

412 Chapter 5 / Integration

80. A rock, dropped from an unknown height, strikes the ground with a speed of 24 m/s. Find the height from which the rock was dropped.

81–88 Evaluate the integrals by making an appropriate substitution. ■

$$81. \int_0^1 (2x + 1)^4 dx$$

$$82. \int_{-5}^0 x\sqrt{4-x} dx$$

$$83. \int_0^1 \frac{dx}{\sqrt{3x+1}}$$

$$84. \int_0^{\sqrt{\pi}} x \sin x^2 dx$$

$$85. \int_0^1 \sin^2(\pi x) \cos(\pi x) dx$$

$$86. \int_e^{e^2} \frac{dx}{x \ln x}$$

$$87. \int_0^1 \frac{dx}{\sqrt{e^x}}$$

$$88. \int_0^{2/\sqrt{3}} \frac{1}{4+9x^2} dx$$

89. Evaluate the limits.

$$(a) \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^{2x} \quad (b) \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{3x}\right)^x$$

90. Find a function f and a number a such that

$$2 + \int_a^x f(t) dt = e^{3x}$$

CHAPTER 5 MAKING CONNECTIONS

1. Consider a Riemann sum

$$\sum_{k=1}^n 2x_k^* \Delta x_k$$

for the integral of $f(x) = 2x$ over an interval $[a, b]$.

(a) Show that if x_k^* is the midpoint of the k th subinterval, the Riemann sum is a telescoping sum. (See Exercises 57–60 of Section 5.4 for other examples of telescoping sums.)

(b) Use part (a), Definition 5.5.1, and Theorem 5.5.2 to evaluate the definite integral of $f(x) = 2x$ over $[a, b]$.

2. The function $f(x) = \sqrt{x}$ is continuous on $[0, 4]$ and therefore integrable on this interval. Evaluate

$$\int_0^4 \sqrt{x} dx$$

by using Definition 5.5.1. Use subintervals of unequal length given by the partition

$$0 < 4(1)^2/n^2 < 4(2)^2/n^2 < \dots < 4(n-1)^2/n^2 < 4$$

and let x_k^* be the right endpoint of the k th subinterval.

3. Make appropriate modifications and repeat Exercise 2 for

$$\int_0^8 \sqrt[3]{x} dx$$

4. Given a continuous function f and a positive real number m , let g denote the function defined by the composition $g(x) = f(mx)$.

(a) Suppose that

$$\sum_{k=1}^n g(x_k^*) \Delta x_k$$

is any Riemann sum for the integral of g over $[0, 1]$. Use the correspondence $u_k = mx_k$, $u_k^* = mx_k^*$ to create a Riemann sum for the integral of f over $[0, m]$. How are the values of the two Riemann sums related?

(b) Use part (a), Definition 5.5.1, and Theorem 5.5.2 to find an equation that relates the integral of g over $[0, 1]$ with the integral of f over $[0, m]$.

(c) How is your answer to part (b) related to Theorem 5.9.1?

5. Given a continuous function f , let g denote the function defined by $g(x) = 2xf(x^2)$.

(a) Suppose that

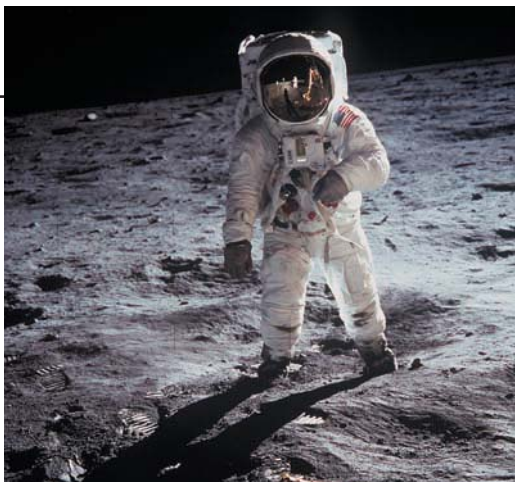
$$\sum_{k=1}^n g(x_k^*) \Delta x_k$$

is any Riemann sum for the integral of g over $[2, 3]$, with $x_k^* = (x_k + x_{k-1})/2$ the midpoint of the k th subinterval. Use the correspondence $u_k = x_k^2$, $u_k^* = (x_k^*)^2$ to create a Riemann sum for the integral of f over $[4, 9]$. How are the values of the two Riemann sums related?

(b) Use part (a), Definition 5.5.1, and Theorem 5.5.2 to find an equation that relates the integral of g over $[2, 3]$ with the integral of f over $[4, 9]$.

(c) How is your answer to part (b) related to Theorem 5.9.1?

6



Courtesy NASA

Calculus is essential for the computations required to land an astronaut on the moon.

APPLICATIONS OF THE DEFINITE INTEGRAL IN GEOMETRY, SCIENCE, AND ENGINEERING

In the last chapter we introduced the definite integral as the limit of Riemann sums in the context of finding areas. However, Riemann sums and definite integrals have applications that extend far beyond the area problem. In this chapter we will show how Riemann sums and definite integrals arise in such problems as finding the volume and surface area of a solid, finding the length of a plane curve, calculating the work done by a force, finding the center of gravity of a planar region, finding the pressure and force exerted by a fluid on a submerged object, and finding properties of suspended cables.

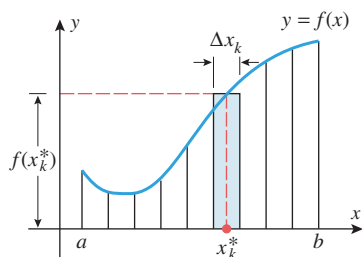
Although these problems are diverse, the required calculations can all be approached by the same procedure that we used to find areas—breaking the required calculation into “small parts,” making an approximation for each part, adding the approximations from the parts to produce a Riemann sum that approximates the entire quantity to be calculated, and then taking the limit of the Riemann sums to produce an exact result.

6.1 AREA BETWEEN TWO CURVES

In the last chapter we showed how to find the area between a curve $y = f(x)$ and an interval on the x -axis. Here we will show how to find the area between two curves.

■ A REVIEW OF RIEMANN SUMS

Before we consider the problem of finding the area between two curves it will be helpful to review the basic principle that underlies the calculation of area as a definite integral. Recall that if f is continuous and nonnegative on $[a, b]$, then the definite integral for the area A under $y = f(x)$ over the interval $[a, b]$ is obtained in four steps (Figure 6.1.1):



▲ Figure 6.1.1

- Divide the interval $[a, b]$ into n subintervals, and use those subintervals to divide the region under the curve $y = f(x)$ into n strips.
- Assuming that the width of the k th strip is Δx_k , approximate the area of that strip by the area $f(x_k^*)\Delta x_k$ of a rectangle of width Δx_k and height $f(x_k^*)$, where x_k^* is a point in the k th subinterval.
- Add the approximate areas of the strips to approximate the entire area A by the Riemann sum:

$$A \approx \sum_{k=1}^n f(x_k^*)\Delta x_k$$

$$\sum_{k=1}^n f(x_k^*) \Delta x_k$$

$$\int_a^b f(x) dx$$

Effect of the limit process on the Riemann sum

▲ Figure 6.1.2

- Take the limit of the Riemann sums as the number of subintervals increases and all their widths approach zero. This causes the error in the approximations to approach zero and produces the following definite integral for the exact area A :

$$A = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k = \int_a^b f(x) dx$$

Figure 6.1.2 illustrates the effect that the limit process has on the various parts of the Riemann sum:

- The quantity x_k^* in the Riemann sum becomes the variable x in the definite integral.
- The interval width Δx_k in the Riemann sum becomes the dx in the definite integral.
- The interval $[a, b]$, which is the union of the subintervals with widths $\Delta x_1, \Delta x_2, \dots, \Delta x_n$, does not appear explicitly in the Riemann sum but is represented by the upper and lower limits of integration in the definite integral.

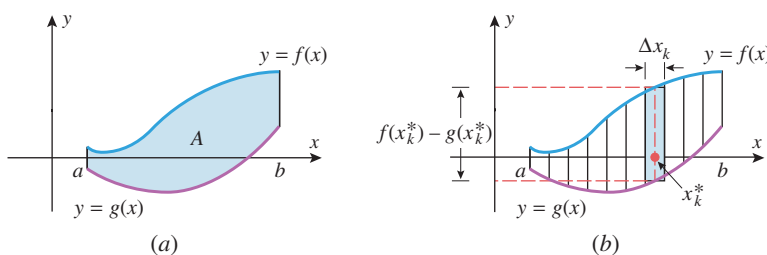
■ AREA BETWEEN $y = f(x)$ AND $y = g(x)$

We will now consider the following extension of the area problem.

6.1.1 FIRST AREA PROBLEM Suppose that f and g are continuous functions on an interval $[a, b]$ and

$$f(x) \geq g(x) \quad \text{for } a \leq x \leq b$$

[This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.1.3a).



► Figure 6.1.3

To solve this problem we divide the interval $[a, b]$ into n subintervals, which has the effect of subdividing the region into n strips (Figure 6.1.3b). If we assume that the width of the k th strip is Δx_k , then the area of the strip can be approximated by the area of a rectangle of width Δx_k and height $f(x_k^*) - g(x_k^*)$, where x_k^* is a point in the k th subinterval. Adding these approximations yields the following Riemann sum that approximates the area A :

$$A \approx \sum_{k=1}^n [f(x_k^*) - g(x_k^*)] \Delta x_k$$

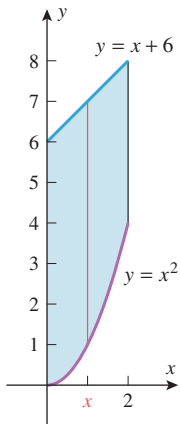
Taking the limit as n increases and the widths of all the subintervals approach zero yields the following definite integral for the area A between the curves:

$$A = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n [f(x_k^*) - g(x_k^*)] \Delta x_k = \int_a^b [f(x) - g(x)] dx$$

In summary, we have the following result.

6.1.2 AREA FORMULA If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)] dx \quad (1)$$



▲ Figure 6.1.4

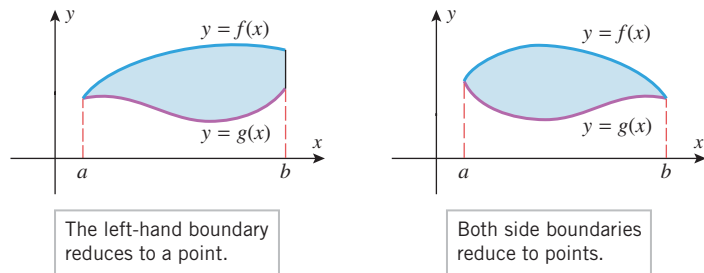
What does the integral in (1) represent if the graphs of f and g cross each other over the interval $[a, b]$? How would you find the area between the curves in this case?

► **Example 1** Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

Solution. The region and a cross section are shown in Figure 6.1.4. The cross section extends from $g(x) = x^2$ on the bottom to $f(x) = x + 6$ on the top. If the cross section is moved through the region, then its leftmost position will be $x = 0$ and its rightmost position will be $x = 2$. Thus, from (1)

$$A = \int_0^2 [(x + 6) - x^2] dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3} \blacktriangleleft$$

It is possible that the upper and lower boundaries of a region may intersect at one or both endpoints, in which case the sides of the region will be points, rather than vertical line segments (Figure 6.1.5). When that occurs you will have to determine the points of intersection to obtain the limits of integration.



► Figure 6.1.5

► **Example 2** Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution. A sketch of the region (Figure 6.1.6) shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. At the endpoints of the region, the upper and lower boundaries have the same y -coordinates; thus, to find the endpoints we equate

$$y = x^2 \quad \text{and} \quad y = x + 6 \quad (2)$$

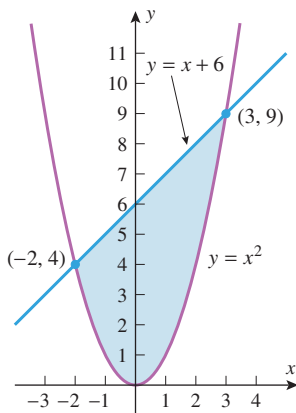
This yields

$$x^2 = x + 6 \quad \text{or} \quad x^2 - x - 6 = 0 \quad \text{or} \quad (x + 2)(x - 3) = 0$$

from which we obtain

$$x = -2 \quad \text{and} \quad x = 3$$

Although the y -coordinates of the endpoints are not essential to our solution, they may be obtained from (2) by substituting $x = -2$ and $x = 3$ in either equation. This yields $y = 4$ and $y = 9$, so the upper and lower boundaries intersect at $(-2, 4)$ and $(3, 9)$.



