothing but sleep and work during a day. There are 1440 minutes in a day. Write a linear function relating minutes of sleep, s, to minutes of work, w.

For the functions in Problems 36–38:

(a) Find the average rate of change between the points
(i) $(-1, f(-1))$ and $(3, f(3))$
(ii) $(a, f(a))$ and $(b, f(b))$
(iii) $(x, f(x))$ and $(x + h, f(x + h))$
(b) What pattern do you see in the average rate of change between the three pairs of points?

36. $f(x) = 5x - 4$

37. $f(x) = \frac{1}{2}x + \frac{5}{2}$

38. $f(x) = x^2 + 1$

[24]Data from the National Hurricane Center, www.nhc.noaa.gov/aboutwindprofile.shtml, accessed October 7, 2004.
[25]http://en.wikipedia.org/wiki/111th United States Congress Members.
[26]T. Parker, *Rules of Thumb* (Boston: Houghton Mifflin, 1983).

39. Table 1.44 gives the average temperature, T, at a depth d, in a borehole in Belleterre, Quebec.[27] Evaluate $\Delta T/\Delta d$ on the following intervals, and explain what your answers tell you about borehole temperature.

(a) $25 \le d \le 150$
(b) $25 \le d \le 75$
(c) $100 \le d \le 200$

Table 1.44

d, depth (m)	25	50	75	100
T, temp (°C)	5.50	5.20	5.10	5.10
d, depth (m)	125	150	175	200
T, temp (°C)	5.30	5.50	5.75	6.00
d, depth (m)	225	250	275	300
T, temp (°C)	6.25	6.50	6.75	7.00

40. The population, $P(t)$, in millions, of a country in year t, is given by the formula $P(t) = 22 + 0.3t$.

(a) Construct a table of values for $t = 0, 10, 20, \ldots, 50$.
(b) Plot the points you found in part (a).
(c) What is the country's initial population?
(d) What is the average rate of change of the population, in millions of people/year?

41. A woodworker sells rocking horses. His start-up costs, including tools, plans, and advertising, total \$5000. Labor and materials for each horse cost \$350.

(a) Calculate the woodworker's total cost, C, to make 1, 2, 5, 10, and 20 rocking horses. Graph C against n, the number of rocking horses that he carves.
(b) Find a formula for C in terms of n.
(c) What is the rate of change of the function C? What does the rate of change tell us about the woodworker's expenses?

42. Outside the US, temperature readings are usually given in degrees Celsius; inside the US, they are often given in degrees Fahrenheit. The exact conversion from Celsius, C, to Fahrenheit, F, uses the formula

$$F = \frac{9}{5}C + 32.$$

An approximate conversion is obtained by doubling the temperature in Celsius and adding 30° to get the equivalent Fahrenheit temperature.

(a) Write a formula using C and F to express the approximate conversion.
(b) How far off is the approximation if the Celsius temperature is $-5°$, $0°$, $15°$, $30°$?
(c) For what temperature (in Celsius) does the approximation agree with the actual formula?

43. Find a formula for the linear function $h(t)$ whose graph intersects the graph of $j(t) = 30(0.2)^t$ at $t = -2$ and $t = 1$.

44. Find the equation of the line l in Figure 1.68. The shapes under the line are squares.

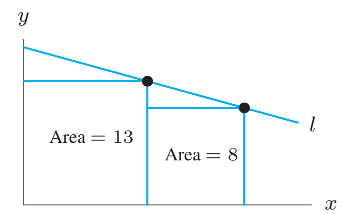


Figure 1.68

45. A bullet is shot straight up into the air from ground level. After t seconds, the velocity of the bullet, in meters per second, is approximated by the formula

$$v = f(t) = 1000 - 9.8t.$$

(a) Evaluate the following: $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$. Compile your results in a table.
(b) Describe in words what is happening to the speed of the bullet. Discuss why you think this is happening.
(c) Evaluate and interpret the slope and both intercepts of $f(t)$.
(d) The gravitational field near the surface of Jupiter is stronger than that near the surface of the earth, which, in turn, is stronger than the field near the surface of the moon. How is the formula for $f(t)$ different for a bullet shot from Jupiter's surface? From the moon?

46. A theater manager graphed weekly profits as a function of the number of patrons and found that the relationship was linear. One week the profit was \$11,328 when 1324 patrons attended. Another week 1529 patrons produced a profit of \$13,275.50.

(a) Find a formula for weekly profit, y, as a function of the number of patrons, x.
(b) Interpret the slope and the y-intercept.
(c) What is the break-even point (the number of patrons for which there is zero profit)?

[27]Hugo Beltrami of St. Francis Xavier University and David Chapman of the University of Utah posted this data at http://geophysics.stfx.ca/public/borehole/borehole.html, accessed November 10, 2005.

(d) Find a formula for the number of patrons as a function of profit.
(e) If the weekly profit was $\$17{,}759.50$, how many patrons attended the theater?

47. Describe a linear (or nearly linear) relationship that you have encountered outside the classroom. Determine the rate of change and interpret it in practical terms.

48. In economics, the *demand* for a product is the amount of that product that consumers are willing to buy at a given price. The quantity demanded of a product usually decreases if the price of that product increases. Suppose that a company believes there is a linear relationship between the demand for its product and its price. The company knows that when the price of its product was $\$3$ per unit, the quantity demanded weekly was 500 units, and that when the unit price was raised to $\$4$, the quantity demanded weekly dropped to 300 units. Let D represent the quantity demanded weekly at a unit price of p dollars.

(a) Calculate D when $p = 5$. Interpret your result.
(b) Find a formula for D in terms of p.
(c) The company raises the price of the good and that the new quantity demanded weekly is 50 units. What is the new price?
(d) Give an economic interpretation of the slope of the function you found in part (b).
(e) Find D when $p = 0$. Find p when $D = 0$. Give economic interpretations of both these results.

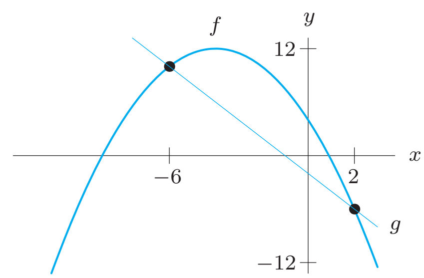
49. In economics, the *supply* of a product is the quantity of that product suppliers are willing to provide at a given price. In theory, the quantity supplied of a product increases if the price of that product increases. Suppose that there is a linear relationship between the quantity supplied, S, of the product described in Problem 48 and its price, p. The quantity supplied weekly is 100 when the price is $\$2$ and the quantity supplied rises by 50 units when the price rises by $\$0.50$.

(a) Find a formula for S in terms of p.
(b) Interpret the slope of your formula in economic terms.
(c) Is there a price below which suppliers will not provide this product?
(d) The *market clearing price* is the price at which supply equals demand. According to theory, the free-market price of a product is its market clearing price. Using the demand function from Problem 48, find the market clearing price for this product.

50. When economists graph demand or supply equations, they place quantity on the horizontal axis and price on the vertical axis.

(a) On the same set of axes, graph the demand and supply equations you found in Problems 48 and 49, with price on the vertical axis.
(b) Indicate how you could estimate the market clearing price from your graph.

51. The figure gives graphs of g, a linear function, and of $f(x) = 12 - 0.5(x + 4)^2$. Find a possible formula for g.



52. Write in slope-intercept form and identify the values of b and m:

$$f(r) = rx^3 + 3rx^2 + 2r + 4sx + 7s + 3.$$

53. Find an equation for the line intersecting the graph of f at $x = -2$ and $x = 5$ given that $f(x) = 2 + \dfrac{3}{x+5}$.

54. A business consultant works 10 hours a day, 6 days a week. She divides her time between meetings with clients and meetings with co-workers. A client meeting requires 3 hours while a co-worker meeting requires 2 hours. Let x be the number of co-worker meetings the consultant holds during a given week. If y is the number of client meetings for which she has time remaining, then y is a function of x. Assume this relationship is linear and that meetings can be split up and continued on different days.

(a) Graph the relationship between y and x. [Hint: Consider the maximum number of client and co-worker meetings that can be held.]
(b) Find a formula for y as a function of x.
(c) Explain what the slope and the x- and y-intercepts represent in the context of the consultant's meeting schedule.
(d) A change is made so that co-worker meetings take 90 minutes instead of 2 hours. Graph this situation. Describe those features of this graph that have changed from the one sketched in part (a) and those that have remained the same.

55. You start 60 miles east of Pittsburgh and drive east at a constant speed of 50 miles per hour. (Assume that the road is straight and permits you to do this.) Find a formula for d, your distance from Pittsburgh as a function of t, the number of hours of travel.

56. Find a formula for the line parallel to the line $y = 20 - 4x$ and containing the point $(3, 12)$.

57. Find the equation of the linear function g whose graph is perpendicular to the line $5x - 3y = 6$; the two lines intersect at $x = 15$.

58. Find the coordinates of point P in Figure 1.69.

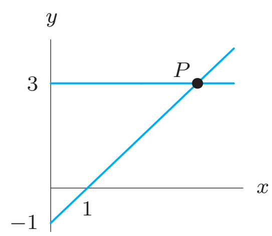


Figure 1.69

59. You want to choose one long-distance telephone company from the following options.

- Company A charges $0.37 per minute.
- Company B charges $13.95 per month plus $0.22 per minute.
- Company C charges a fixed rate of $50 per month.

Let Y_A, Y_B, Y_C represent the monthly charges using Company A, B, and C, respectively. Let x be the number of minutes per month spent on long-distance calls.

(a) Find formulas for Y_A, Y_B, Y_C as functions of x.
(b) Figure 1.70 gives the graphs of the functions in part (a). Which function corresponds to which graph?
(c) Find the x-values for which Company B is cheapest.

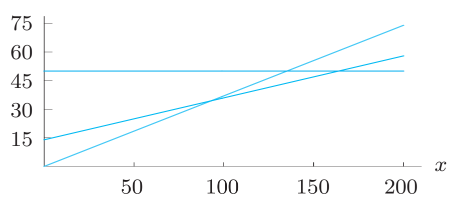


Figure 1.70

60. A commission is a payment made to an employee based on a percentage of sales made. For example, car salespeople earn commission on the selling price of a car. In parts (a)–(d), explain how to choose between the options for different levels of sales.

(a) A weekly salary of $100 or a weekly salary of $50 plus 10% commission.
(b) A weekly salary of $175 plus 7% commission or a weekly salary of $175 plus 8% commission.
(c) A weekly salary of $145 plus 7% commission or a weekly salary of $165 plus 7% commission.
(d) A weekly salary of $225 plus 3% commission or a weekly salary of $180 plus 6% commission.

61. Table 1.45 shows the IQ of ten students and the number of hours of TV each watches per week.

(a) Make a scatter plot of the data.
(b) By eye, make a rough estimate of the correlation coefficient.
(c) Use a calculator or computer to find the least squares regression line and the correlation coefficient. Your values should be correct to four decimal places.

Table 1.45

IQ	110	105	120	140	100	125	130	105	115	110
TV	10	12	8	2	12	10	5	6	13	3

62. For 35 years, major league baseball Hall of Fame member Henry Aaron held the record for the greatest number of career home runs. His record was broken by Barry Bonds in 2007. Table 1.46 shows Aaron's cumulative yearly record [28] from the start of his career, 1954, until 1973.