▲ Figure 6.1.6

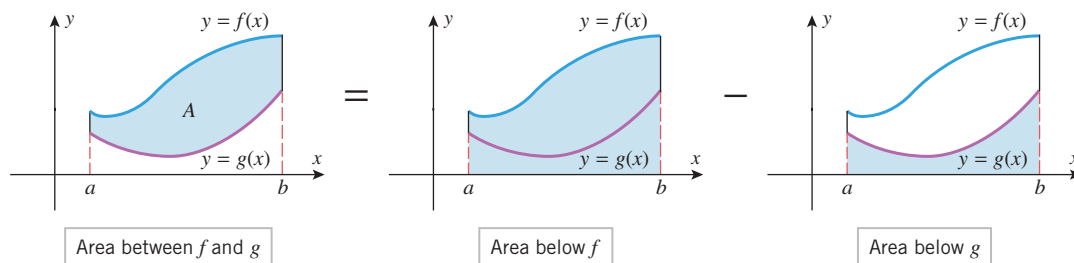
From (1) with $f(x) = x + 6$, $g(x) = x^2$, $a = -2$, and $b = 3$, we obtain the area

$$A = \int_{-2}^3 [(x + 6) - x^2] dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^3 = \frac{27}{2} - \left(-\frac{22}{3} \right) = \frac{125}{6} \blacktriangleleft$$

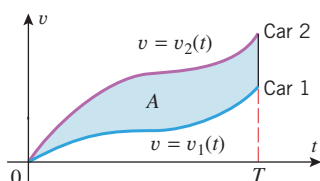
In the case where f and g are *nonnegative* on the interval $[a, b]$, the formula

$$A = \int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

states that the area A between the curves can be obtained by subtracting the area under $y = g(x)$ from the area under $y = f(x)$ (Figure 6.1.7).



▲ Figure 6.1.7



▲ Figure 6.1.8

► **Example 3** Figure 6.1.8 shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same time. Give a physical interpretation of the area A between the curves over the interval $0 \leq t \leq T$.

Solution. From (1)

$$A = \int_0^T [v_2(t) - v_1(t)] dt = \int_0^T v_2(t) dt - \int_0^T v_1(t) dt$$

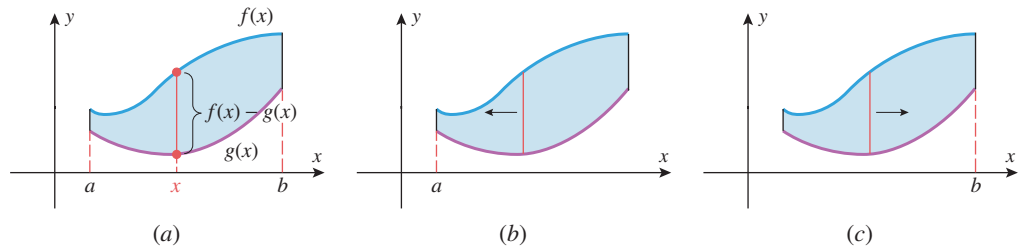
Since v_1 and v_2 are nonnegative functions on $[0, T]$, it follows from Formula (4) of Section 5.7 that the integral of v_1 over $[0, T]$ is the distance traveled by car 1 during the time interval $0 \leq t \leq T$, and the integral of v_2 over $[0, T]$ is the distance traveled by car 2 during the same time interval. Since $v_1(t) \leq v_2(t)$ on $[0, T]$, car 2 travels farther than car 1 does over the time interval $0 \leq t \leq T$, and the area A represents the distance by which car 2 is ahead of car 1 at time T . ◀

Some regions may require careful thought to determine the integrand and limits of integration in (1). Here is a systematic procedure that you can follow to set up this formula.

It is not necessary to make an extremely accurate sketch in Step 1; the only purpose of the sketch is to determine which curve is the upper boundary and which is the lower boundary.

Finding the Limits of Integration for the Area Between Two Curves

- Step 1.** Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x -axis, connecting the top and bottom boundaries (Figure 6.1.9a).
- Step 2.** The y -coordinate of the top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).
- Step 3.** To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$ (Figures 6.1.9b and 6.1.9c).



▲ Figure 6.1.9

There is a useful way of thinking about this procedure:

If you view the vertical line segment as the “cross section” of the region at the point x , then Formula (1) states that the area between the curves is obtained by integrating the length of the cross section over the interval $[a, b]$.

It is possible for the upper or lower boundary of a region to consist of two or more different curves, in which case it will be convenient to subdivide the region into smaller pieces in order to apply Formula (1). This is illustrated in the next example.

► **Example 4** Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

Solution. To determine the appropriate boundaries of the region, we need to know where the curves $x = y^2$ and $y = x - 2$ intersect. In Example 2 we found intersections by equating the expressions for y . Here it is easier to rewrite the latter equation as $x = y + 2$ and equate the expressions for x , namely,

$$x = y^2 \quad \text{and} \quad x = y + 2 \tag{3}$$

This yields

$$y^2 = y + 2 \quad \text{or} \quad y^2 - y - 2 = 0 \quad \text{or} \quad (y + 1)(y - 2) = 0$$

from which we obtain $y = -1, y = 2$. Substituting these values in either equation in (3) we see that the corresponding x -values are $x = 1$ and $x = 4$, respectively, so the points of intersection are $(1, -1)$ and $(4, 2)$ (Figure 6.1.10a).

To apply Formula (1), the equations of the boundaries must be written so that y is expressed explicitly as a function of x . The upper boundary can be written as $y = \sqrt{x}$ (rewrite $x = y^2$ as $y = \pm\sqrt{x}$ and choose the $+$ for the upper portion of the curve). The lower boundary consists of two parts:

$$y = -\sqrt{x} \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{and} \quad y = x - 2 \quad \text{for} \quad 1 \leq x \leq 4$$

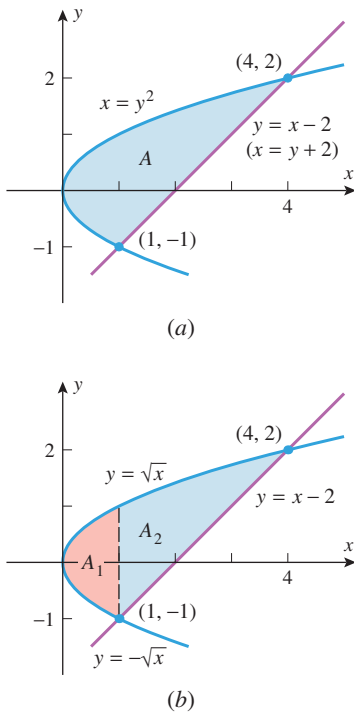
(Figure 6.1.10b). Because of this change in the formula for the lower boundary, it is necessary to divide the region into two parts and find the area of each part separately.

From (1) with $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x}$, $a = 0$, and $b = 1$, we obtain

$$A_1 = \int_0^1 [\sqrt{x} - (-\sqrt{x})] dx = 2 \int_0^1 \sqrt{x} dx = 2 \left[\frac{2}{3} x^{3/2} \right]_0^1 = \frac{4}{3} - 0 = \frac{4}{3}$$

From (1) with $f(x) = \sqrt{x}$, $g(x) = x - 2$, $a = 1$, and $b = 4$, we obtain

$$\begin{aligned} A_2 &= \int_1^4 [\sqrt{x} - (x - 2)] dx = \int_1^4 (\sqrt{x} - x + 2) dx \\ &= \left[\frac{2}{3} x^{3/2} - \frac{1}{2} x^2 + 2x \right]_1^4 = \left(\frac{16}{3} - 8 + 8 \right) - \left(\frac{2}{3} - \frac{1}{2} + 2 \right) = \frac{19}{6} \end{aligned}$$



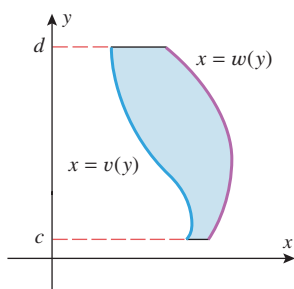
▲ Figure 6.1.10

Thus, the area of the entire region is

$$A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2} \blacktriangleleft$$

REVERSING THE ROLES OF x AND y

Sometimes it is much easier to find the area of a region by integrating with respect to y rather than x . We will now show how this can be done.



▲ Figure 6.1.11

6.1.3 SECOND AREA PROBLEM Suppose that w and v are continuous functions of y on an interval $[c, d]$ and that

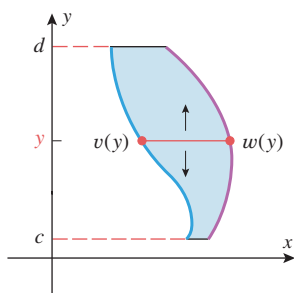
$$w(y) \geq v(y) \quad \text{for } c \leq y \leq d$$

[This means that the curve $x = w(y)$ lies to the right of the curve $x = v(y)$ and that the two can touch but not cross.] Find the area A of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, and above and below by the lines $y = d$ and $y = c$ (Figure 6.1.11).

Proceeding as in the derivation of (1), but with the roles of x and y reversed, leads to the following analog of 6.1.2.

6.1.4 AREA FORMULA If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)] dy \quad (4)$$



▲ Figure 6.1.12

The guiding principle in applying this formula is the same as with (1): The integrand in (4) can be viewed as the length of the horizontal cross section at an arbitrary point y on the y -axis, in which case Formula (4) states that the area can be obtained by integrating the length of the horizontal cross section over the interval $[c, d]$ on the y -axis (Figure 6.1.12).

In Example 4, we split the region into two parts to facilitate integrating with respect to x . In the next example we will see that splitting this region can be avoided if we integrate with respect to y .

► **Example 5** Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y .

Solution. As indicated in Figure 6.1.10 the left boundary is $x = y^2$, the right boundary is $y = x - 2$, and the region extends over the interval $-1 \leq y \leq 2$. However, to apply (4) the equations for the boundaries must be written so that x is expressed explicitly as a function of y . Thus, we rewrite $y = x - 2$ as $x = y + 2$. It now follows from (4) that

$$A = \int_{-1}^2 [(y + 2) - y^2] dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 = \frac{9}{2}$$

which agrees with the result obtained in Example 4. ◀

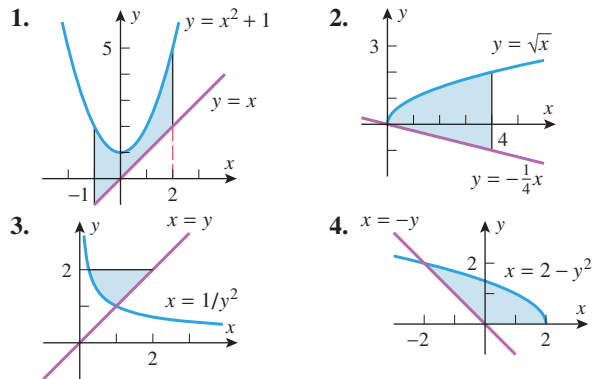
The choice between Formulas (1) and (4) is usually dictated by the shape of the region and which formula requires the least amount of splitting. However, sometimes one might choose the formula that requires more splitting because it is easier to evaluate the resulting integrals.

QUICK CHECK EXERCISES 6.1 (See page 421 for answers.)