(a) Plot Aaron's cumulative number of home runs H on the vertical axis, and the time t in years along the horizontal axis, where $t = 1$ corresponds to 1954.
(b) By eye, draw a straight line that fits these data well and find its equation.
(c) Use a calculator or computer to find the equation of the regression line for these data. What is the correlation coefficient, r, to 4 decimal places? To 3 decimal places? What does this tell you?
(d) What does the slope of the regression line mean in terms of Henry Aaron's home-run record?
(e) From your answer to part (d), how many home runs do you estimate Henry Aaron hit in each of the years 1974, 1975, 1976, and 1977? If you were told that Henry Aaron retired at the end of the 1976 season, would this affect your answers?

Table 1.46 *Henry Aaron's cumulative home-run record, H, from 1954 to 1973, with t in years since 1953*

t	1	2	3	4	5	6	7	8	9	10
H	13	40	66	110	140	179	219	253	298	342
t	11	12	13	14	15	16	17	18	19	20
H	366	398	442	481	510	554	592	639	673	713

[28] Adapted from "Graphing Henry Aaron's home-run output" by H. Ringel, *The Physics Teacher*, January 1974, page 43.

63. The graph of a linear function $y = f(x)$ passes through the two points $(a, f(a))$ and $(b, f(b))$, where $a < b$ and $f(a) < f(b)$.

(a) Graph the function labeling the two points.
(b) Find the slope of the line in terms of f, a, and b.

64. Let $f(x) = 0.003 - (1.246x + 0.37)$.

(a) Calculate the following average rates of change:

(i) $\dfrac{f(2) - f(1)}{2 - 1}$ (ii) $\dfrac{f(1) - f(2)}{1 - 2}$

(iii) $\dfrac{f(3) - f(4)}{3 - 4}$

(b) Rewrite $f(x)$ in the form $f(x) = b + mx$.

Write the linear function $y = -3 - x/2$ in the forms given in Problems 65–66, assuming all constants are positive.

65. $y = \dfrac{p}{p-1} - r^2x$ **66.** $y = \dfrac{x + k}{z}$

67. You spend c dollars on x apples and y bananas. In Figure 1.71, line l gives y as a function of x.

(a) If apples cost p dollars each and bananas cost q each, label the x- and y-intercepts of l. [Note: Your labels will involve the constants p, q or c.]
(b) What is the slope of l?

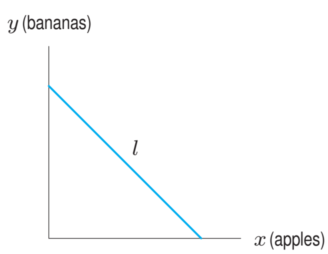


Figure 1.71: Axes not necessarily to scale

68. The apples in Problem 67 cost more than bananas, so $p > q$. Which of the two lines, l_1 or l_2, in Figure 1.72 could represent $y = f(x)$?

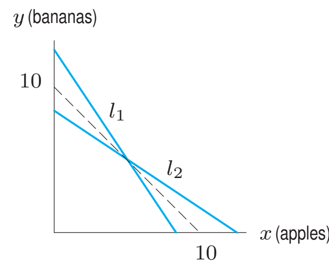


Figure 1.72

69. Many people think that hair growth is stimulated by haircuts. In fact, there is no difference in the rate hair grows after a haircut, but there *is* a difference in the rate at which hair's ends break off. A haircut eliminates dead and split ends, thereby slowing the rate at which hair breaks. However, even with regular haircuts, hair will not grow to an indefinite length. The average life cycle of human scalp hair is 3-5 years, after which the hair is shed.[29]

Judy trims her hair once a year, when its growth is slowed by split ends. She cuts off just enough to eliminate dead and split ends, and then lets it grow another year. After 5 years, she realizes her hair won't grow any longer. Graph the length of her hair as a function of time. Indicate when she receives her haircuts.

70. Academics have suggested that loss of worker productivity can result from sleep deprivation. An article in the September 26, 1993, *New York Times* quotes David Poltrack, the senior vice president for planning and research at CBS, as saying that seven million Americans are staying up an hour later than usual to watch talk show host David Letterman. The article goes on to quote Timothy Monk, a professor at the University of Pittsburgh School of Medicine, as saying, ". . . my hunch is that the effect [on productivity due to sleep deprivation among this group] would be in the area of a 10 percent decrement." The article next quotes Robert Solow, a Nobel prize-winning professor of economics at MIT, who suggests the following procedure to estimate the impact that this loss in productivity will have on the US economy—an impact he dubbed "the Letterman loss." First, Solow says, we find the percentage of the work force who watch the program. Next, we determine this group's contribution to the gross domestic product (GDP). Then we reduce the group's contribution by 10% to account for the loss in productivity due to sleep deprivation. The amount of this reduction is "the Letterman loss."

(a) The article estimated that the GDP is \$6.325 trillion, and that 7 million Americans watch the show. Assume that the nation's work force is 118 million

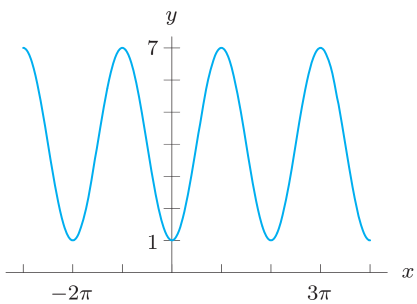
[29] *Britannica Micropaedia* vol. 5 (Chicago: Encyclopaedia Britannica, Inc., 1989).

people and that 75% of David Letterman's audience belongs to this group. What percentage of the work force is in Dave's audience?

(b) What percent of the GDP would be expected to come from David Letterman's audience? How much money would they have contributed if they had not watched the show?

(c) How big is "the Letterman loss"?

71. Judging from the graph of $y = f(x)$ in the figure, find a possible formula for the line intersecting it at $x = -2\pi$ and $x = 3\pi$.



Problems 72–74 refer to Table 1.47, which describes a *boustrophedonic pairing function.* [30] We can use this table to list all the positive roots of the positive integers, such as $\sqrt{2}, \sqrt{3}, \sqrt[3]{2}, \sqrt[3]{3}, \sqrt[4]{2}, \sqrt[4]{3}, \ldots$, without omitting any. If n is an entry in the table, we let c stand for the column number and r the row number of the entry and define the function g by

$$g(n) = c^{1/r}.$$

For instance, if $n = 6$, then $c = 2$ and $r = 3$ so we have $g(6) = 2^{1/3}$, or $\sqrt[3]{2}$.

Table 1.47

	1	2	3	4	5	6	⋯
1	1	2	9	10	25	26	⋯
2	4	3	8	11	24	27	⋯
3	5	6	7	12	23	28	⋯
4	16	15	14	13	22	29	⋯
5	17	18	19	20	21	30	⋯
6	36	35	34	33	32	31	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

72. Evaluate $g(22)$.

73. Evaluate $g(54)$.

74. Find a solution to $g(n) = \sqrt{3}$. Is this solution unique? Explain your reasoning.

Problems 75–76 ask about the A series of paper. Many countries use A4 paper, which is somewhat different from the 8.5 by 11 inch paper standard in the US.[31] A4 is part of a series of paper sizes specified in ISO 216, an international standard. Two sheets of A4 paper, if laid side by side (not end to end), are the same size as one sheet of A3 paper. Likewise, two sheets of A3 are the same as one sheet of A2, and so on. See Figure 1.73. Each sheet in the series has the same proportions, length to width, and the largest sheet, A0, has an area of exactly 1 m^2.

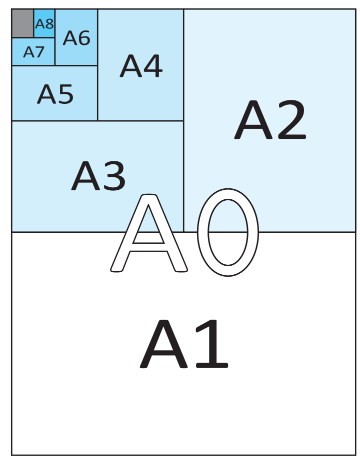


Figure 1.73

75. Let $f(n)$ be the area in cm^2 of the size n sheet in the A-series, so that $f(1)$ is the area of A1 paper, $f(2)$ of A2 paper, and so on. Since 1 m equals 100 cm, we know that $f(0) = (100 \text{ cm})^2 = 10{,}000 \text{ cm}^2$. Complete the table of values of f.

Table 1.48

n	0	1	2	3	4	5
$f(n)$	10,000					

[30] The term *boustrophedonic* means "ox-plowing": Notice how the entries in the table, $1, 2, 3, 4 \ldots$, turn back and forth, like an ox plowing a field.

[31] http://en.wikipedia.org/wiki/ISO_216, accessed January 30, 2008. Note that the actual sizes of A-series paper are rounded to the nearest millimeter.

76. We know that A1 paper has the same proportion (length over width) as A0. Further, the length of A1 paper is the same as the width W of A0, and the width of A1 is half the length L of A0. (See the figure.)

(a) Find a formula for L in terms of W. What does your formula tell you about the shape of A-series paper? How does the shape compare to US letter paper?

(b) Given that A4 paper has area 625 cm^2, find its width and height.

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–54 true or false? Give an explanation for your answer.

1. $Q = f(t)$ means Q is equal to f times t.

2. A function must be defined by a formula.

3. If $P = f(x)$ then P is called the dependent variable.

4. Independent variables are always denoted by the letter x or t.

5. It is possible for two quantities to be related and yet neither be a function of the other.

6. A function is a rule that takes certain values as inputs and assigns to each input value exactly one output value.

7. It is possible for a table of values to represent a function.

8. If Q is a function of P, then P is a function of Q.

9. The graph of a circle is not the graph of a function.

10. If $n = f(A)$ is the number of angels that can dance on the head of a pin whose area is A square millimeters, then $f(10) = 100$ tells us that 10 angels can dance on the head of a pin whose area is 100 square millimeters.

11. Average speed can be computed by dividing the distance traveled by the time elapsed.

12. The average rate of change of a function Q with respect to t over an interval can be symbolically represented as $\dfrac{\Delta t}{\Delta Q}$.

13. If $y = f(x)$ and as x increases, y increases, then f is an increasing function.

14. If f is a decreasing function, then the average rate of change of f on any interval is negative.

15. The average rate of change of a function over an interval is the slope of a line connecting two points of the graph of the function.

16. The average rate of change of $y = 3x - 4$ between $x = 2$ and $x = 6$ is 7.

17. The average rate of change of $f(x) = 10 - x^2$ between $x = 1$ and $x = 2$ is the ratio $\dfrac{10 - 2^2 - 10 - 1^2}{2 - 1}$.

18. If $y = x^2$ then the slope of the line connecting the point $(2, 4)$ to the point $(3, 9)$ is the same as the slope of the line connecting the point $(-2, 4)$ to the point $(-3, 9)$.

19. A linear function can have different rates of change over different intervals.

20. The graph of a linear function is a straight line.

21. If a line has the equation $3x + 2y = 7$, then the slope of the line is 3.

22. A table of values represents a linear function if $\dfrac{\text{Change in output}}{\text{Change in input}} = \text{constant}$.

23. If a linear function is decreasing, then its slope is negative.

24. If $y = f(x)$ is linear and its slope is negative, then in the expression $\dfrac{\Delta y}{\Delta x}$ either Δx or Δy is negative, but not both.

25. A linear function can have a slope that is zero.

26. If a line has slope 2 and y-intercept -3, then its equation may be written $y = -3x + 2$.

27. The line $3x + 5y = 7$ has slope $3/5$.

28. A line that goes through the point $(-2, 3)$ and whose slope is 4 has the equation $y = 4x + 5$.

29. The line $4x + 3y = 52$ intersects the x-axis at $x = 13$.

30. If $f(x) = -2x + 7$ then $f(2) = 3$.

31. The line that passes through the points $(1, 2)$ and $(4, -10)$ has slope 4.

32. The linear equation $y - 5 = 4(x + 1)$ is equivalent to the equation $y = 4x + 6$.

33. The line $y - 4 = -2(x + 3)$ goes through the point $(4, -3)$.

34. The line whose equation is $y = 3 - 7x$ has slope -7.

35. The line $y = -5x + 8$ intersects the y-axis at $y = 8$.

36. The equation $y = -2 - \frac{2}{3}x$ represents a linear function.

37. The lines $y = 8 - 3x$ and $-2x + 16y = 8$ both cross the y-axis at $y = 8$.

38. The graph of $f(x) = 6$ is a line whose slope is six.

39. The lines $y = -\frac{4}{5}x + 7$ and $4x - 5y = 8$ are parallel.

40. The lines $y = 7 + 9x$ and $y - 4 = -\frac{1}{9}(x + 5)$ are perpendicular.

41. The lines $y = -2x + 5$ and $y = 6x - 3$ intersect at the point $(1, 3)$.

42. If two lines never intersect then their slopes are equal.

43. The equation of a line parallel to the y-axis could be $y = -\frac{3}{4}$.

44. A line parallel to the x-axis has slope zero.

45. The slope of a vertical line is undefined.

46. Fitting the best line to a set of data is called linear regression.

47. The process of estimating a value within the range for which we have data is called interpolation.

48. Extrapolation tends to be more reliable than interpolation.

49. If two quantities have a high correlation then one quantity causes the other.

50. If the correlation coefficient is zero, there is not a relationship between the two quantities.

51. A correlation coefficient can have a value of $-\frac{3}{7}$.

52. A value of a correlation coefficient is always between negative and positive one.

53. A correlation coefficient of one indicates that all the data points lie on a straight line.

54. A regression line is also referred to as a least squares line.

SKILLS REFRESHER FOR CHAPTER ONE: LINEAR EQUATIONS AND THE COORDINATE PLANE

Solving Linear Equations

To solve a linear equation, we isolate the variable.

Example 1 Solve $22 + 1.3t = 31.1$ for t.

Solution We subtract 22 from both sides. Since $31.1 - 22 = 9.1$, we have

$$1.3t = 9.1.$$

We divide both sides by 1.3, so

$$t = \frac{9.1}{1.3} = 7.$$

Example 2 Solve $3 - [5.4 + 2(4.3 - x)] = 2 - (0.3x - 0.8)$ for x.

Solution We begin by clearing the innermost parentheses on each side. Using the distributive law, this gives

$$3 - [5.4 + 8.6 - 2x] = 2 - 0.3x + 0.8.$$

Then

$$3 - 14 + 2x = 2 - 0.3x + 0.8$$
$$2.3x = 13.8,$$
$$x = 6.$$

Example 3 Solve $ax = c + bx$ for x. Assume $a \neq b$.

Solution To solve for x, we first get all the terms involving x on the left side by subtracting bx from both sides

$$ax - bx = c.$$

Factoring on the left, $ax - bx = (a - b)x$, enables us to solve for x by dividing both sides by $(a - b)$:

$$x(a - b) = c$$
$$x = \frac{c}{(a - b)}.$$

Since $a \neq b$, division by $(a - b)$ is possible.

Example 4 Solve for q if $p^2q + r(-q - 1) = 4(p + r)$.

Solution We first collect all the terms containing q on the left side of the equation.

$$p^2q - rq - r = 4p + 4r$$
$$p^2q - rq = 4p + 5r.$$

To solve for q, we factor and then divide by the coefficient of q.

$$q(p^2 - r) = 4p + 5r$$
$$q = \frac{4p + 5r}{p^2 - r}.$$

Solving Exactly Versus Solving Approximately

Some equations can be solved exactly, often by using algebra. For example, the equation $7x - 1 = 0$ has the exact solution $x = 1/7$. Other equations can be hard or even impossible to solve exactly. However, it is often possible, and sometimes easier, to find an approximate solution to an equation by using a graph or a numerical method on a calculator. The equation $7x - 1 = 0$ has the approximate solution $x \approx 0.14$ (since $1/7 = 0.142857\ldots$). We use the sign $\approx$, meaning approximately equal, when we want to emphasize that we are making an approximation.

Systems of Linear Equations

To solve for two unknowns, we must have two equations—that is, two relationships between the unknowns. Similarly, three unknowns require three equations, and n unknowns (n an integer) require n equations. The group of equations is known as a *system* of equations. To solve the system, we find the *simultaneous* solutions to all equations in the system.

We can solve these equations either by *substitution* (see Example 5) or by *elimination* (see Example 6).

Example 5 Solve for x and y in the following system of equations using substitution.

$$\begin{cases} y + \dfrac{x}{2} = 3 \\ 2(x + y) = 1 - y \end{cases}$$

Solution Solving the first equation for y, we write $y = 3 - x/2$. Substituting for y in the second equation gives

$$2\left(x + \left(3 - \frac{x}{2}\right)\right) = 1 - \left(3 - \frac{x}{2}\right).$$

Then

$$\begin{aligned} 2x + 6 - x &= -2 + \frac{x}{2} \\ x + 6 &= -2 + \frac{x}{2} \\ 2x + 12 &= -4 + x \\ x &= -16. \end{aligned}$$

Using $x = -16$ in the first equation to find the corresponding y, we have

$$\begin{aligned} y - \frac{16}{2} &= 3 \\ y &= 3 + 8 = 11. \end{aligned}$$

Thus, the solution that simultaneously solves both equations is $x = -16$, $y = 11$.

Example 6 Solve for x and y in the following system of equations using elimination.

$$\begin{cases} 8x - 5y = 11 \\ -2x + 10y = -1. \end{cases}$$

Solution To eliminate y, we observe that if we multiply the first equation by 2, the coefficients of y are -10 and 10:

$$\begin{cases} 16x - 10y = 22 \\ -2x + 10y = -1. \end{cases}$$

Adding these two equations gives

$$\begin{aligned} 14x &= 21 \\ x &= 3/2. \end{aligned}$$

We can substitute this value for x in either of the original equations to find y. For example,

$$\begin{aligned} 8\left(\frac{3}{2}\right) - 5y &= 11 \\ 12 - 5y &= 11 \\ -5y &= -1 \\ y &= 1/5 \end{aligned}$$

Thus, the solution is $x = 3/2$, $y = 1/5$.

Intersection of Two Lines

The coordinates of the point of intersection of two lines satisfy the equations of both lines. Thus, the point can be found by solving the equations simultaneously.

Example 7 Find the point of intersection of the lines $y = 3 - \frac{2}{3}x$ and $y = -4 + \frac{3}{2}x$.

Solution Since the y-values of the two lines are equal at the point of intersection, we have

$$-4 + \frac{3}{2}x = 3 - \frac{2}{3}x.$$

Notice that we have converted a pair of equations into a single equation by eliminating one of the two variables. This equation can be simplified by multiplying both sides by 6:

$$\begin{aligned} 6\left(-4 + \frac{3}{2}x\right) &= 6\left(3 - \frac{2}{3}x\right) \\ -24 + 9x &= 18 - 4x \\ 13x &= 42 \\ x &= \frac{42}{13}. \end{aligned}$$

We can evaluate either of the original equations at $x = \frac{42}{13}$ to find y. For example, $y = -4 + \frac{3}{2}x$ gives

$$y = -4 + \frac{3}{2}\left(\frac{42}{13}\right) = \frac{11}{13}.$$

Therefore, the point of intersection is $\left(\frac{42}{13}, \frac{11}{13}\right)$. You can check that this point also satisfies the other equation. The lines and their point of intersection are shown in Figure 1.74.

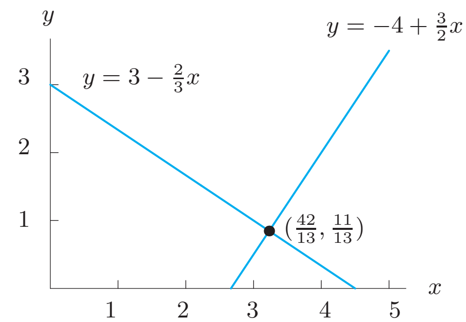


Figure 1.74: Intersection of lines is solution to simultaneous equations

Exercises Skills for Chapter 1

Solve the equations in Exercises 1–12.

1. $3x = 15$

2. $-2y = 12$

3. $4z = 22$

4. $x + 3 = 10$

5. $w - 23 = -34$

6. $7 - 3y = -14$

7. $13t + 2 = 47$

8. $0.5x - 3 = 7$

9. $3t - \dfrac{2(t-1)}{3} = 4$

10. $2(r+5) - 3 = 3(r-8) + 21$

11. $B - 4[B - 3(1-B)] = 42$

12. $1.06s - 0.01(248.4 - s) = 22.67s$

In Exercises 13–22, solve for the indicated variable.

13. $A = l \cdot w,$ for l.

14. $l = l_0 + \dfrac{k}{2}w$, for w.

15. $h = v_0 t + \dfrac{1}{2}at^2$, for a.

16. $3xy + 1 = 2y - 5x,$ for y.

17. $u(v+2) + w(v-3) = z(v-1)$, for v.

18. $S = \dfrac{rL - a}{r - 1}$, for r.

19. $\dfrac{a - cx}{b + dx} + a = 0$, for x.

20. $\dfrac{At - B}{C - B(1-2t)} = 3$, for t.

21. $y'y^2 + 2xyy' = 4y$, for y'.

22. $2x - (xy' + yy') + 2yy' = 0$, for y'.

Solve the systems of equations in Exercises 23–27.

23. $\begin{cases} 3x - 2y = 6 \\ y = 2x - 5 \end{cases}$

24. $\begin{cases} x = 7y - 9 \\ 4x - 15y = 26 \end{cases}$

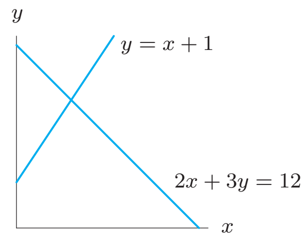
25. $\begin{cases} 2x + 3y = 7 \\ y = -\frac{3}{5}x + 6 \end{cases}$

26. $\begin{cases} 3x - y = 17 \\ -2x - 3y = -4 \end{cases}$

27. $\begin{cases} ax + y = 2a \\ x + ay = 1 + a^2 \end{cases}$

Determine the points of intersection for Exercises 28–29.

28.