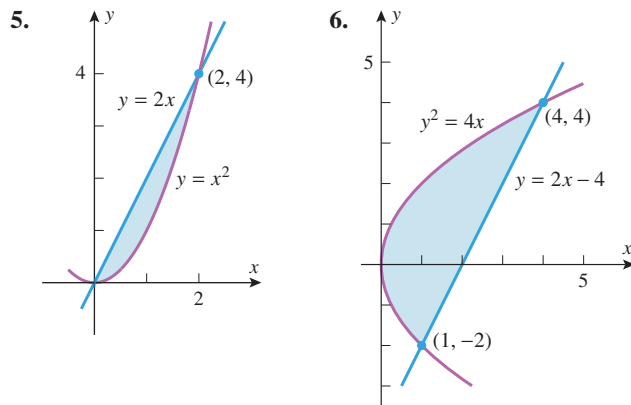
- An integral expression for the area of the region between the curves $y = 20 - 3x^2$ and $y = e^x$ and bounded on the sides by $x = 0$ and $x = 2$ is _____.
- An integral expression for the area of the parallelogram bounded by $y = 2x + 8$, $y = 2x - 3$, $x = -1$, and $x = 5$ is _____. The value of this integral is _____.
- (a) The points of intersection for the circle $x^2 + y^2 = 4$ and the line $y = x + 2$ are _____ and _____.
- (b) Expressed as a definite integral with respect to x , _____ gives the area of the region inside the circle $x^2 + y^2 = 4$ and above the line $y = x + 2$.
- (c) Expressed as a definite integral with respect to y , _____ gives the area of the region described in part (b).
- The area of the region enclosed by the curves $y = x^2$ and $y = \sqrt[3]{x}$ is _____.

EXERCISE SET 6.1  Graphing Utility  CAS

1–4 Find the area of the shaded region. ■




5–6 Find the area of the shaded region by (a) integrating with respect to x and (b) integrating with respect to y . ■



7–18 Sketch the region enclosed by the curves and find its area. ■

- $y = x^2$, $y = \sqrt{x}$, $x = \frac{1}{4}$, $x = 1$
- $y = x^3 - 4x$, $y = 0$, $x = 0$, $x = 2$
- $y = \cos 2x$, $y = 0$, $x = \pi/4$, $x = \pi/2$
- $y = \sec^2 x$, $y = 2$, $x = -\pi/4$, $x = \pi/4$
- $x = \sin y$, $x = 0$, $y = \pi/4$, $y = 3\pi/4$
- $x^2 = y$, $x = y - 2$
- $y = e^x$, $y = e^{2x}$, $x = 0$, $x = \ln 2$
- $x = 1/y$, $x = 0$, $y = 1$, $y = e$
- $y = \frac{2}{1+x^2}$, $y = |x|$
- $y = \frac{1}{\sqrt{1-x^2}}$, $y = 2$
- $y = 2 + |x - 1|$, $y = -\frac{1}{3}x + 7$
- $y = x$, $y = 4x$, $y = -x + 2$

 **19–26** Use a graphing utility, where helpful, to find the area of the region enclosed by the curves. ■

- $y = x^3 - 4x^2 + 3x$, $y = 0$
- $y = x^3 - 2x^2$, $y = 2x^2 - 3x$
- $y = \sin x$, $y = \cos x$, $x = 0$, $x = 2\pi$
- $y = x^3 - 4x$, $y = 0$
- $x = y^3 - y$, $x = 0$
- $x = y^3 - 4y^2 + 3y$, $x = y^2 - y$
- $y = xe^{x^2}$, $y = 2|x|$
- $y = \frac{1}{x\sqrt{1-(\ln x)^2}}$, $y = \frac{3}{x}$

27–30 True-False Determine whether the statement is true or false. Explain your answer. [In each exercise, assume that f and g are distinct continuous functions on $[a, b]$ and that A denotes the area of the region bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$.] ■

- If f and g differ by a positive constant c , then $A = c(b - a)$.
- If $\int_a^b [f(x) - g(x)] dx = -3$ then $A = 3$.
- If $\int_a^b [f(x) - g(x)] dx = 0$ then the graphs of $y = f(x)$ and $y = g(x)$ cross at least once on $[a, b]$.
- If $A = \left| \int_a^b [f(x) - g(x)] dx \right|$

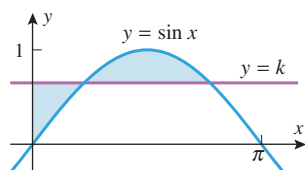
then the graphs of $y = f(x)$ and $y = g(x)$ don't cross on $[a, b]$.

31. Estimate the value of k ($0 < k < 1$) so that the region enclosed by $y = 1/\sqrt{1-x^2}$, $y = x$, $x = 0$, and $x = k$ has an area of 1 square unit.
32. Estimate the area of the region in the first quadrant enclosed by $y = \sin 2x$ and $y = \sin^{-1} x$.
33. Use a CAS to find the area enclosed by $y = 3 - 2x$ and $y = x^6 + 2x^5 - 3x^4 + x^2$.
34. Use a CAS to find the exact area enclosed by the curves $y = x^5 - 2x^3 - 3x$ and $y = x^3$.
35. Find a horizontal line $y = k$ that divides the area between $y = x^2$ and $y = 9$ into two equal parts.
36. Find a vertical line $x = k$ that divides the area enclosed by $x = \sqrt{y}$, $x = 2$, and $y = 0$ into two equal parts.
37. (a) Find the area of the region enclosed by the parabola $y = 2x - x^2$ and the x -axis.
 (b) Find the value of m so that the line $y = mx$ divides the region in part (a) into two regions of equal area.
38. Find the area between the curve $y = \sin x$ and the line segment joining the points $(0, 0)$ and $(5\pi/6, 1/2)$ on the curve.

39–43 Use Newton's Method (Section 4.7), where needed, to approximate the x -coordinates of the intersections of the curves to at least four decimal places, and then use those approximations to approximate the area of the region. ■

39. The region that lies below the curve $y = \sin x$ and above the line $y = 0.2x$, where $x \geq 0$.
40. The region enclosed by the graphs of $y = x^2$ and $y = \cos x$.
41. The region enclosed by the graphs of $y = (\ln x)/x$ and $y = x - 2$.
42. The region enclosed by the graphs of $y = 3 - 2\cos x$ and $y = 2/(1 + x^2)$.
43. The region enclosed by the graphs of $y = x^2 - 1$ and $y = 2\sin x$.
44. Referring to the accompanying figure, use a CAS to estimate the value of k so that the areas of the shaded regions are equal.

Source: This exercise is based on Problem A1 that was posed in the Fifty-Fourth Annual William Lowell Putnam Mathematical Competition.



◀ Figure Ex-44

FOCUS ON CONCEPTS

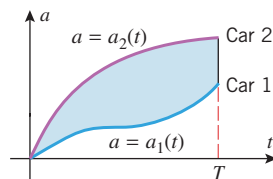
45. Two racers in adjacent lanes move with velocity functions $v_1(t)$ m/s and $v_2(t)$ m/s, respectively. Suppose that the racers are even at time $t = 60$ s. Interpret the

value of the integral

$$\int_0^{60} [v_2(t) - v_1(t)] dt$$

in this context.

46. The accompanying figure shows acceleration versus time curves for two cars that move along a straight track, accelerating from rest at the starting line. What does the area A between the curves over the interval $0 \leq t \leq T$ represent? Justify your answer.



◀ Figure Ex-46

47. Suppose that f and g are integrable on $[a, b]$, but neither $f(x) \geq g(x)$ nor $g(x) \geq f(x)$ holds for all x in $[a, b]$ [i.e., the curves $y = f(x)$ and $y = g(x)$ are intertwined].
- (a) What is the geometric significance of the integral

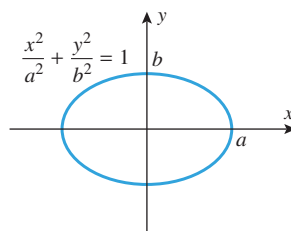
$$\int_a^b [f(x) - g(x)] dx?$$

- (b) What is the geometric significance of the integral

$$\int_a^b |f(x) - g(x)| dx?$$

48. Let $A(n)$ be the area in the first quadrant enclosed by the curves $y = \sqrt[n]{x}$ and $y = x$.
- (a) By considering how the graph of $y = \sqrt[n]{x}$ changes as n increases, make a conjecture about the limit of $A(n)$ as $n \rightarrow +\infty$.
- (b) Confirm your conjecture by calculating the limit.

49. Find the area of the region enclosed between the curve $x^{1/2} + y^{1/2} = a^{1/2}$ and the coordinate axes.
50. Show that the area of the ellipse in the accompanying figure is πab . [Hint: Use a formula from geometry.]



◀ Figure Ex-50

51. **Writing** Suppose that f and g are continuous on $[a, b]$ but that the graphs of $y = f(x)$ and $y = g(x)$ cross several times. Describe a step-by-step procedure for determining the area bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$.

52. **Writing** Suppose that R and S are two regions in the xy -plane that lie between a pair of lines L_1 and L_2 that are parallel to the y -axis. Assume that each line between L_1 and L_2 that is parallel to the y -axis intersects R and S in

line segments of equal length. Give an informal argument that the area of R is equal to the area of S . (Make reasonable assumptions about the boundaries of R and S .)

QUICK CHECK ANSWERS 6.1

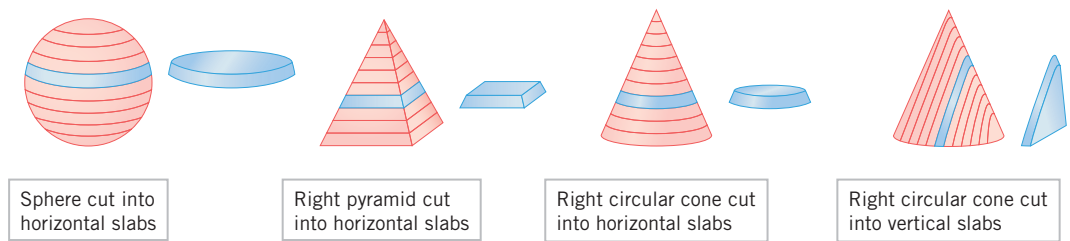
1. $\int_0^2 [(20 - 3x^2) - e^x] dx$ 2. $\int_{-1}^5 [(2x + 8) - (2x - 3)] dx$; 66 3. (a) $(-2, 0)$; $(0, 2)$ (b) $\int_{-2}^0 [\sqrt{4 - x^2} - (x + 2)] dx$
 (c) $\int_0^2 [(y - 2) + \sqrt{4 - y^2}] dy$ 4. $\frac{5}{12}$

6.2 VOLUMES BY SLICING; DISKS AND WASHERS

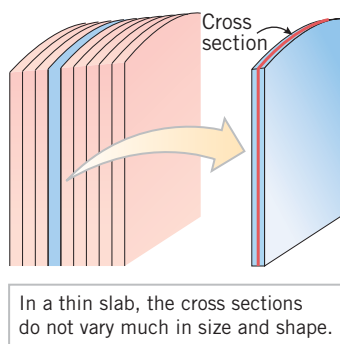
In the last section we showed that the area of a plane region bounded by two curves can be obtained by integrating the length of a general cross section over an appropriate interval. In this section we will see that the same basic principle can be used to find volumes of certain three-dimensional solids.

VOLUMES BY SLICING

Recall that the underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the area. Under appropriate conditions, the same strategy can be used to find the volume of a solid. The idea is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the volume (Figure 6.2.1).



▲ Figure 6.2.1



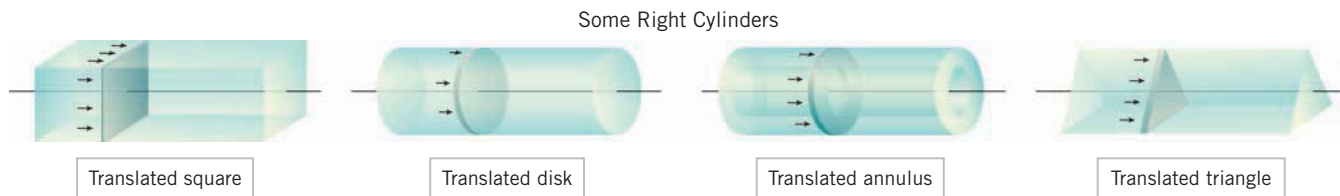
▲ Figure 6.2.2

What makes this method work is the fact that a *thin* slab has a cross section that does not vary much in size or shape, which, as we will see, makes its volume easy to approximate (Figure 6.2.2). Moreover, the thinner the slab, the less variation in its cross sections and the better the approximation. Thus, once we approximate the volumes of the slabs, we can set up a Riemann sum whose limit is the volume of the entire solid. We will give the details shortly, but first we need to discuss how to find the volume of a solid whose cross sections do not vary in size and shape (i.e., are congruent).