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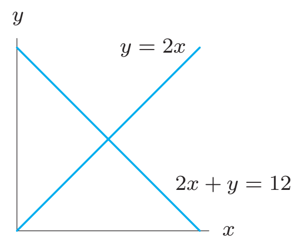
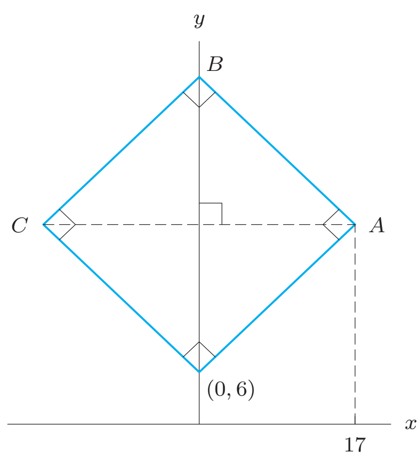


The figures in Problems 30–31 are parallelograms. Find the coordinates of the labeled point(s).

30.



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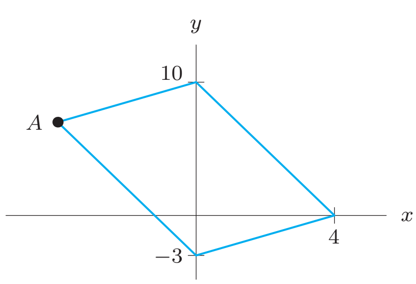
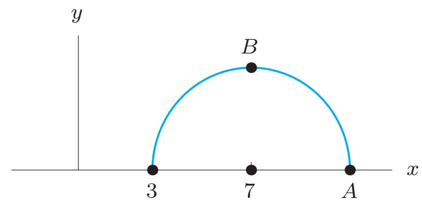


The figures in Problems 32–35 contain a semicircle with the center marked. Find the coordinates of A, a point on the diameter, and B, an extreme point (highest, lowest, or farthest to the right).

32.



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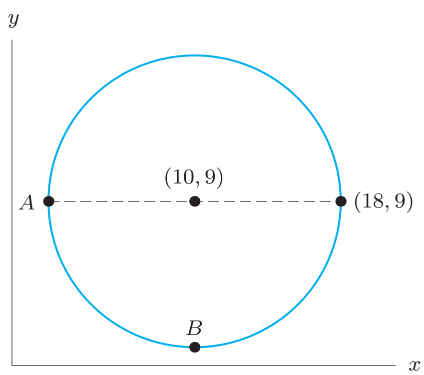

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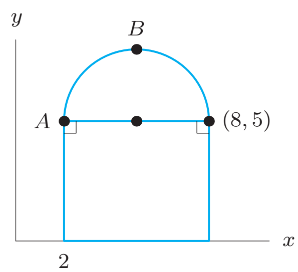

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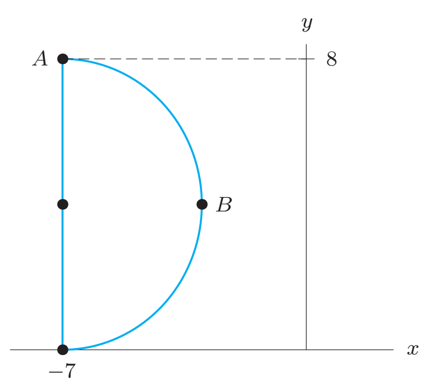

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Chapter Two

FUNCTIONS

Contents

2.1 INPUT AND OUTPUT

Finding Output Values: Evaluating a Function

Evaluating a function means calculating the value of a function's output from a particular value of the input.

In the housepainting example on page 4, the notation $n = f(A)$ indicates that the number of gallons of paint n is a function of area A. The expression $f(A)$ represents the output of the function—specifically, the amount of paint required to cover an area of A ft^2. For example, $f(20{,}000)$ represents the number of gallons of paint required to cover an area of 20,000 ft^2.

Example 1 Using the fact that 1 gallon of paint covers 250 ft^2, evaluate the expression $f(20{,}000)$.

Solution To evaluate $f(20{,}000)$, calculate the number of gallons required to cover 20,000 ft^2:

$$f(20{,}000) = \frac{20{,}000 \text{ ft}^2}{250 \text{ ft}^2/\text{gallon}} = 80 \text{ gallons of paint.}$$

Evaluating a Function Using a Formula

If we have a formula for a function, we evaluate it by substituting the input value into the formula.

Example 2 The formula for the area of a circle of radius r is $A = q(r) = \pi r^2$. Use the formula to evaluate $q(10)$ and $q(20)$. What do your results tell you about circles?

Solution In the expression $q(10)$, the value of r is 10, so

$$q(10) = \pi \cdot 10^2 = 100\pi \approx 314.$$

Similarly, substituting $r = 20$, we have

$$q(20) = \pi \cdot 20^2 = 400\pi \approx 1257.$$

The statements $q(10) \approx 314$ and $q(20) \approx 1257$ tell us that a circle of radius 10 cm has an area of approximately 314 cm^2 and a circle of radius 20 cm has an area of approximately 1257 cm^2.

Example 3 Let $g(x) = \dfrac{x^2+1}{5+x}$. Evaluate the following expressions.

(a) $g(3)$ (b) $g(-1)$ (c) $g(a)$

Solution (a) To evaluate $g(3)$, replace every x in the formula with 3:

$$g(3) = \frac{3^2+1}{5+3} = \frac{10}{8} = 1.25.$$

(b) To evaluate $g(-1)$, replace every x in the formula with (-1):

$$g(-1) = \frac{(-1)^2+1}{5+(-1)} = \frac{2}{4} = 0.5.$$

(c) To evaluate $g(a)$, replace every x in the formula with a:

$$g(a) = \frac{a^2+1}{5+a}.$$

Evaluating a function may involve algebraic simplification, as the following example shows.

Example 4 Let $h(x) = x^2 - 3x + 5$. Evaluate and simplify the following expressions.

(a) $h(2)$ (b) $h(a-2)$ (c) $h(a) - 2$ (d) $h(a) - h(2)$

Solution Notice that x is the input and $h(x)$ is the output. It is helpful to rewrite the formula as

$$\text{Output} = h(\text{Input}) = (\text{Input})^2 - 3\cdot(\text{Input}) + 5.$$

(a) For $h(2)$, we have Input $= 2$, so

$$h(2) = (2)^2 - 3\cdot(2) + 5 = 3.$$

(b) In this case, Input $= a - 2$. We substitute and multiply out

$$\begin{aligned} h(a-2) &= (a-2)^2 - 3(a-2) + 5 \\ &= a^2 - 4a + 4 - 3a + 6 + 5 \\ &= a^2 - 7a + 15. \end{aligned}$$

(c) First input a, then subtract 2:

$$\begin{aligned} h(a) - 2 &= a^2 - 3a + 5 - 2 \\ &= a^2 - 3a + 3. \end{aligned}$$

(d) Since we found $h(2) = 3$ in part (a), we subtract from $h(a)$:

$$\begin{aligned} h(a) - h(2) &= a^2 - 3a + 5 - 3 \\ &= a^2 - 3a + 2. \end{aligned}$$

Finding Input Values: Solving Equations

Given an input, we evaluate the function to find the output. Sometimes the situation is reversed; we know the output and we want to find a corresponding input. If the function is given by a formula, the input values are solutions to an equation.

Example 5 Use the cricket function $T = \frac{1}{4}R + 40$, introduced on page 3, to find the rate, R, at which the snowy tree cricket chirps when the temperature, T, is 76°F.

Solution We want to find R when $T = 76$. Substitute $T = 76$ into the formula and solve the equation

$$\begin{aligned} 76 &= \frac{1}{4}R + 40 \\ 36 &= \frac{1}{4}R \qquad \text{subtract 40 from both sides} \\ 144 &= R. \qquad \text{multiply both sides by 4} \end{aligned}$$

The cricket chirps at a rate of 144 chirps per minute when the temperature is 76°F.

Example 6 Suppose $f(x) = \dfrac{1}{\sqrt{x-4}}$.

(a) Find an x-value that results in $f(x) = 2$.
(b) Is there an x-value that results in $f(x) = -2$?

Solution (a) To find an x-value that results in $f(x) = 2$, solve the equation

$$2 = \frac{1}{\sqrt{x-4}}.$$

Square both sides:

$$4 = \frac{1}{x-4}.$$

Now multiply by $(x-4)$:

$$\begin{aligned} 4(x-4) &= 1 \\ 4x - 16 &= 1 \\ x &= \frac{17}{4} = 4.25. \end{aligned}$$

The x-value is 4.25. (Note that the simplification $(x-4)/(x-4) = 1$ in the second step was valid because $x - 4 \neq 0$.)

(b) Since $\sqrt{x-4}$ is nonnegative if it is defined, its reciprocal, $f(x) = \dfrac{1}{\sqrt{x-4}}$ is also nonnegative if it is defined. Thus, $f(x)$ is not negative for any x input, so there is no x-value that results in $f(x) = -2$.

In the next example, we solve an equation for a quantity that is being used to model a physical quantity; we must choose the solutions that make sense in the context of the model.

Example 7 Let $A = q(r)$ be the area of a circle of radius r, where r is in cm. What is the radius of a circle whose area is 100 cm^2?

Solution The output $q(r)$ is an area. Solving the equation $q(r) = 100$ for r gives the radius of a circle whose area is 100 cm^2. Since the formula for the area of a circle is $q(r) = \pi r^2$, we solve

$$\begin{aligned} q(r) = \pi r^2 &= 100 \\ r^2 &= \frac{100}{\pi} \\ r &= \pm\sqrt{\frac{100}{\pi}} = \pm 5.642. \end{aligned}$$

We have two solutions for r, one positive and one negative. Since a circle cannot have a negative radius, we take $r = 5.642$ cm. A circle of area 100 cm^2 has a radius of 5.642 cm.

Finding Output and Input Values From Tables and Graphs

The following two examples use function notation with a table and a graph respectively.

Example 8 Table 2.1 shows the revenue, $R = f(t)$, received, by the National Football League,[1] NFL, from network TV as a function of the year, t, since 1975.

(a) Evaluate and interpret $f(25)$. (b) Solve and interpret $f(t) = 1159$.

Table 2.1

Year, t (since 1975)	0	5	10	15	20	25	30
Revenue, R (million \$)	201	364	651	1075	1159	2200	2200

Solution (a) Table 2.1 shows $f(25) = 2200$. Since $t = 25$ in the year 2000, we know that NFL's revenue from TV was \$2200 million in the year 2000.

(b) Solving $f(t) = 1159$ means finding the year in which TV revenues were \$1159 million; it is $t = 20$. In 1995, NFL's TV revenues were \$1159 million.

[1] *Newsweek*, January 26, 1998.

Example 9 A man drives from his home to a store and back. The entire trip takes 30 minutes. Figure 2.1 gives his velocity $v(t)$ (in mph) as a function of the time t (in minutes) since he left home. A negative velocity indicates that he is traveling away from the store back to his home.