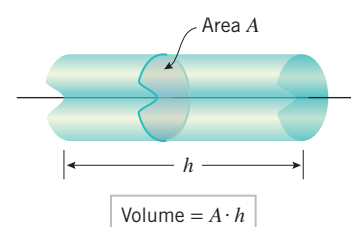
One of the simplest examples of a solid with congruent cross sections is a right circular cylinder of radius r , since all cross sections taken perpendicular to the central axis are circular regions of radius r . The volume V of a right circular cylinder of radius r and height h can be expressed in terms of the height and the area of a cross section as

$$V = \pi r^2 h = [\text{area of a cross section}] \times [\text{height}] \tag{1}$$

This is a special case of a more general volume formula that applies to solids called right cylinders. A **right cylinder** is a solid that is generated when a plane region is translated along a line or **axis** that is perpendicular to the region (Figure 6.2.3).



▲ Figure 6.2.3



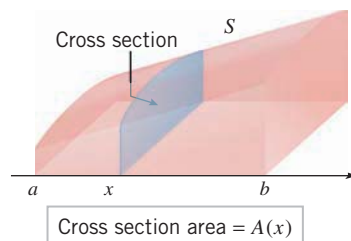
▲ Figure 6.2.4

If a right cylinder is generated by translating a region of area A through a distance h , then h is called the **height** (or sometimes the **width**) of the cylinder, and the volume V of the cylinder is defined to be

$$V = A \cdot h = [\text{area of a cross section}] \times [\text{height}] \quad (2)$$

(Figure 6.2.4). Note that this is consistent with Formula (1) for the volume of a right *circular* cylinder.

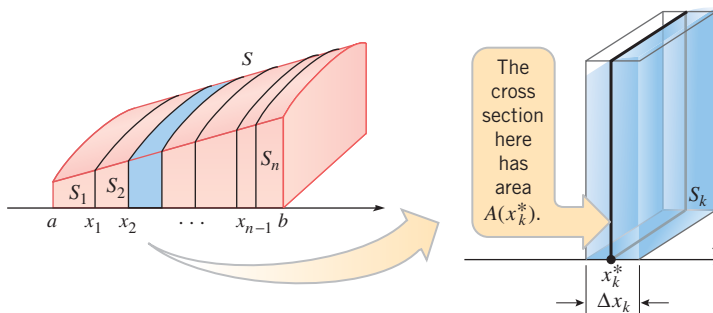
We now have all of the tools required to solve the following problem.



▲ Figure 6.2.5

6.2.1 PROBLEM Let S be a solid that extends along the x -axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the x -axis at $x = a$ and $x = b$ (Figure 6.2.5). Find the volume V of the solid, assuming that its cross-sectional area $A(x)$ is known at each x in the interval $[a, b]$.

To solve this problem we begin by dividing the interval $[a, b]$ into n subintervals, thereby dividing the solid into n slabs as shown in the left part of Figure 6.2.6. If we assume that the width of the k th subinterval is Δx_k , then the volume of the k th slab can be approximated by the volume $A(x_k^*)\Delta x_k$ of a right cylinder of width (height) Δx_k and cross-sectional area $A(x_k^*)$, where x_k^* is a point in the k th subinterval (see the right part of Figure 6.2.6).



► Figure 6.2.6

Adding these approximations yields the following Riemann sum that approximates the volume V :

$$V \approx \sum_{k=1}^n A(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n A(x_k^*) \Delta x_k = \int_a^b A(x) dx$$

In summary, we have the following result.

It is understood in our calculations of volume that the units of volume are the cubed units of length [e.g., cubic inches (in^3) or cubic meters (m^3)].

6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x -axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x -axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x) dx \quad (3)$$

provided $A(x)$ is integrable.

There is a similar result for cross sections perpendicular to the y -axis.

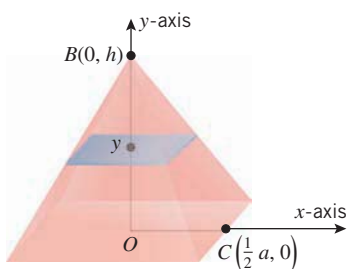
6.2.3 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the y -axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y -axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y) dy \quad (4)$$

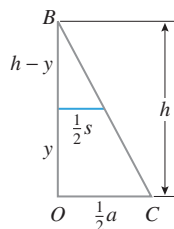
provided $A(y)$ is integrable.

In words, these formulas state:

The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.



(a)



(b)

▲ Figure 6.2.7

► **Example 1** Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a .

Solution. As illustrated in Figure 6.2.7a, we introduce a rectangular coordinate system in which the y -axis passes through the apex and is perpendicular to the base, and the x -axis passes through the base and is parallel to a side of the base.

At any y in the interval $[0, h]$ on the y -axis, the cross section perpendicular to the y -axis is a square. If s denotes the length of a side of this square, then by similar triangles (Figure 6.2.7b)

$$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h} \quad \text{or} \quad s = \frac{a}{h}(h-y)$$

Thus, the area $A(y)$ of the cross section at y is

$$A(y) = s^2 = \frac{a^2}{h^2}(h-y)^2$$

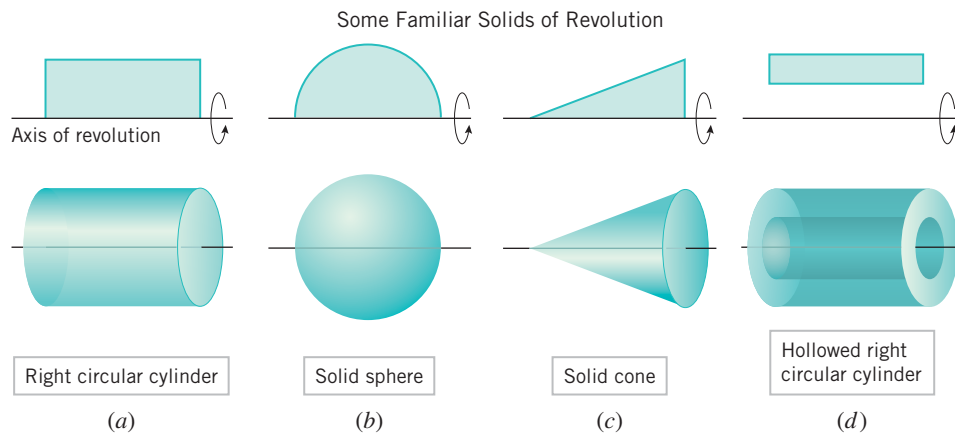
and by (4) the volume is

$$\begin{aligned}
 V &= \int_0^h A(y) dy = \int_0^h \frac{a^2}{h^2} (h - y)^2 dy = \frac{a^2}{h^2} \int_0^h (h - y)^2 dy \\
 &= \frac{a^2}{h^2} \left[-\frac{1}{3} (h - y)^3 \right]_{y=0}^h = \frac{a^2}{h^2} \left[0 + \frac{1}{3} h^3 \right] = \frac{1}{3} a^2 h
 \end{aligned}$$

That is, the volume is $\frac{1}{3}$ of the area of the base times the altitude. ◀

■ SOLIDS OF REVOLUTION

A *solid of revolution* is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the *axis of revolution*. Many familiar solids are of this type (Figure 6.2.8).

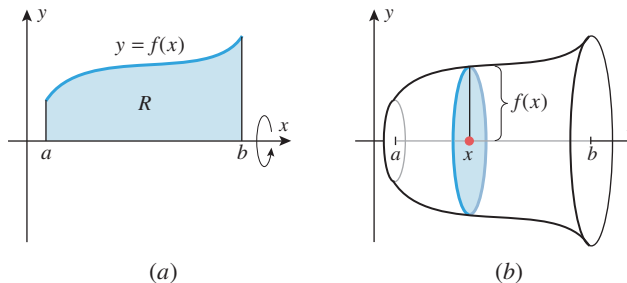


► Figure 6.2.8

■ VOLUMES BY DISKS PERPENDICULAR TO THE x-AXIS

We will be interested in the following general problem.

6.2.4 PROBLEM Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x -axis, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region R about the x -axis.



► Figure 6.2.9

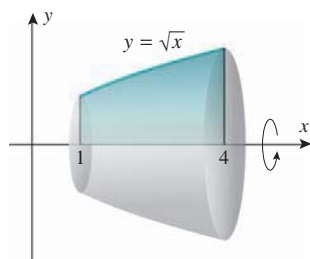
We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x -axis at the point x is a circular disk of radius $f(x)$ (Figure 6.2.9b). The area of this region is

$$A(x) = \pi[f(x)]^2$$

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2 dx \quad (5)$$

Because the cross sections are disk shaped, the application of this formula is called the *method of disks*.

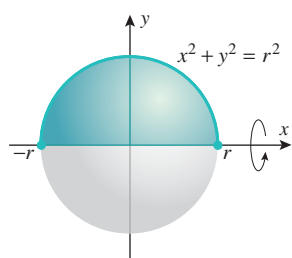


▲ Figure 6.2.10

► **Example 2** Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x -axis (Figure 6.2.10).

Solution. From (5), the volume is

$$V = \int_a^b \pi[f(x)]^2 dx = \int_1^4 \pi x dx = \left. \frac{\pi x^2}{2} \right|_1^4 = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2} \blacktriangleleft$$



▲ Figure 6.2.11

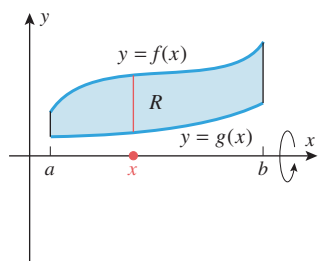
► **Example 3** Derive the formula for the volume of a sphere of radius r .

Solution. As indicated in Figure 6.2.11, a sphere of radius r can be generated by revolving the upper semicircular disk enclosed between the x -axis and

$$x^2 + y^2 = r^2$$

about the x -axis. Since the upper half of this circle is the graph of $y = f(x) = \sqrt{r^2 - x^2}$, it follows from (5) that the volume of the sphere is

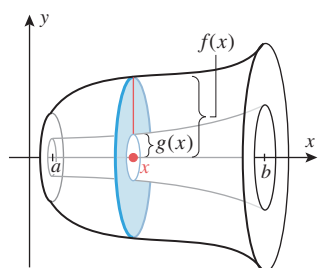
$$V = \int_a^b \pi[f(x)]^2 dx = \int_{-r}^r \pi(r^2 - x^2) dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r = \frac{4}{3} \pi r^3 \blacktriangleleft$$



(a)

■ VOLUMES BY WASHERS PERPENDICULAR TO THE x -AXIS

Not all solids of revolution have solid interiors; some have holes or channels that create interior surfaces, as in Figure 6.2.8d. So we will also be interested in problems of the following type.



(b)

6.2.5 PROBLEM Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.12a). Find the volume of the solid of revolution that is generated by revolving the region R about the x -axis (Figure 6.2.12b).

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x -axis at the point x is the annular or “washer-shaped”

▲ Figure 6.2.12

region with inner radius $g(x)$ and outer radius $f(x)$ (Figure 6.2.12b); its area is

$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi([f(x)]^2 - [g(x)]^2)$$

Thus, from (3) the volume of the solid is

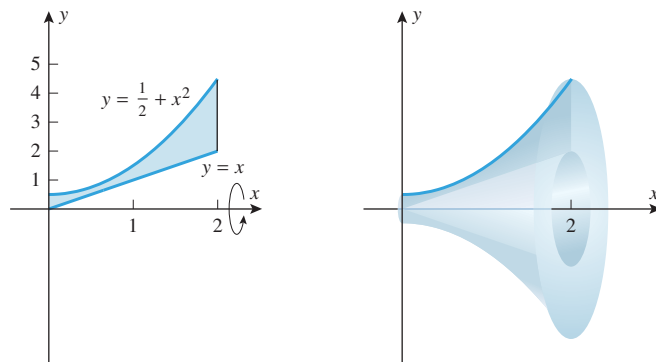
$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2) dx \tag{6}$$

Because the cross sections are washer shaped, the application of this formula is called the *method of washers*.

► **Example 4** Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x -axis.

Solution. First sketch the region (Figure 6.2.13a); then imagine revolving it about the x -axis (Figure 6.2.13b). From (6) the volume is

$$\begin{aligned} V &= \int_a^b \pi([f(x)]^2 - [g(x)]^2) dx = \int_0^2 \pi \left(\left[\frac{1}{2} + x^2 \right]^2 - x^2 \right) dx \\ &= \int_0^2 \pi \left(\frac{1}{4} + x^4 \right) dx = \pi \left[\frac{x}{4} + \frac{x^5}{5} \right]_0^2 = \frac{69\pi}{10} \blacktriangleleft \end{aligned}$$



Unequal scales on axes

Region defined by f and g

The resulting solid of revolution

► Figure 6.2.13

(a)

(b)

■ **VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE y -AXIS**

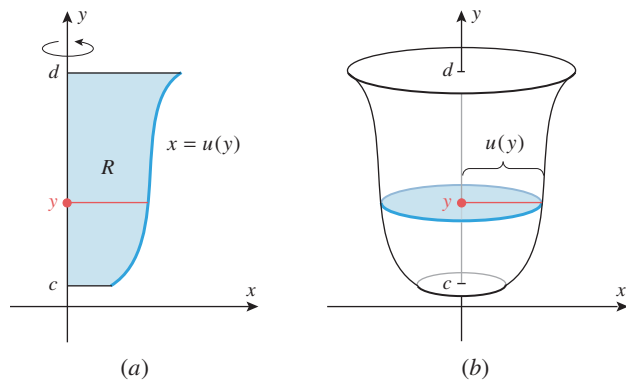
The methods of disks and washers have analogs for regions that are revolved about the y -axis (Figures 6.2.14 and 6.2.15). Using the method of slicing and Formula (4), you should be able to deduce the following formulas for the volumes of the solids in the figures.

$$V = \int_c^d \pi[u(y)]^2 dy$$

Disks

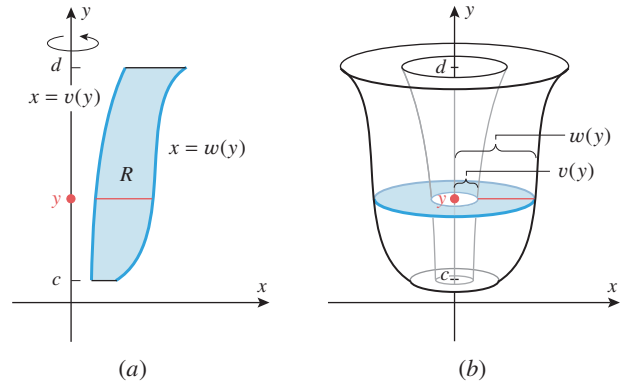
$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2) dy \tag{7-8}$$

Washers



Disks

▲ Figure 6.2.14



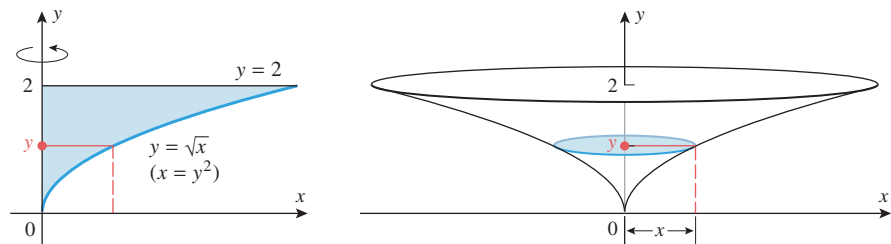
Washers

▲ Figure 6.2.15

► **Example 5** Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y -axis.

Solution. First sketch the region and the solid (Figure 6.2.16). The cross sections taken perpendicular to the y -axis are disks, so we will apply (7). But first we must rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, from (7) with $u(y) = y^2$, the volume is

$$V = \int_c^d \pi[u(y)]^2 dy = \int_0^2 \pi y^4 dy = \left. \frac{\pi y^5}{5} \right]_0^2 = \frac{32\pi}{5} \quad \blacktriangleleft$$



► Figure 6.2.16

■ OTHER AXES OF REVOLUTION

It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. Instead of developing a new formula for each situation, we will appeal to Formulas (3) and (4) and integrate an appropriate cross-sectional area to find the volume.

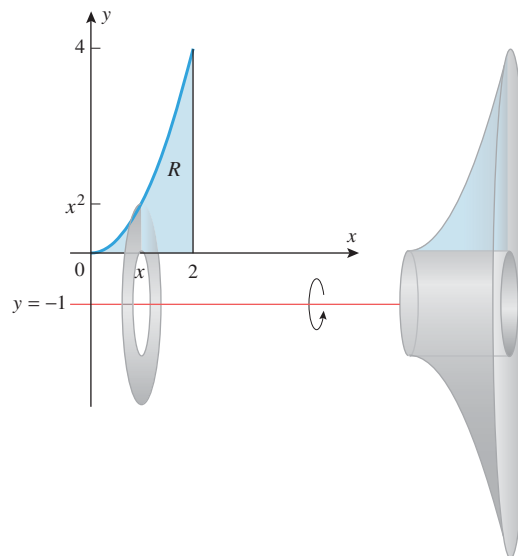
► **Example 6** Find the volume of the solid generated when the region under the curve $y = x^2$ over the interval $[0, 2]$ is rotated about the line $y = -1$.

Solution. First sketch the region and the axis of revolution; then imagine revolving the region about the axis (Figure 6.2.17). At each x in the interval $0 \leq x \leq 2$, the cross section of the solid perpendicular to the axis $y = -1$ is a washer with outer radius $x^2 + 1$ and inner radius 1. Since the area of this washer is

$$A(x) = \pi[(x^2 + 1)^2 - 1^2] = \pi(x^4 + 2x^2)$$

it follows by (3) that the volume of the solid is

$$V = \int_0^2 A(x) dx = \int_0^2 \pi (x^4 + 2x^2) dx = \pi \left[\frac{1}{5}x^5 + \frac{2}{3}x^3 \right]_0^2 = \frac{176\pi}{15} \blacktriangleleft$$



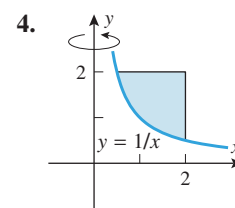
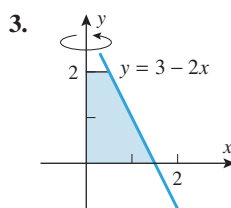
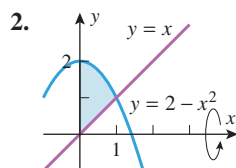
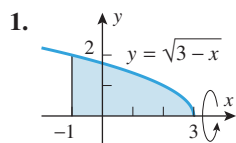
► Figure 6.2.17

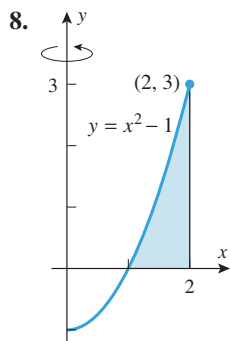
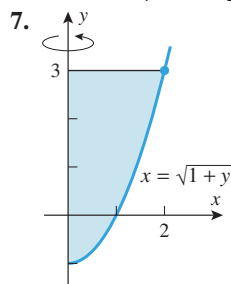
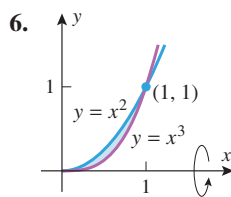
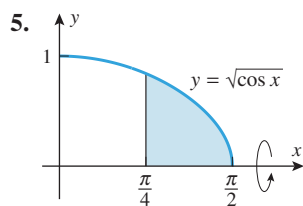
✓ QUICK CHECK EXERCISES 6.2 (See page 431 for answers.)

- A solid S extends along the x -axis from $x = 1$ to $x = 3$. For x between 1 and 3, the cross-sectional area of S perpendicular to the x -axis is $3x^2$. An integral expression for the volume of S is _____. The value of this integral is _____.
- A solid S is generated by revolving the region between the x -axis and the curve $y = \sqrt{\sin x}$ ($0 \leq x \leq \pi$) about the x -axis.
 - For x between 0 and π , the cross-sectional area of S perpendicular to the x -axis at x is $A(x) =$ _____.
 - An integral expression for the volume of S is _____.
 - The value of the integral in part (b) is _____.
- A solid S is generated by revolving the region enclosed by the line $y = 2x + 1$ and the curve $y = x^2 + 1$ about the x -axis.
 - For x between _____ and _____, the cross-sectional area of S perpendicular to the x -axis at x is $A(x) =$ _____.
 - An integral expression for the volume of S is _____.
- A solid S is generated by revolving the region enclosed by the line $y = x + 1$ and the curve $y = x^2 + 1$ about the y -axis.
 - For y between _____ and _____, the cross-sectional area of S perpendicular to the y -axis at y is $A(y) =$ _____.
 - An integral expression for the volume of S is _____.

EXERCISE SET 6.2 C CAS

1–8 Find the volume of the solid that results when the shaded region is revolved about the indicated axis. ■





9. Find the volume of the solid whose base is the region bounded between the curve $y = x^2$ and the x -axis from $x = 0$ to $x = 2$ and whose cross sections taken perpendicular to the x -axis are squares.
10. Find the volume of the solid whose base is the region bounded between the curve $y = \sec x$ and the x -axis from $x = \pi/4$ to $x = \pi/3$ and whose cross sections taken perpendicular to the x -axis are squares.

11–18 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x -axis. ■

11. $y = \sqrt{25 - x^2}$, $y = 3$
12. $y = 9 - x^2$, $y = 0$ 13. $x = \sqrt{y}$, $x = y/4$
14. $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$
[Hint: Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.]
15. $y = e^x$, $y = 0$, $x = 0$, $x = \ln 3$
16. $y = e^{-2x}$, $y = 0$, $x = 0$, $x = 1$
17. $y = \frac{1}{\sqrt{4 + x^2}}$, $x = -2$, $x = 2$, $y = 0$
18. $y = \frac{e^{3x}}{\sqrt{1 + e^{6x}}}$, $x = 0$, $x = 1$, $y = 0$
19. Find the volume of the solid whose base is the region bounded between the curve $y = x^3$ and the y -axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y -axis are squares.
20. Find the volume of the solid whose base is the region enclosed between the curve $x = 1 - y^2$ and the y -axis and whose cross sections taken perpendicular to the y -axis are squares.

21–26 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the y -axis. ■

21. $x = \csc y$, $y = \pi/4$, $y = 3\pi/4$, $x = 0$
22. $y = x^2$, $x = y^2$

23. $x = y^2$, $x = y + 2$
24. $x = 1 - y^2$, $x = 2 + y^2$, $y = -1$, $y = 1$
25. $y = \ln x$, $x = 0$, $y = 0$, $y = 1$
26. $y = \sqrt{\frac{1 - x^2}{x^2}}$ ($x > 0$), $x = 0$, $y = 0$, $y = 2$

27–30 True–False Determine whether the statement is true or false. Explain your answer. [In these exercises, assume that a solid S of volume V is bounded by two parallel planes perpendicular to the x -axis at $x = a$ and $x = b$ and that for each x in $[a, b]$, $A(x)$ denotes the cross-sectional area of S perpendicular to the x -axis.] ■

27. If each cross section of S perpendicular to the x -axis is a square, then S is a rectangular parallelepiped (i.e., is box shaped).
28. If each cross section of S is a disk or a washer, then S is a solid of revolution.
29. If x is in centimeters (cm), then $A(x)$ must be a quadratic function of x , since units of $A(x)$ will be square centimeters (cm^2).
30. The average value of $A(x)$ on the interval $[a, b]$ is given by $V/(b - a)$.
31. Find the volume of the solid that results when the region above the x -axis and below the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$$

is revolved about the x -axis.

32. Let V be the volume of the solid that results when the region enclosed by $y = 1/x$, $y = 0$, $x = 2$, and $x = b$ ($0 < b < 2$) is revolved about the x -axis. Find the value of b for which $V = 3$.
33. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x + 1}$, $y = \sqrt{2x}$, and $y = 0$ is revolved about the x -axis. [Hint: Split the solid into two parts.]
34. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$ is revolved about the x -axis. [Hint: Split the solid into two parts.]