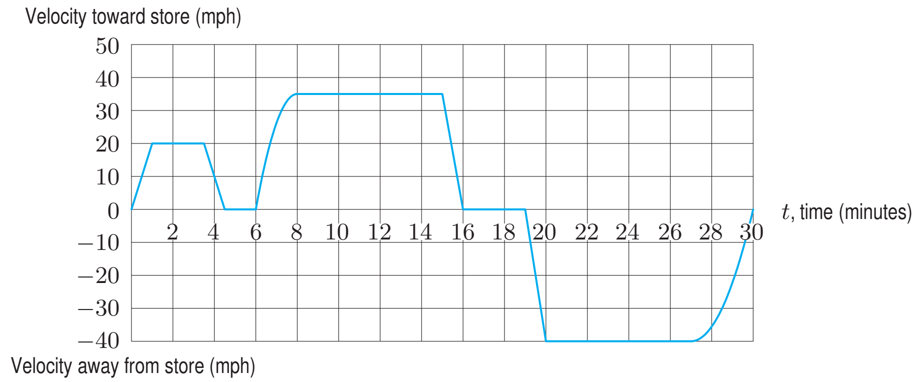


Figure 2.1: Velocity of a man on a trip to the store and back

Evaluate and interpret:

(a) $v(5)$ (b) $v(24)$ (c) $v(8) - v(6)$ (d) $v(-3)$

Solve for t and interpret:

(e) $v(t) = 15$ (f) $v(t) = -20$ (g) $v(t) = v(7)$

Solution

(a) To evaluate $v(5)$, look on the graph where $t = 5$ minutes. Five minutes after he left home, his velocity is 0 mph. Thus, $v(5) = 0$. Perhaps he had to stop at a light.
(b) The graph shows that $v(24) = -40$ mph. After 24 minutes, he is traveling at 40 mph away from the store, back to his home.
(c) From the graph, $v(8) = 35$ mph and $v(6) = 0$ mph. Thus, $v(8) - v(6) = 35 - 0 = 35$. This shows that the man's speed increased by 35 mph in the interval between $t = 6$ minutes and $t = 8$ minutes.
(d) The quantity $v(-3)$ is not defined since the graph only gives velocities for nonnegative times.
(e) To solve for t when $v(t) = 15$, look on the graph where the velocity is 15 mph. This occurs at $t \approx 0.75$ minute, 3.75 minutes, 6.5 minutes, and 15.5 minutes. At each of these four times the man's velocity was 15 mph.
(f) To solve $v(t) = -20$ for t, we see that the velocity is -20 mph (that is, 20 mph toward home) at $t \approx 19.5$ and $t \approx 29$ minutes.
(g) First we evaluate $v(7) \approx 27$. To solve $v(t) = 27$, we look for the values of t making the velocity 27 mph. One such t is of course $t = 7$; the other t is $t \approx 15$ minutes. These are the two times when the velocity is the same as it is at 7 minutes.

Exercises and Problems for Section 2.1

Skill Refresher

For Exercises S1–S6, expand and simplify.

S1. $5(x-3)$

S2. $a(2a+5)$

S3. $(m-5)(4(m-5)+2)$

S4. $(x+2)(3x-8)$

S5. $3\left(1+\frac{1}{x}\right)$

S6. $3+2\left(\frac{1}{x}\right)^2 - x$

Solve the equations in Exercises S7–S10.

S7. $x^2 - 9 = 0$

S8. $\sqrt{2x-1} + 3 = 9$

S9. $\frac{21}{z-5} - \frac{13}{z^2-5z} = 3$

S10. $2x^{\frac{3}{2}} - 1 = 7$

Exercises

1. If $f(t) = t^2 - 4$, **(a)** Find $f(0)$ **(b)** Solve $f(t) = 0$.

2. If $g(x) = x^2 - 5x + 6$, **(a)** Find $g(0)$ **(b)** Solve $g(x) = 0$.

3. If $g(t) = \dfrac{1}{t+2} - 1$, **(a)** Find $g(0)$ **(b)** Solve $g(t) = 0$.

4. If $h(x) = ax^2 + bx + c$, find $h(0)$.

5. If $g(x) = -\frac{1}{2}x^{1/3}$, find $g(-27)$.

6. Let $f(x) = \dfrac{2x+1}{x+1}$. For what value of x is $f(x) = 0.3$?

If $p(r) = r^2 + 5$, evaluate the expressions in Exercises 7–8.

7. $p(7)$

8. $p(x) + p(8)$

In Figure 2.2, mark the point(s) representing the statements in Exercises 9–12 and label their coordinates.

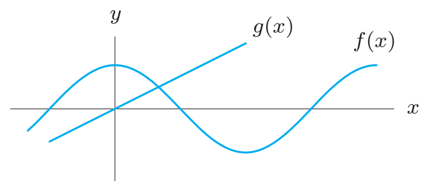


Figure 2.2

9. $f(0) = 2$

10. $f(-3) = f(3) = f(9) = 0$

11. $f(2) = g(2)$

12. $g(x) > f(x)$ for $x > 2$

Problems

13. Let $F = g(t)$ be the number of foxes in a park as a function of t, the number of months since January 1. Evaluate $g(9)$ using Table 1.3 on page 5. What does this tell us about the fox population?

14. Let $F = g(t)$ be the number of foxes in month t in the national park described in Example 5 on page 5. Solve the equation $g(t) = 75$. What does your solution tell you about the fox population?

15. Let $f(x) = 3 + 2x^2$. Find $f\left(\dfrac{1}{3}\right)$ and $\dfrac{f(1)}{f(3)}$. Are they equal?

16. Let $g(x) = x^2 + x$. Find formulas for the following functions. Simplify your answers.

(a) $g(-3x)$ **(b)** $g(1-x)$ **(c)** $g(x+\pi)$
(d) $g(\sqrt{x})$ **(e)** $g(1/(x+1))$ **(f)** $g(x^2)$

17. Let $f(x) = \dfrac{x}{x-1}$.

(a) Find and simplify

(i) $f\left(\dfrac{1}{t}\right)$ (ii) $f\left(\dfrac{1}{t+1}\right)$

(b) Solve $f(x) = 3$.

18. **(a)** Using Figure 2.3, fill in Table 2.2.

Table 2.2

x	-2	-1	0	1	2	3
$h(x)$						

(b) Evaluate $h(3) - h(1)$ (c) Evaluate $h(2) - h(0)$
(d) Evaluate $2h(0)$ (e) Evaluate $h(1) + 3$

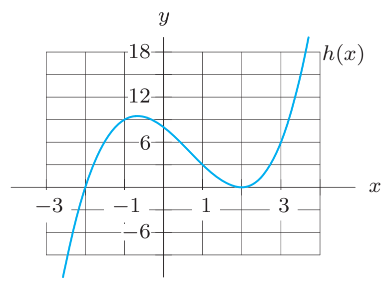


Figure 2.3

19. A ball is thrown up from the ground with initial velocity 64 ft/sec. Its height at time t is

$$h(t) = -16t^2 + 64t.$$

(a) Evaluate $h(1)$ and $h(3)$. What does this tell us about the height of the ball?
(b) Sketch this function. Using a graph, determine when the ball hits the ground and the maximum height of the ball.

20. Let $v(t) = t^2 - 2t$ be the velocity, in ft/sec, of an object at time t, in seconds.

(a) What is the initial velocity, $v(0)$?
(b) When does the object have a velocity of zero?
(c) What is the meaning of the quantity $v(3)$? What are its units?

21. Let $s(t) = 11t^2 + t + 100$ be the position, in miles, of a car driving on a straight road at time t, in hours. The car's velocity at any time t is given by $v(t) = 22t + 1$.

(a) Use function notation to express the car's position after 2 hours. Where is the car then?

(b) Use function notation to express the question, "When is the car going 65 mph?"
(c) Where is the car when it is going 67 mph?

22. Use the letters a, b, c, d, e, h in Figure 2.4 to answer the following questions.

(a) What are the coordinates of the points P and Q?
(b) Evaluate $f(b)$.
(c) Solve $f(x) = e$ for x.
(d) Suppose $c = f(z)$ and $z = f(x)$. What is x?
(e) Suppose $f(b) = -f(d)$. What additional information does this give you?

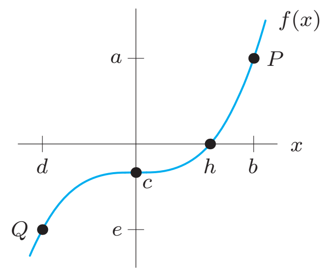


Figure 2.4

23. New York state income tax is based on taxable income, which is part of a person's total income. The tax owed to the state is calculated using the taxable income (not total income). In 2009, for a single person with a taxable income between \$20,000 and \$200,000, the tax owed was \$973 plus 6.85% of the taxable income over \$20,000.[2]

(a) Compute the tax owed by a lawyer whose taxable income is \$68,000.
(b) Consider a lawyer whose taxable income is 80% of her total income, $\$x$, where x is between \$85,000 and \$120,000. Write a formula for $T(x)$, the taxable income.
(c) Write a formula for $L(x)$, the amount of tax owed by the lawyer in part (b).
(d) Use $L(x)$ to evaluate the tax liability for $x = 85{,}000$ and compare your results to part (a).

24. **(a)** Complete Table 2.3 using

$$f(x) = 2x(x-3) - x(x-5) \quad \text{and} \quad g(x) = x^2 - x.$$

What do you notice? Graph these two functions. Are the two functions the same? Explain.

(b) Complete Table 2.4 using

$$h(x) = x^5 - 5x^3 + 6x + 1 \quad \text{and} \quad j(x) = 2x + 1.$$

What do you notice? Graph these two functions. Are the two functions the same? Explain.

[2] www.nystax.gov, accessed January 4, 2010.

Table 2.3

x	-2	-1	0	1	2
$f(x)$					
$g(x)$					

Table 2.4

x	-2	-1	0	1	2
$h(x)$					
$j(x)$					

Problems 25–26 concern $v = r(s)$, the eyewall wind profile of a hurricane at landfall, where v is the eyewall wind speed (in mph) as a function of s, the height (in meters) above the ground. (The eyewall is the band of clouds that surrounds the eye of the storm.) Let s_0 be the height at which the wind speed is greatest, and let $v_0 = r(s_0)$. Interpret the following in terms of the hurricanes.

25. $r(0.5s_0)$

26. $r(s) = 0.75v_0$

27. Let $h(x) = x^2 + bx + c$. Evaluate and simplify:

(a) $h(1)$ **(b)** $h(b+1)$

28. If $g(x) = x\sqrt{x} + 100x$, evaluate without a calculator

(a) $g(100)$ **(b)** $g(4/25)$ **(c)** $g(1.21 \cdot 10^4)$

In Problems 29–31, if $f(x) = \dfrac{ax}{a+x}$, find and simplify

29. $f(a)$

30. $f(1-a)$

31. $f\left(\dfrac{1}{1-a}\right)$

32. Values of f and g are given in Table 2.5.

(a) Evaluate $f(1)$ and $g(3)$.
(b) Describe in full sentences the patterns you see in the values for each function.
(c) Assuming that the patterns you observed in part (b) hold true for all values of x, calculate $f(5)$, $f(-2)$, $g(5)$, and $g(-2)$.
(d) Find possible formulas for $f(x)$ and $g(x)$.

Table 2.5

x	-1	0	1	2	3	4
$f(x)$	-4	-1	2	5	8	11
$g(x)$	4	1	0	1	4	9

33. Table 2.6 shows $N(s)$, the number of sections of Economics 101, as a function of s, the number of students in the course. If s is between two numbers listed in the table, then $N(s)$ is the higher number of sections.

Table 2.6

s	50	75	100	125	150	175	200
$N(s)$	4	4	5	5	6	6	7

(a) Evaluate and interpret:
(i) $N(150)$ (ii) $N(80)$ (iii) $N(55.5)$

(b) Solve for s and interpret:
(i) $N(s) = 4$ (ii) $N(s) = N(125)$

34. Figure 2.5 shows $y = f(x)$. Label the coordinates of any points on the graph where

(a) $f(c) = 0$.
(b) $f(0) = d$.