FOCUS ON CONCEPTS

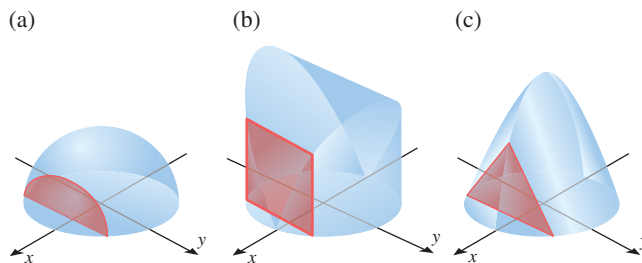
35. Suppose that f is a continuous function on $[a, b]$, and let R be the region between the curve $y = f(x)$ and the line $y = k$ from $x = a$ to $x = b$. Using the method of disks, derive with explanation a formula for the volume of a solid generated by revolving R about the line $y = k$. State and explain additional assumptions, if any, that you need about f for your formula.
36. Suppose that v and w are continuous functions on $[c, d]$, and let R be the region between the curves $x = v(y)$ and $x = w(y)$ from $y = c$ to $y = d$. Using the method of washers, derive with explanation a formula for the volume of a solid generated by revolving R about the line

$x = k$. State and explain additional assumptions, if any, that you need about v and w for your formula.

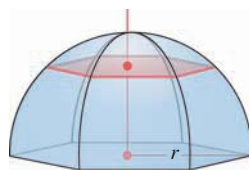
37. Consider the solid generated by revolving the shaded region in Exercise 1 about the line $y = 2$.
- Make a conjecture as to which is larger: the volume of this solid or the volume of the solid in Exercise 1. Explain the basis of your conjecture.
 - Check your conjecture by calculating this volume and comparing it to the volume obtained in Exercise 1.
38. Consider the solid generated by revolving the shaded region in Exercise 4 about the line $x = 2.5$.
- Make a conjecture as to which is larger: the volume of this solid or the volume of the solid in Exercise 4. Explain the basis of your conjecture.
 - Check your conjecture by calculating this volume and comparing it to the volume obtained in Exercise 4.

39. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 9$ is revolved about the line $x = 9$.
40. Find the volume of the solid that results when the region in Exercise 39 is revolved about the line $y = 3$.
41. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $y = -1$.
42. Find the volume of the solid that results when the region in Exercise 41 is revolved about the line $x = -1$.
43. Find the volume of the solid that results when the region enclosed by $y = x^2$ and $y = x^3$ is revolved about the line $x = 1$.
44. Find the volume of the solid that results when the region in Exercise 43 is revolved about the line $y = -1$.
45. A nose cone for a space reentry vehicle is designed so that a cross section, taken x ft from the tip and perpendicular to the axis of symmetry, is a circle of radius $\frac{1}{4}x^2$ ft. Find the volume of the nose cone given that its length is 20 ft.
46. A certain solid is 1 ft high, and a horizontal cross section taken x ft above the bottom of the solid is an annulus of inner radius x^2 ft and outer radius \sqrt{x} ft. Find the volume of the solid.
47. Find the volume of the solid whose base is the region bounded between the curves $y = x$ and $y = x^2$, and whose cross sections perpendicular to the x -axis are squares.
48. The base of a certain solid is the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 4$. Every cross section perpendicular to the x -axis is a semicircle with its diameter across the base. Find the volume of the solid.

49. In parts (a)–(c) find the volume of the solid whose base is enclosed by the circle $x^2 + y^2 = 1$ and whose cross sections taken perpendicular to the x -axis are
- semicircles
 - squares
 - equilateral triangles.



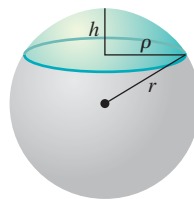
50. As shown in the accompanying figure, a cathedral dome is designed with three semicircular supports of radius r so that each horizontal cross section is a regular hexagon. Show that the volume of the dome is $r^3\sqrt{3}$.



◀ Figure Ex-50

- ◻ 51–54 Use a CAS to estimate the volume of the solid that results when the region enclosed by the curves is revolved about the stated axis. ■

51. $y = \sin^8 x$, $y = 2x/\pi$, $x = 0$, $x = \pi/2$; x -axis
52. $y = \pi^2 \sin x \cos^3 x$, $y = 4x^2$, $x = 0$, $x = \pi/4$; x -axis
53. $y = e^x$, $x = 1$, $y = 1$; y -axis
54. $y = x\sqrt{\tan^{-1} x}$, $y = x$; x -axis
55. The accompanying figure shows a *spherical cap* of radius ρ and height h cut from a sphere of radius r . Show that the volume V of the spherical cap can be expressed as
- $V = \frac{1}{3}\pi h^2(3r - h)$
 - $V = \frac{1}{6}\pi h(3\rho^2 + h^2)$.

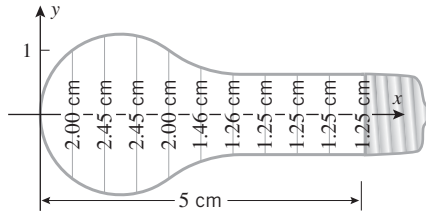


◀ Figure Ex-55

56. If fluid enters a hemispherical bowl with a radius of 10 ft at a rate of $\frac{1}{2}$ ft³/min, how fast will the fluid be rising when the depth is 5 ft? [Hint: See Exercise 55.]
57. The accompanying figure (on the next page) shows the dimensions of a small lightbulb at 10 equally spaced points.
- Use formulas from geometry to make a rough estimate of the volume enclosed by the glass portion of the bulb.

(cont.)

(b) Use the average of left and right endpoint approximations to approximate the volume.



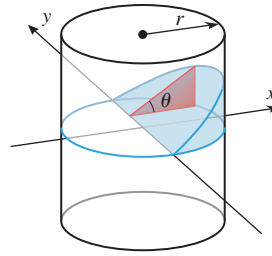
▲ Figure Ex-57

58. Use the result in Exercise 55 to find the volume of the solid that remains when a hole of radius $r/2$ is drilled through the center of a sphere of radius r , and then check your answer by integrating.
59. As shown in the accompanying figure, a cocktail glass with a bowl shaped like a hemisphere of diameter 8 cm contains a cherry with a diameter of 2 cm. If the glass is filled to a depth of h cm, what is the volume of liquid it contains? [Hint: First consider the case where the cherry is partially submerged, then the case where it is totally submerged.]



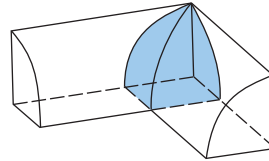
◀ Figure Ex-59

60. Find the volume of the torus that results when the region enclosed by the circle of radius r with center at $(h, 0)$, $h > r$, is revolved about the y -axis. [Hint: Use an appropriate formula from plane geometry to help evaluate the definite integral.]
61. A wedge is cut from a right circular cylinder of radius r by two planes, one perpendicular to the axis of the cylinder and the other making an angle θ with the first. Find the volume of the wedge by slicing perpendicular to the y -axis as shown in the accompanying figure.

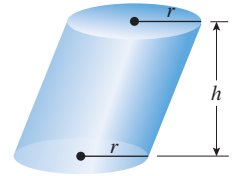


◀ Figure Ex-61

62. Find the volume of the wedge described in Exercise 61 by slicing perpendicular to the x -axis.
63. Two right circular cylinders of radius r have axes that intersect at right angles. Find the volume of the solid common to the two cylinders. [Hint: One-eighth of the solid is sketched in the accompanying figure.]
64. In 1635 Bonaventura Cavalieri, a student of Galileo, stated the following result, called **Cavalieri's principle**: *If two solids have the same height, and if the areas of their cross sections taken parallel to and at equal distances from their bases are always equal, then the solids have the same volume.* Use this result to find the volume of the oblique cylinder in the accompanying figure. (See Exercise 52 of Section 6.1 for a planar version of Cavalieri's principle.)



▲ Figure Ex-63



▲ Figure Ex-64

65. **Writing** Use the results of this section to derive Cavalieri's principle (Exercise 64).
66. **Writing** Write a short paragraph that explains how Formulas (4)–(8) may all be viewed as consequences of Formula (3).

✓ QUICK CHECK ANSWERS 6.2

1. $\int_1^3 3x^2 dx$; 26 2. (a) $\pi \sin x$ (b) $\int_0^\pi \pi \sin x dx$ (c) 2π 3. (a) 0; 2; $\pi[(2x+1)^2 - (x^2+1)^2] = \pi[-x^4 + 2x^2 + 4x]$
 (b) $\int_0^2 \pi[-x^4 + 2x^2 + 4x] dx$ 4. (a) 1; 2; $\pi[(y-1) - (y-1)^2] = \pi[-y^2 + 3y - 2]$ (b) $\int_1^2 \pi[-y^2 + 3y - 2] dy$

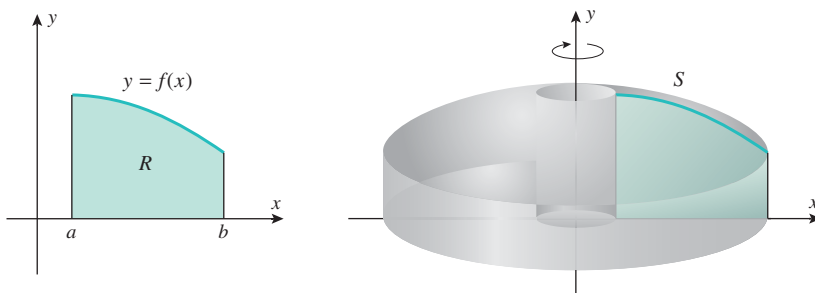
6.3 VOLUMES BY CYLINDRICAL SHELLS

The methods for computing volumes that have been discussed so far depend on our ability to compute the cross-sectional area of the solid and to integrate that area across the solid. In this section we will develop another method for finding volumes that may be applicable when the cross-sectional area cannot be found or the integration is too difficult.

■ CYLINDRICAL SHELLS

In this section we will be interested in the following problem.

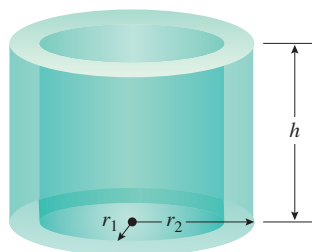
6.3.1 PROBLEM Let f be continuous and nonnegative on $[a, b]$ ($0 \leq a < b$), and let R be the region that is bounded above by $y = f(x)$, below by the x -axis, and on the sides by the lines $x = a$ and $x = b$. Find the volume V of the solid of revolution S that is generated by revolving the region R about the y -axis (Figure 6.3.1).



► Figure 6.3.1

Sometimes problems of the above type can be solved by the method of disks or washers perpendicular to the y -axis, but when that method is not applicable or the resulting integral is difficult, the *method of cylindrical shells*, which we will discuss here, will often work.

A **cylindrical shell** is a solid enclosed by two concentric right circular cylinders (Figure 6.3.2). The volume V of a cylindrical shell with inner radius r_1 , outer radius r_2 , and height h can be written as



▲ Figure 6.3.2

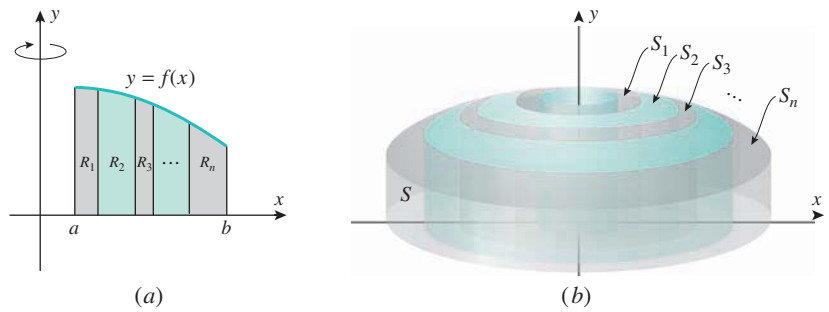
$$\begin{aligned} V &= [\text{area of cross section}] \cdot [\text{height}] \\ &= (\pi r_2^2 - \pi r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \cdot \left[\frac{1}{2}(r_1 + r_2)\right] \cdot h \cdot (r_2 - r_1) \end{aligned}$$

But $\frac{1}{2}(r_1 + r_2)$ is the average radius of the shell and $r_2 - r_1$ is its thickness, so

$$V = 2\pi \cdot [\text{average radius}] \cdot [\text{height}] \cdot [\text{thickness}] \quad (1)$$

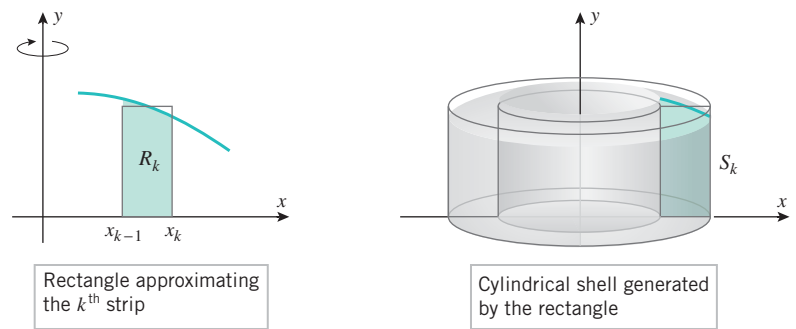
We will now show how this formula can be used to solve Problem 6.3.1. The underlying idea is to divide the interval $[a, b]$ into n subintervals, thereby subdividing the region R into n strips, R_1, R_2, \dots, R_n (Figure 6.3.3a). When the region R is revolved about the y -axis, these strips generate “tube-like” solids S_1, S_2, \dots, S_n that are nested one inside the other and together comprise the entire solid S (Figure 6.3.3b). Thus, the volume V of the solid can be obtained by adding together the volumes of the tubes; that is,

$$V = V(S_1) + V(S_2) + \cdots + V(S_n)$$



► Figure 6.3.3 (a) (b)

As a rule, the tubes will have curved upper surfaces, so there will be no simple formulas for their volumes. However, if the strips are thin, then we can approximate each strip by a rectangle (Figure 6.3.4a). These rectangles, when revolved about the y-axis, will produce cylindrical shells whose volumes closely approximate the volumes of the tubes generated by the original strips (Figure 6.3.4b). We will show that by adding the volumes of the cylindrical shells we can obtain a Riemann sum that approximates the volume V , and by taking the limit of the Riemann sums we can obtain an integral for the exact volume V .



► Figure 6.3.4 (a) (b)

To implement this idea, suppose that the k th strip extends from x_{k-1} to x_k and that the width of this strip is

$$\Delta x_k = x_k - x_{k-1}$$

If we let x_k^* be the *midpoint* of the interval $[x_{k-1}, x_k]$, and if we construct a rectangle of height $f(x_k^*)$ over the interval, then revolving this rectangle about the y-axis produces a cylindrical shell of average radius x_k^* , height $f(x_k^*)$, and thickness Δx_k (Figure 6.3.5). From (1), the volume V_k of this cylindrical shell is

$$V_k = 2\pi x_k^* f(x_k^*) \Delta x_k$$

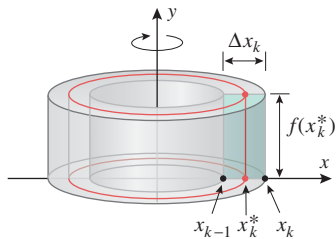
Adding the volumes of the n cylindrical shells yields the following Riemann sum that approximates the volume V :

$$V \approx \sum_{k=1}^n 2\pi x_k^* f(x_k^*) \Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n 2\pi x_k^* f(x_k^*) \Delta x_k = \int_a^b 2\pi x f(x) dx$$

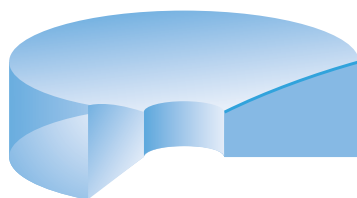
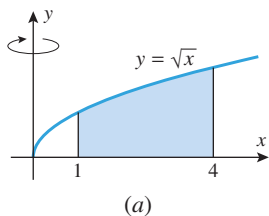
In summary, we have the following result.



▲ Figure 6.3.5

6.3.2 VOLUME BY CYLINDRICAL SHELLS ABOUT THE y-AXIS Let f be continuous and nonnegative on $[a, b]$ ($0 \leq a < b$), and let R be the region that is bounded above by $y = f(x)$, below by the x -axis, and on the sides by the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution that is generated by revolving the region R about the y -axis is given by

$$V = \int_a^b 2\pi x f(x) dx \quad (2)$$



Cutaway view of the solid
(b)

▲ Figure 6.3.6

► **Example 1** Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x -axis is revolved about the y -axis.

Solution. First sketch the region (Figure 6.3.6a); then imagine revolving it about the y -axis (Figure 6.3.6b). Since $f(x) = \sqrt{x}$, $a = 1$, and $b = 4$, Formula (2) yields

$$V = \int_1^4 2\pi x \sqrt{x} dx = 2\pi \int_1^4 x^{3/2} dx = \left[2\pi \cdot \frac{2}{5} x^{5/2} \right]_1^4 = \frac{4\pi}{5} [32 - 1] = \frac{124\pi}{5} \blacktriangleleft$$

■ **VARIATIONS OF THE METHOD OF CYLINDRICAL SHELLS**

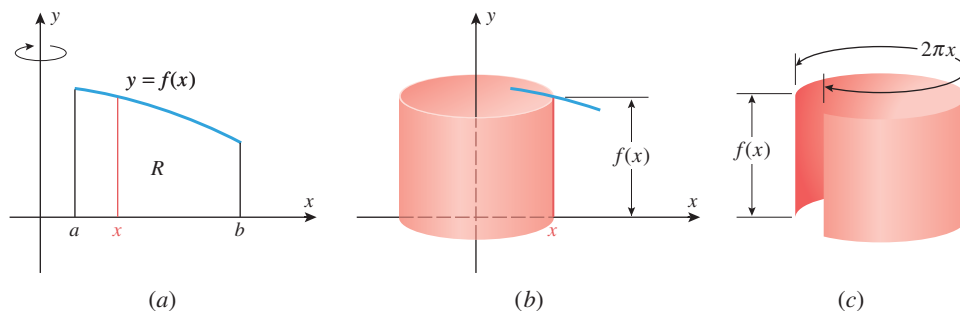
The method of cylindrical shells is applicable in a variety of situations that do not fit the conditions required by Formula (2). For example, the region may be enclosed between two curves, or the axis of revolution may be some line other than the y -axis. However, rather than develop a separate formula for every possible situation, we will give a general way of thinking about the method of cylindrical shells that can be adapted to each new situation as it arises.

For this purpose, we will need to reexamine the integrand in Formula (2): At each x in the interval $[a, b]$, the vertical line segment from the x -axis to the curve $y = f(x)$ can be viewed as the cross section of the region R at x (Figure 6.3.7a). When the region R is revolved about the y -axis, the cross section at x sweeps out the *surface* of a right circular cylinder of height $f(x)$ and radius x (Figure 6.3.7b). The area of this surface is

$$2\pi x f(x)$$

(Figure 6.3.7c), which is the integrand in (2). Thus, Formula (2) can be viewed informally in the following way.

6.3.3 AN INFORMAL VIEWPOINT ABOUT CYLINDRICAL SHELLS The volume V of a solid of revolution that is generated by revolving a region R about an axis can be obtained by integrating the area of the surface generated by an arbitrary cross section of R taken parallel to the axis of revolution.



► Figure 6.3.7

The following examples illustrate how to apply this result in situations where Formula (2) is not applicable.

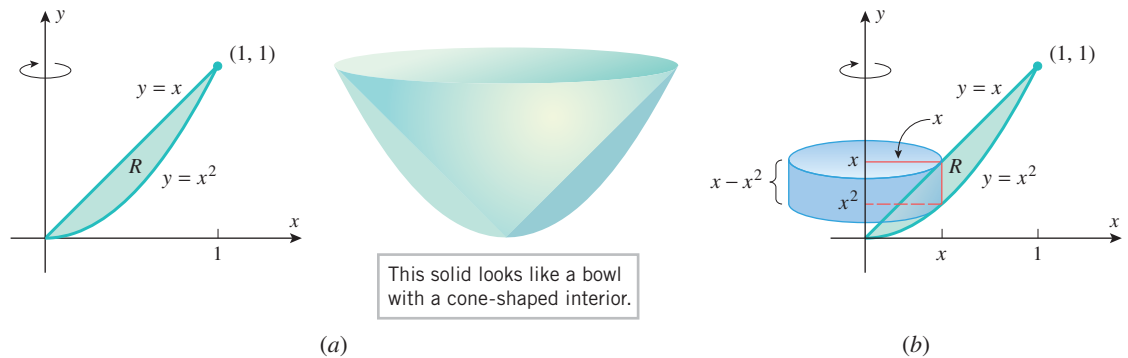
► **Example 2** Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed between $y = x$ and $y = x^2$ is revolved about the y -axis (Figure 6.3.8a).

Solution. As illustrated in part (b) of Figure 6.3.8, at each x in $[0, 1]$ the cross section of R parallel to the y -axis generates a cylindrical surface of height $x - x^2$ and radius x . Since the area of this surface is

$$2\pi x(x - x^2)$$

the volume of the solid is

$$\begin{aligned} V &= \int_0^1 2\pi x(x - x^2) dx = 2\pi \int_0^1 (x^2 - x^3) dx \\ &= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = 2\pi \left[\frac{1}{3} - \frac{1}{4} \right] = \frac{\pi}{6} \quad \blacktriangleleft \end{aligned}$$



▲ Figure 6.3.8

► **Example 3** Use cylindrical shells to find the volume of the solid generated when the region R under $y = x^2$ over the interval $[0, 2]$ is revolved about the line $y = -1$.

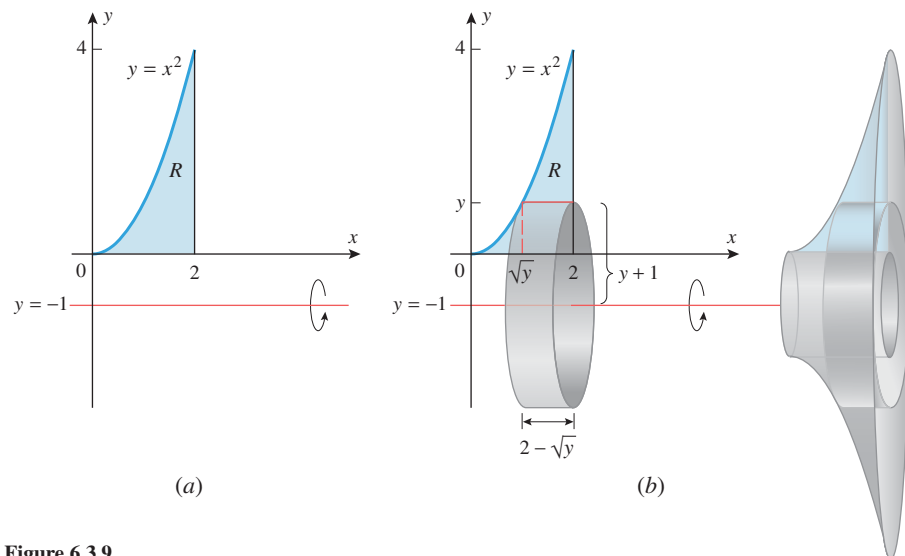
Solution. First draw the axis of revolution; then imagine revolving the region about the axis (Figure 6.3.9a). As illustrated in Figure 6.3.9b, at each y in the interval $0 \leq y \leq 4$, the cross section of R parallel to the x -axis generates a cylindrical surface of height $2 - \sqrt{y}$ and radius $y + 1$. Since the area of this surface is

$$2\pi(y + 1)(2 - \sqrt{y})$$

it follows that the volume of the solid is

$$\begin{aligned} \int_0^4 2\pi(y + 1)(2 - \sqrt{y}) dy &= 2\pi \int_0^4 (2y - y^{3/2} + 2 - y^{1/2}) dy \\ &= 2\pi \left[y^2 - \frac{2}{5}y^{5/2} + 2y - \frac{2}{3}y^{3/2} \right]_0^4 = \frac{176\pi}{15} \quad \blacktriangleleft \end{aligned}$$

Note that the volume found in Example 3 agrees with the volume of the same solid found by the method of washers in Example 6 of Section 6.2. Confirm that the volume in Example 2 found by the method of cylindrical shells can also be obtained by the method of washers.