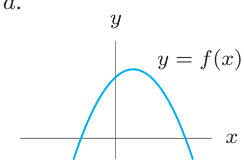


Figure 2.5

35. An epidemic of influenza spreads through a city. Figure 2.6 is the graph of $I = f(w)$, where I is the number of individuals (in thousands) infected w weeks after the epidemic begins.

(a) Evaluate $f(2)$ and explain its meaning in terms of the epidemic.

(b) Approximately how many people were infected at the height of the epidemic? When did that occur? Write your answer in the form $f(a) = b$.

(c) Solve $f(w) = 4.5$ and explain what the solutions mean in terms of the epidemic.

(d) The graph used $f(w) = 6w(1.3)^{-w}$. Use the graph to estimate the solution of the inequality $6w(1.3)^{-w} \geq 6$. Explain what the solution means in terms of the epidemic.

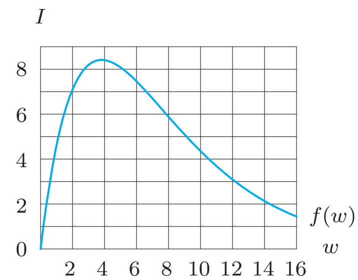


Figure 2.6

Problems 36–37 concern studies which indicate that as carbon dioxide (CO_2) levels rise, hurricanes will become more intense.[3] Hurricane intensity is measured in terms of the minimum central pressure P (in mb): the lower the pressure, the more powerful the storm. Since warm ocean waters fuel hurricanes, P is a decreasing function of H, sea surface temperature in °C. Let $P = n(H)$ be the hurricane-intensity function for present-day CO_2 levels, and let $P = N(H)$ be the hurricane-intensity function for future projected CO_2 levels. If H_0 is the average temperature in the Caribbean Sea, what do the following quantities tell you about hurricane intensity?

36. $N(H_0) - n(H_0)$

37. $n(H_0 + 1) - n(H_0)$

2.2 DOMAIN AND RANGE

In Example 4 on page 5, we defined R to be the average monthly rainfall at Chicago's O'Hare airport in month t. Although R is a function of t, the value of R is not defined for every possible value of t. For instance, it makes no sense to consider the value of R for $t = -3$, or $t = 8.21$, or $t = 13$ (since a year has 12 months). Thus, although R is a function of t, this function is defined only for certain values of t. Notice also that R, the output value of this function, takes only the values $\{1.8, 2.1, 2.4, 2.5, 2.7, 3.1, 3.2, 3.4, 3.5, 3.7\}$.

A function is often defined only for certain values of the independent variable. Also, the dependent variable often takes on only certain values. This leads to the following definitions:

> If $Q = f(t)$, then
> - the **domain** of f is the set of input values, t, which yield an output value.
> - the **range** of f is the corresponding set of output values, Q.

[3] *Journal of Climate*, September 14, 2004, pages 3477–3495.

Thus, the domain of a function is the set of input values, and the range is the set of output values.

If the domain of a function is not specified, we usually assume that it is as large as possible—that is, all numbers that make sense as inputs for the function. For example, if there are no restrictions, the domain of the function $f(x) = x^2$ is the set of all real numbers, because we can substitute any real number into the formula $f(x) = x^2$. Sometimes, however, we may restrict the domain to suit a particular application. If the function $f(x) = x^2$ is used to represent the area of a square of side x, we restrict the domain to positive numbers.

If a function is being used to model a real-world situation, the domain and range of the function are often determined by the constraints of the situation being modeled, as in the next example.

Example 1 The house-painting function $n = f(A)$ in Example 2 on page 4 has domain $A > 0$ because all houses have some positive paintable area. There is a practical upper limit to A because houses cannot be infinitely large, but in principle, A can be as large or as small as we like, as long as it is positive. Therefore we take the domain of f to be $A > 0$.

The range of this function is $n \geq 0$, because we cannot use a negative amount of paint.

Choosing Realistic Domains and Ranges

When a function is used to model a real situation, it may be necessary to modify the domain and range.

Example 2 Algebraically speaking, the formula

$$T = \frac{1}{4}R + 40$$

can be used for all values of R. If we know nothing more about this function than its formula, its domain is all real numbers. The formula for $T = \frac{1}{4}R + 40$ can return any value of T when we choose an appropriate R-value. (See Figure 2.7.) Thus, the range of the function is also all real numbers. However, if we use this formula to represent the temperature, T, as a function of a cricket's chirp rate, R, as we did in Example 1 on page 2, some values of R cannot be used. For example, it does not make sense to talk about a negative chirp rate. Also, there is some maximum chirp rate $R_{\max}$ that no cricket can physically exceed. Thus, to use this formula to express T as a function of R, we must restrict R to the interval $0 \leq R \leq R_{\max}$ shown in Figure 2.8.

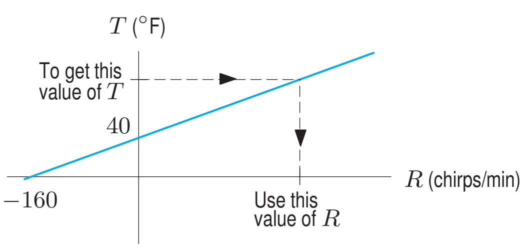


Figure 2.7: Graph showing that any T value can be obtained from some R value

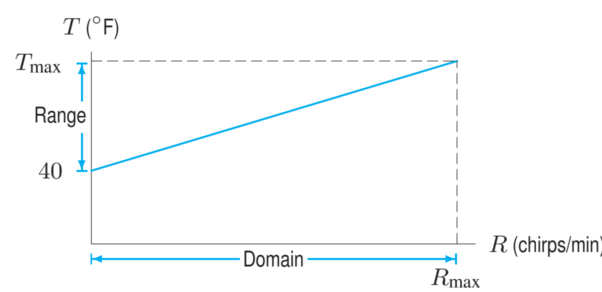


Figure 2.8: Graph showing that if $0 \leq R \leq R_{\max}$, then $40 \leq T \leq T_{\max}$

The range of the cricket function is also restricted. Since the chirp rate is nonnegative, the smallest value of T occurs when $R = 0$. This happens at $T = 40$. On the other hand, if the temperature gets too hot, the cricket will not be able to keep chirping faster. If the temperature $T_{\max}$ corresponds to the chirp rate $R_{\max}$, then the values of T are restricted to the interval $40 \leq T \leq T_{\max}$.

Using a Graph to Find the Domain and Range of a Function

A good way to estimate the domain and range of a function is to examine its graph. The domain is the set of input values on the horizontal axis that give rise to a point on the graph; the range is the corresponding set of output values on the vertical axis.

Example 3 A sunflower plant is measured every day t, for $t \geq 0$. The height, $h(t)$ centimeters, of the plant[4] can be modeled by using the *logistic function*

$$h(t) = \frac{260}{1 + 24(0.9)^t}.$$

(a) Using a graphing calculator or computer, graph the height over 80 days.
(b) What is the domain of this function? What is the range? What does this tell you about the height of the sunflower?

Solution (a) The logistic function is graphed in Figure 2.9.

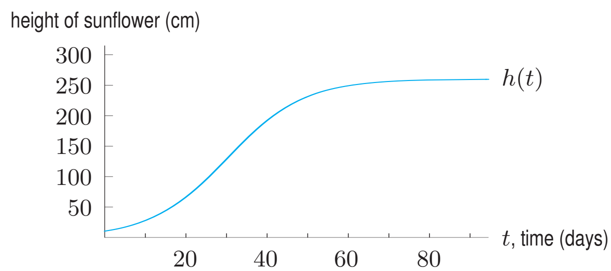


Figure 2.9: Height of sunflower as a function of time

(b) The domain of this function is $t \geq 0$. If we consider the fact that the sunflower dies at some point, then there is an upper bound on the domain, $0 \leq t \leq T$, where T is the day on which the sunflower dies.

To find the range, notice that the smallest value of h occurs at $t = 0$. Evaluating gives $h(0) = 10.4$ cm. This means that the plant was 10.4 cm high when it was first measured on day $t = 0$. Tracing along the graph, $h(t)$ increases. As t-values get large, $h(t)$-values approach, but never reach, 260. This suggests that the range is $10.4 \leq h(t) < 260$. This information tells us that sunflowers typically grow to a height of about 260 cm.

Using a Formula to Find the Domain and Range of a Function

When a function is defined by a formula, its domain and range can often be determined by examining the formula algebraically.

Example 4 State the domain and range of g, where

$$g(x) = \frac{1}{x}.$$

Solution The domain is all real numbers except those which do not yield an output value. The expression $1/x$ is defined for any real number x except 0 (division by 0 is undefined). Therefore,

$$\text{Domain: all real } x, \quad x \neq 0.$$

[4] Adapted from H.S. Reed and R.H. Holland, "Growth of an Annual Plant Helianthus," *Proc. Nat. Acad. Sci.*, 5, 1919.

The range is all real numbers that the formula can return as output values. It is not possible for $g(x)$ to equal zero, since 1 divided by a real number is never zero. All real numbers except 0 are possible output values, since all nonzero real numbers have reciprocals. Thus

$$\text{Range: all real values,} \quad g(x) \neq 0.$$

The graph in Figure 2.10 indicates agreement with these values for the domain and range.

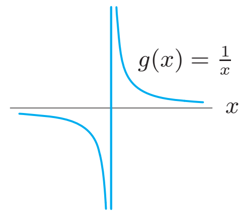


Figure 2.10: Domain and range of $g(x) = 1/x$

Example 5 Find the domain of the function $f(x) = \dfrac{1}{\sqrt{x-4}}$ by examining its formula.

Solution The domain is all real numbers except those for which the function is undefined. The square root of a negative number is undefined (if we restrict ourselves to real numbers), and so is division by zero. Therefore we need

$$x - 4 > 0.$$

Thus, the domain is all real numbers greater than 4.

$$\text{Domain:} \quad x > 4.$$

In Example 6 on page 69, we saw that for $f(x) = 1/\sqrt{x-4}$, the output, $f(x)$, cannot be negative. Note that $f(x)$ cannot be zero either. (Why?) The range of $f(x) = 1/\sqrt{x-4}$ is $f(x) > 0$. See Exercise 9.

Exercises and Problems for Section 2.2

Skill Refresher

In Exercises S1–S4, for what value(s), if any, are the functions undefined?

S1. $f(x) = \dfrac{x-2}{x-3}$

S2. $g(x) = \dfrac{1}{x(x-3)}$

S3. $h(x) = \sqrt{x-15}$

S4. $k(x) = \sqrt{15-x}$

Solve the inequalities in Exercises S5–S10.

S5. $x - 8 > 0$

S6. $-x + 5 > 0$

S7. $-3(n-4) > 12$

S8. $12 \leq 24 - 4a$

S9. $x^2 - 25 > 0$

S10. $36 - x^2 \geq 0$

Exercises

In Exercises 1–4, use a graph to find the range of the function on the given domain.

1. $f(x) = \dfrac{1}{x}, \quad -2 \leq x \leq 2$

2. $f(x) = \dfrac{1}{x^2}, \quad -1 \leq x \leq 1$

3. $f(x) = x^2 - 4, \quad -2 \leq x \leq 3$

4. $f(x) = \sqrt{9 - x^2}, \quad -3 \leq x \leq 1$

Find the domain of the functions in Exercises 5–14 algebraically.

5. $f(x) = \dfrac{1}{x+3}$

6. $p(t) = \dfrac{1}{t^2-4}$

7. $f(t) = \dfrac{t-3}{3t+9}$

8. $n(q) = \dfrac{1}{q^4+2}$

9. $f(x) = \dfrac{1}{\sqrt{x-4}}$

10. $y(t) = \dfrac{1}{t^4}$

11. $f(x) = \sqrt{x^2-4}$

12. $q(r) = \sqrt[3]{r^2-16}$

13. $m(x) = x^2 - 9$

14. $t(a) = \sqrt[4]{a-2}$

Find the domain and range of the functions in Exercises 15–16 algebraically.

15. $m(q) = \dfrac{1}{5}q - 4$

16. $f(x) = \sqrt{15-4x}$

Given the domain D of the functions in Exercises 17–20, find possible values for the unknowns a and b (where applicable).

17. $f(x) = \dfrac{1}{x-a}$, D: all real numbers $\neq 3$

18. $p(t) = \dfrac{1}{(2t-a)(t+b)}$, D: all real numbers except 4 and 5.

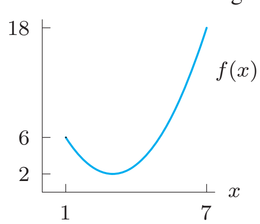
19. $m(r) = \sqrt{r-a}$, D: all real numbers ≥ -3

20. $n(q) = \sqrt{r^2+a}$, D: all real numbers

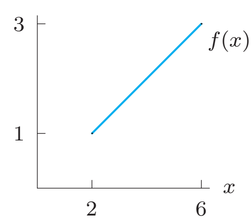
Problems

In Problems 21–22, estimate the domain and range of the function. Assume the entire graph is shown.

21.



22.



23. Give a formula for a function whose domain is all negative values of x except $x = -5$.

24. Give a formula for a function that is undefined for $x = -2$ and for $x < -4$, but is defined everywhere else.

25. A restaurant is open from 2 pm to 2 am each day, and a maximum of 200 clients can fit inside. If $f(t)$ is the number of clients in the restaurant t hours after 2 pm each day, what are a reasonable domain and range for $f(t)$?

26. What is the domain of the function f giving average monthly rainfall at Chicago's O'Hare airport? (See Table 1.2 on page 5.)

27. A movie theater seats 200 people. For any particular show, the amount of money the theater makes is a function of the number of people, n, in attendance. If a ticket costs \$4.00, find the domain and range of this function. Sketch its graph.

28. A car gets the best mileage at intermediate speeds. Graph the gas mileage as a function of speed. Determine a reasonable domain and range for the function and justify your reasoning.

29. **(a)** Use Table 2.7 to determine the number of calories that a person weighing 200 lb uses in a half–hour of walking.[5]

(b) Table 2.7 illustrates a relationship between the number of calories used per minute walking and a person's weight in pounds. Describe in words what is true about this relationship. Identify the dependent and independent variables. Specify whether it is an increasing or decreasing function.

(c) (i) Graph the linear function for walking, as described in part (b), and estimate its equation.

(ii) Interpret the meaning of the vertical intercept of the graph of the function.

(iii) Specify a meaningful domain and range for your function.

(iv) Use your function to determine how many calories per minute a person who weighs 135 lb uses per minute of walking.

Table 2.7 *Calories per minute as a function of weight*

Activity	100 lb	120 lb	150 lb	170 lb	200 lb	220 lb
Walking	2.7	3.2	4.0	4.6	5.4	5.9
Bicycling	5.4	6.5	8.1	9.2	10.8	11.9
Swimming	5.8	6.9	8.7	9.8	11.6	12.7

In Problems 30–31, find the domain and range of the function.

30. $h(x) = \dfrac{a}{\sqrt{x}}$, where a is a constant

31. $p(x) = |x-b| + 6$, where b is a constant

[5] Source: 1993 World Almanac. Speeds assumed are 3 mph for walking, 10 mph for bicycling, and 2 mph for swimming.

32. The last digit, d, of a phone number is a function of n, its position in the phone book. Table 2.8 gives d for the first 10 listings in the 2009 New York State telephone directory.[6] The table shows that the last digit of the first listing is 1, the last digit of the second listing is 5, and so on. In principle we could use a phone book to figure out other values of d. For instance, if $n = 300$, we could count down to the 300^{th} listing in order to determine d. So we write $d = f(n)$.

(a) What is the value of $f(6)$?
(b) Explain how you could use the phone book to find the domain of f.
(c) What is the range of f?

Table 2.8

n	1	2	3	4	5	6	7	8	9	10
d	1	2	1	5	9	9	0	1	1	7

33. In month $t = 0$, a small group of rabbits escapes from a ship onto an island where there are no rabbits. The island rabbit population, $p(t)$, in month t is given by

$$p(t) = \frac{1000}{1 + 19(0.9)^t}, \quad t \geq 0.$$

(a) Evaluate $p(0)$, $p(10)$, $p(50)$, and explain their meaning in terms of rabbits.
(b) Graph $p(t)$ for $0 \leq t \leq 100$. Describe the graph in words. Does it suggest the growth in population you would expect among rabbits on an island?
(c) Estimate the range of $p(t)$. What does this tell you about the rabbit population?
(d) Explain how you can find the range of $p(t)$ from its formula.

34. Bronze is an alloy or mixture of the metals copper and tin. The properties of bronze depend on the percentage of copper in the mix. A chemist decides to study the properties of a given alloy of bronze as the proportion of copper is varied. She starts with 9 kg of bronze that contain 3 kg of copper and 6 kg of tin and either adds or removes copper. Let $f(x)$ be the percentage of copper in the mix if x kg of copper are added ($x > 0$) or removed ($x < 0$).

(a) State the domain and range of f. What does your answer mean in the context of bronze?
(b) Find a formula in terms of x for $f(x)$.
(c) If the formula you found in part (b) was not intended to represent the percentage of copper in an alloy of bronze, but instead simply defined an abstract mathematical function, what would be the domain and range of this function?

35. Let t be time in seconds and let $r(t)$ be the rate, in gallons/second, that water enters a reservoir:

$$r(t) = 800 - 40t.$$

(a) Evaluate the expressions $r(0), r(15), r(25)$, and explain their physical significance.
(b) Graph $y = r(t)$ for $0 \leq t \leq 30$, labeling the intercepts. What is the physical significance of the slope and the intercepts?
(c) For $0 \leq t \leq 30$, when does the reservoir have the most water? When does it have the least water?
(d) What are the domain and range of $r(t)$?

36. The surface area of a cylindrical aluminum can is a measure of how much aluminum the can requires. If the can has radius r and height h, its surface area A and its volume V are given by the equations:

$$A = 2\pi r^2 + 2\pi rh \quad \text{and} \quad V = \pi r^2 h.$$

(a) The volume, V, of a 12 oz cola can is 355 cm^3. A cola can is approximately cylindrical. Express its surface area A as a function of its radius r, where r is measured in centimeters. [Hint: First solve for h in terms of r.]
(b) Graph $A = s(r)$, the surface area of a cola can whose volume is 355 cm^3, for $0 \leq r \leq 10$.
(c) What is the domain of $s(r)$? Based on your graph, what, approximately, is the range of $s(r)$?
(d) The manufacturers wish to use the smallest amount of aluminum (in cm^2) necessary to make a 12-oz cola can. Use your answer in (c) to find the minimum amount of aluminum needed. State the values of r and h that minimize the amount of aluminum used.
(e) The radius of a real 12-oz cola can is about 3.25 cm. Show that real cola cans use more aluminum than necessary to hold 12 oz of cola. Why do you think real cola cans are made in this way?

[6] www6.oft.state.ny.us/telecom/phones/techSubRange.do?type=INDIVIDUAL&prefix=A$D-1340452-P=1, accessed February 10, 2010.

2.3 PIECEWISE-DEFINED FUNCTIONS

A function may employ different formulas on different parts of its domain. Such a function is said to be *piecewise defined*. For example, the function graphed in Figure 2.11 has the following formulas:

$$y = x^2 \quad \text{for} \quad x \le 2 \qquad\qquad y = \begin{cases} x^2 & \text{for } x \le 2 \\ 6 - x & \text{for } x > 2. \end{cases}$$
$$y = 6 - x \quad \text{for} \quad x > 2 \quad \text{or more compactly}$$

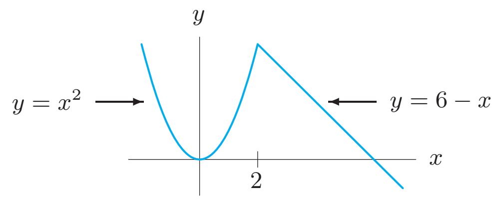


Figure 2.11: Piecewise defined function

Example 1 Graph the function $y = g(x)$ given by the following formulas:

$$g(x) = x + 1 \quad \text{for} \quad x \le 2 \qquad \text{and} \qquad g(x) = 1 \quad \text{for} \quad x > 2.$$

Using bracket notation, this function is written:

$$g(x) = \begin{cases} x + 1 & \text{for } x \le 2 \\ 1 & \text{for } x > 2. \end{cases}$$

Solution For $x \le 2$, graph the line $y = x + 1$. The solid dot at the point $(2, 3)$ shows that it is included in the graph. For $x > 2$, graph the horizontal line $y = 1$. See Figure 2.12. The open circle at the point $(2, 1)$ shows that it is not included in the graph. (Note that $g(2) = 3$, and $g(2)$ cannot have more than one value.)

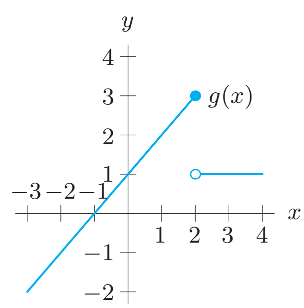


Figure 2.12: Graph of the piecewise defined function g

Example 2 A long-distance calling plan charges 99 cents for any call up to 20 minutes in length and 7 cents for each additional minute or part of a minute.

(a) Use bracket notation to write a formula for the cost, C, of a call as a function of its length t in minutes.
(b) Graph the function.
(c) State the domain and range of the function.

Solution

(a) For $0 < t \leq 20$, the value of C is 99 cents. If $t > 20$, we subtract 20 to find the additional minutes and multiply by the rate, 7 cents per minute.[7] The cost function in cents is thus

$$C = f(t) = \begin{cases} 99 & \text{for } 0 < t \leq 20 \\ 99 + 7(t - 20) & \text{for } t > 20, \end{cases}$$

or, after simplifying,

$$C = f(t) = \begin{cases} 99 & \text{for } 0 < t \leq 20 \\ 7t - 41 & \text{for } t > 20. \end{cases}$$

(b) See Figure 2.13.

(c) Because negative and zero call lengths do not make sense, the domain is $t > 0$. From the graph, we see that the range is $C \geq 99$.

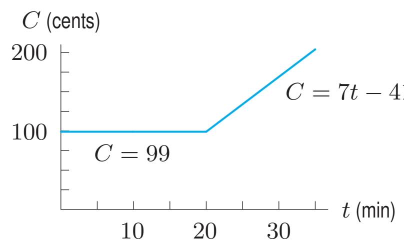


Figure 2.13: Cost of a long-distance phone call

Example 3 The Ironman Triathlon is a race that consists of three parts: a 2.4-mile swim followed by a 112-mile bike race and then a 26.2-mile marathon. A participant swims steadily at 2 mph, cycles steadily at 20 mph, and then runs steadily at 9 mph.[8] Assuming that no time is lost during the transition from one stage to the next, find a formula for the distance covered, d, in miles, as a function of the elapsed time t in hours, from the beginning of the race. Graph the function.

Solution

For each leg of the race, we use the formula Distance = Rate · Time. First, we calculate how long it took for the participant to cover each of the three parts of the race. The first leg took $2.4/2 = 1.2$ hours, the second leg took $112/20 = 5.6$ hours, and the final leg took $26.2/9 \approx 2.91$ hours. Thus, the participant finished the race in $1.2 + 5.6 + 2.91 = 9.71$ hours.

During the first leg, $t \leq 1.2$ and the speed is 2 mph, so

$$d = 2t \qquad \text{for} \qquad 0 \leq t \leq 1.2.$$

During the second leg, $1.2 < t \leq 1.2 + 5.6 = 6.8$ and the speed is 20 mph. The length of time spent in the second leg is $(t - 1.2)$ hours. Thus, by time t,

$$\text{Distance covered in the second leg} = 20(t - 1.2) \quad \text{for } 1.2 < t \leq 6.8.$$

When the participant is in the second leg, the total distance covered is the sum of the distance covered in the first leg (2.4 miles) plus the part of the second leg that has been covered by time t:

$$\begin{aligned} d &= 2.4 + 20(t - 1.2) \\ &= 20t - 21.6 \qquad \text{for } 1.2 < t \leq 6.8. \end{aligned}$$

In the third leg, $6.8 < t \leq 9.71$ and the speed is 9 mph. Since 6.8 hours were spent on the first two parts of the race, the length of time spent on the third leg is $(t - 6.8)$ hours. Thus, by time t,

$$\text{Distance covered in the third leg} = 9(t - 6.8) \quad \text{for } 6.8 < t \leq 9.71.$$

[7] In actuality, most calling plans round the call length to whole minutes or specified fractions of a minute.
[8] Personal communication Susan Reid, Athletics Department, University of Arizona.

When the participant is in the third leg, the total distance covered is the sum of the distances covered in the first leg (2.4 miles) and the second leg (112 miles), plus the part of the third leg that has been covered by time t:

$$\begin{aligned} d &= 2.4 + 112 + 9(t - 6.8) \\ &= 9t + 53.2 \qquad \text{for } 6.8 < t \le 9.71. \end{aligned}$$

The formula for d is different on different intervals of t:

$$d = \begin{cases} 2t & \text{for} \quad 0 \le t \le 1.2 \\ 20t - 21.6 & \text{for} \quad 1.2 < t \le 6.8 \\ 9t + 53.2 & \text{for} \quad 6.8 < t \le 9.71. \end{cases}$$

Figure 2.14 gives a graph of the distance covered, d, as a function of time, t. Notice the three pieces.

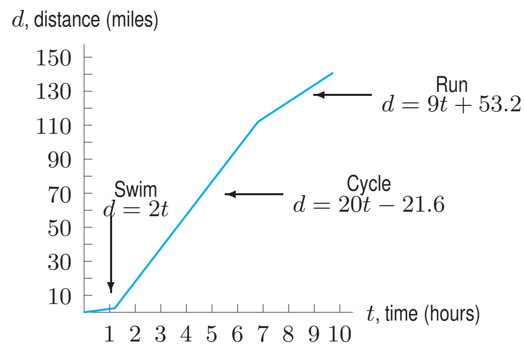


Figure 2.14: Ironman Triathlon: d as a function of t

The Absolute Value Function

The absolute value of a x, written $|x|$, is defined piecewise:

$$\text{For positive } x, \quad |x| = x.$$

$$\text{For negative } x, \quad |x| = -x.$$

(Remember that $-x$ is a positive number if x is a negative number.) For example, if $x = -3$, then

$$|-3| = -(-3) = 3.$$

For $x = 0$, we have $|0| = 0$. This leads to the following two-part definition:

The **Absolute Value Function** is defined by

$$f(x) = |x| = \begin{cases} x & \text{for} \quad x \ge 0 \\ -x & \text{for} \quad x < 0 \end{cases}.$$

Table 2.9 gives values of $f(x) = |x|$ and Figure 2.15 shows a graph of $f(x)$.

Table 2.9 *Absolute value function*

x	$\|x\|$
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

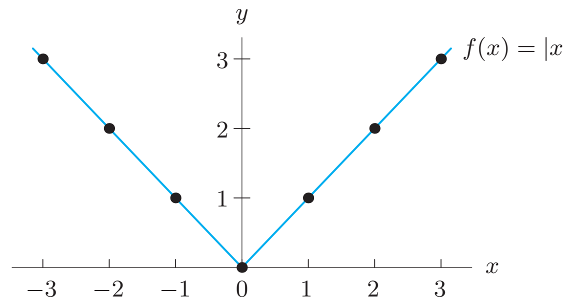
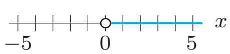


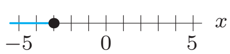
Figure 2.15: Graph of absolute value function

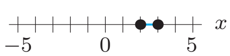
Exercises and Problems for Section 2.3

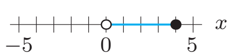
Skill Refresher

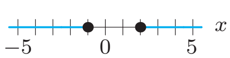
In Exercises S1–S6, write all the possible values for x that match the graph.

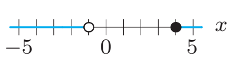
S1.


S2.


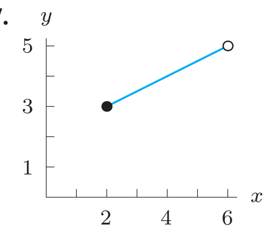
S3.


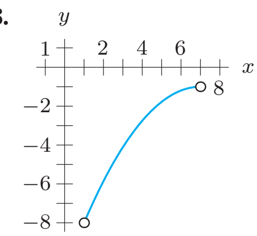
S4.


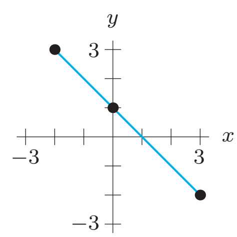
S5.


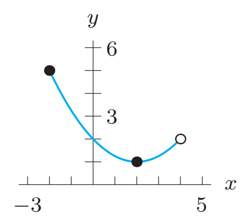
S6.


In Exercises S7–S10, determine the domain and range of the function.

S7.


S8.


S9.


S10.


Exercises

Graph the piecewise defined functions in Exercises 1–4. Use an open circle to represent a point that is not included and a solid dot to indicate a point that is on the graph.

1. $f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

2. $f(x) = \begin{cases} x+1, & -2 \le x < 0 \\ x-1, & 0 \le x < 2 \\ x-3, & 2 \le x < 4 \end{cases}$

3. $f(x) = \begin{cases} x+4, & x \le -2 \\ 2, & -2 < x < 2 \\ 4-x, & x \ge 2 \end{cases}$

4. $f(x) = \begin{cases} x^2, & x \le 0 \\ \sqrt{x}, & 0 < x < 4 \\ x/2, & x \ge 4 \end{cases}$

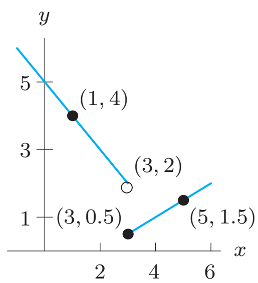
For Exercises 5–6, find the domain and range.

5. $G(x) = \begin{cases} x+1 & \text{for} \quad x < -1 \\ x^2+3 & \text{for} \quad x \ge -1 \end{cases}$

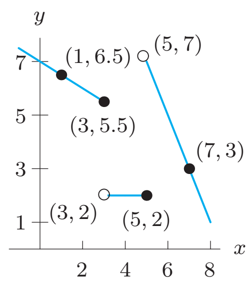
6. $F(x) = \begin{cases} x^3 & \text{for} \quad x \le 1 \\ 1/x & \text{for} \quad x > 1 \end{cases}$

In Exercises 7–10, write formulas for the functions.

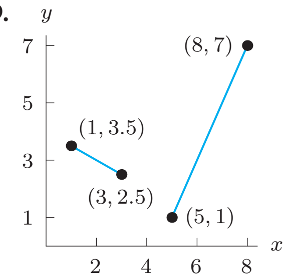
7.



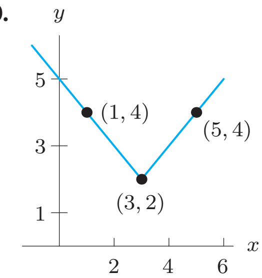
8.



9.



10.



Problems

11. Consider the graph in Figure 2.16. An open circle represents a point that is not included.

(a) Is y a function of x? Explain.
(b) Is x a function of y? Explain.
(c) The domain of $y = f(x)$ is $0 \le x < 4$. What is the range of $y = f(x)$?

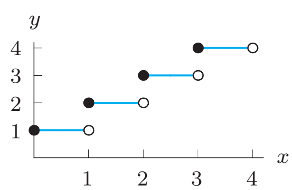


Figure 2.16

12. Many people believe that $\sqrt{x^2} = x$. We will investigate this claim graphically and numerically.

(a) Graph the two functions x and $\sqrt{x^2}$ in the window $-5 \le x \le 5$, $-5 \le y \le 5$. Based on what you see, do you believe that $\sqrt{x^2} = x$? What function does the graph of $\sqrt{x^2}$ remind you of?
(b) Complete Table 2.10. Based on this table, do you believe that $\sqrt{x^2} = x$? What function does the table for $\sqrt{x^2}$ remind you of? Is this the same function you found in part (a)?

Table 2.10

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\sqrt{x^2}$											

(c) Explain how you know that $\sqrt{x^2}$ is the same as the function $|x|$.
(d) Graph the function $\sqrt{x^2} - |x|$ in the window $-5 \le x \le 5$, $-5 \le y \le 5$. Explain what you see.

13. **(a)** Graph $u(x) = |x|/x$ in the window $-5 \le x \le 5$, $-5 \le y \le 5$. Explain what you see.
(b) Complete Table 2.11. Does this table agree with what you found in part (a)?

Table 2.11

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\lvert x\rvert/x$											

(c) Identify the domain and range of $u(x)$.
(d) Comment on the claim that $u(x)$ can be written as

$$u(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

14. The charge for a taxi ride in New York City is \$2.50 upon entry and \$0.40 for each $1/5$ of a mile traveled (rounded up to the nearest $1/5$ mile), when the taxicab is traveling at 6 mph or more. In addition, a New York State Tax Surcharge of \$0.50 is added to the fare. [9]

(a) Make a table showing the cost of a trip as a function of its length. Your table should start at zero and go up to two miles in $1/5$-mile intervals.
(b) What is the cost for a 1.2-mile trip?
(c) How far can you go for \$5.80?
(d) Graph the cost function in part (a).

15. A museum charges \$40 for a group of 10 or fewer people. A group of more than 10 people must, in addition to the \$40, pay \$2 per person for the number of people above 10. For example, a group of 12 pays \$44 and a group of 15 pays \$50. The maximum group size is 50.

(a) Draw a graph that represents this situation.
(b) What are the domain and range of the cost function?

[9] www.nyc.gov/html/tlc/html/passenger/taxicab_rate.shtml, accessed February 11, 2010.