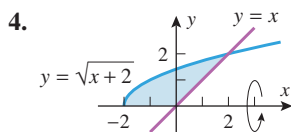
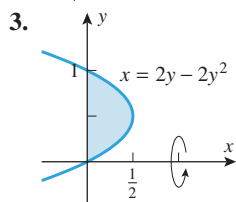
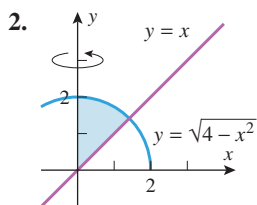
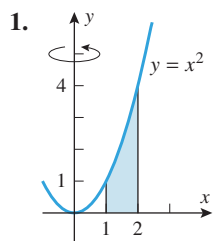
▶ Figure 6.3.9

✓ QUICK CHECK EXERCISES 6.3 (See page 438 for answers.)

- Let R be the region between the x -axis and the curve $y = 1 + \sqrt{x}$ for $1 \leq x \leq 4$.
 - For x between 1 and 4, the area of the cylindrical surface generated by revolving the vertical cross section of R at x about the y -axis is _____.
 - Using cylindrical shells, an integral expression for the volume of the solid generated by revolving R about the y -axis is _____.
- Let R be the region described in Quick Check Exercise 1.
 - For x between 1 and 4, the area of the cylindrical surface generated by revolving the vertical cross section of R at x about the line $x = 5$ is _____.
 - Using cylindrical shells, an integral expression for the volume of the solid generated by revolving R about the line $x = 5$ is _____.
- A solid S is generated by revolving the region enclosed by the curves $x = (y - 2)^2$ and $x = 4$ about the x -axis. Using cylindrical shells, an integral expression for the volume of S is _____.

EXERCISE SET 6.3 C CAS

1–4 Use cylindrical shells to find the volume of the solid generated when the shaded region is revolved about the indicated axis. ■



5–12 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y -axis. ■

- $y = x^3$, $x = 1$, $y = 0$
- $y = \sqrt{x}$, $x = 4$, $x = 9$, $y = 0$
- $y = 1/x$, $y = 0$, $x = 1$, $x = 3$
- $y = \cos(x^2)$, $x = 0$, $x = \frac{1}{2}\sqrt{\pi}$, $y = 0$
- $y = 2x - 1$, $y = -2x + 3$, $x = 2$
- $y = 2x - x^2$, $y = 0$
- $y = \frac{1}{x^2 + 1}$, $x = 0$, $x = 1$, $y = 0$
- $y = e^{x^2}$, $x = 1$, $x = \sqrt{3}$, $y = 0$

13–16 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the x -axis. ■

13. $y^2 = x$, $y = 1$, $x = 0$
 14. $x = 2y$, $y = 2$, $y = 3$, $x = 0$
 15. $y = x^2$, $x = 1$, $y = 0$ 16. $xy = 4$, $x + y = 5$

17–20 True-False Determine whether the statement is true or false. Explain your answer. ■

17. The volume of a cylindrical shell is equal to the product of the thickness of the shell with the surface area of a cylinder whose height is that of the shell and whose radius is equal to the average of the inner and outer radii of the shell.
 18. The method of cylindrical shells is a special case of the method of integration of cross-sectional area that was discussed in Section 6.2.
 19. In the method of cylindrical shells, integration is over an interval on a coordinate axis that is *perpendicular* to the axis of revolution of the solid.
 20. The Riemann sum approximation

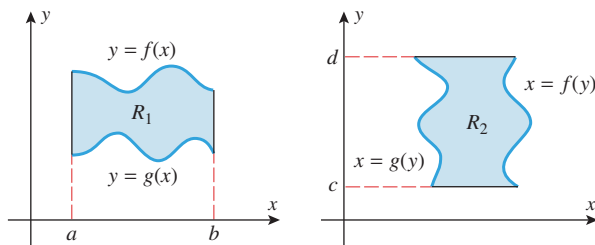
$$V \approx \sum_{k=1}^n 2\pi x_k^* f(x_k^*) \Delta x_k \quad \left(\text{where } x_k^* = \frac{x_k + x_{k-1}}{2} \right)$$

for the volume of a solid of revolution is exact when f is a constant function.

21. Use a CAS to find the volume of the solid generated when the region enclosed by $y = e^x$ and $y = 0$ for $1 \leq x \leq 2$ is revolved about the y -axis.
 22. Use a CAS to find the volume of the solid generated when the region enclosed by $y = \cos x$, $y = 0$, and $x = 0$ for $0 \leq x \leq \pi/2$ is revolved about the y -axis.
 23. Consider the region to the right of the y -axis, to the left of the vertical line $x = k$ ($0 < k < \pi$), and between the curve $y = \sin x$ and the x -axis. Use a CAS to estimate the value of k so that the solid generated by revolving the region about the y -axis has a volume of 8 cubic units.

FOCUS ON CONCEPTS

24. Let R_1 and R_2 be regions of the form shown in the accompanying figure. Use cylindrical shells to find a formula for the volume of the solid that results when
 (a) region R_1 is revolved about the y -axis
 (b) region R_2 is revolved about the x -axis.



▲ Figure Ex-24

25. (a) Use cylindrical shells to find the volume of the solid that is generated when the region under the curve

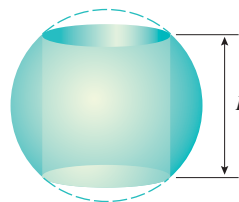
$$y = x^3 - 3x^2 + 2x$$

over $[0, 1]$ is revolved about the y -axis.

- (b) For this problem, is the method of cylindrical shells easier or harder than the method of slicing discussed in the last section? Explain.
 26. Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is enclosed by $y = f(x)$ and $y = 0$ for $a \leq x \leq b$. Using the method of cylindrical shells, derive with explanation a formula for the volume of the solid generated by revolving R about the line $x = k$, where $k \leq a$.

27–28 Using the method of cylindrical shells, set up but do not evaluate an integral for the volume of the solid generated when the region R is revolved about (a) the line $x = 1$ and (b) the line $y = -1$. ■

27. R is the region bounded by the graphs of $y = x$, $y = 0$, and $x = 1$.
 28. R is the region in the first quadrant bounded by the graphs of $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$.
 29. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = 1/x^3$, $x = 1$, $x = 2$, $y = 0$ is revolved about the line $x = -1$.
 30. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = x^3$, $y = 1$, $x = 0$ is revolved about the line $y = 1$.
 31. Use cylindrical shells to find the volume of the cone generated when the triangle with vertices $(0, 0)$, $(0, r)$, $(h, 0)$, where $r > 0$ and $h > 0$, is revolved about the x -axis.
 32. The region enclosed between the curve $y^2 = kx$ and the line $x = \frac{1}{4}k$ is revolved about the line $x = \frac{1}{2}k$. Use cylindrical shells to find the volume of the resulting solid. (Assume $k > 0$.)
 33. As shown in the accompanying figure, a cylindrical hole is drilled all the way through the center of a sphere. Show that the volume of the remaining solid depends only on the length L of the hole, not on the size of the sphere.



◀ Figure Ex-33

34. Use cylindrical shells to find the volume of the torus obtained by revolving the circle $x^2 + y^2 = a^2$ about the line

$x = b$, where $b > a > 0$. [Hint: It may help in the integration to think of an integral as an area.]

35. Let V_x and V_y be the volumes of the solids that result when the region enclosed by $y = 1/x$, $y = 0$, $x = \frac{1}{2}$, and $x = b$ ($b > \frac{1}{2}$) is revolved about the x -axis and y -axis, respectively. Is there a value of b for which $V_x = V_y$?
36. (a) Find the volume V of the solid generated when the region bounded by $y = 1/(1+x^4)$, $y = 0$, $x = 1$, and $x = b$ ($b > 1$) is revolved about the y -axis.
 (b) Find $\lim_{b \rightarrow +\infty} V$.

37. **Writing** Faced with the problem of computing the volume of a solid of revolution, how would you go about deciding whether to use the method of disks/washers or the method of cylindrical shells?

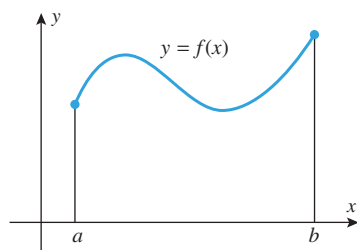
38. **Writing** With both the method of disks/washers and with the method of cylindrical shells, we integrate an “area” to get the volume of a solid of revolution. However, these two approaches differ in very significant ways. Write a brief paragraph that discusses these differences.

✓ QUICK CHECK ANSWERS 6.3

1. (a) $2\pi x(1 + \sqrt{x})$ (b) $\int_1^4 2\pi x(1 + \sqrt{x}) dx$ 2. (a) $2\pi(5-x)(1 + \sqrt{x})$ (b) $\int_1^4 2\pi(5-x)(1 + \sqrt{x}) dx$
3. $\int_0^4 2\pi y[4 - (y-2)^2] dy$

6.4 LENGTH OF A PLANE CURVE

In this section we will use the tools of calculus to study the problem of finding the length of a plane curve.



▲ Figure 6.4.1

■ ARC LENGTH

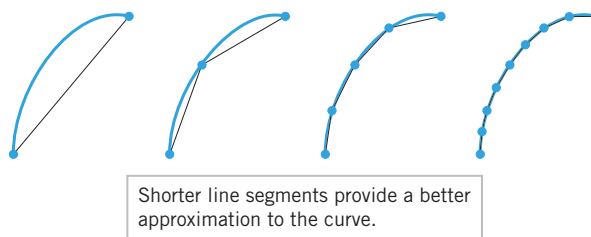
Our first objective is to define what we mean by the **length** (also called the **arc length**) of a plane curve $y = f(x)$ over an interval $[a, b]$ (Figure 6.4.1). Once that is done we will be able to focus on the problem of computing arc lengths. To avoid some complications that would otherwise occur, we will impose the requirement that f' be continuous on $[a, b]$, in which case we will say that $y = f(x)$ is a **smooth curve** on $[a, b]$ or that f is a **smooth function** on $[a, b]$. Thus, we will be concerned with the following problem.

6.4.1 ARC LENGTH PROBLEM Suppose that $y = f(x)$ is a smooth curve on the interval $[a, b]$. Define and find a formula for the arc length L of the curve $y = f(x)$ over the interval $[a, b]$.

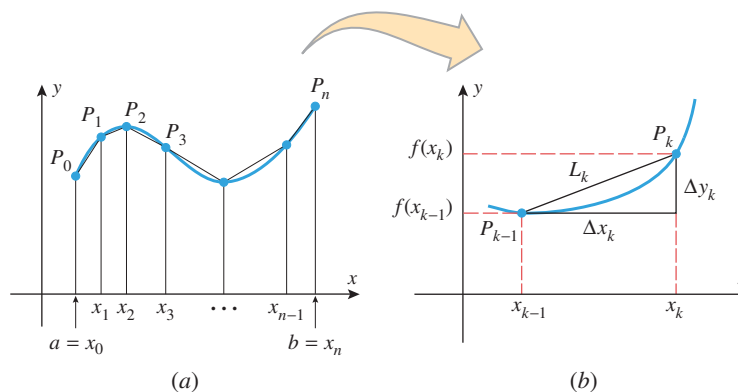
Intuitively, you might think of the arc length of a curve as the number obtained by aligning a piece of string with the curve and then measuring the length of the string after it is straightened out.

To define the arc length of a curve we start by breaking the curve into small segments. Then we approximate the curve segments by line segments and add the lengths of the line segments to form a Riemann sum. Figure 6.4.2 illustrates how such line segments tend to become better and better approximations to a curve as the number of segments increases. As the number of segments increases, the corresponding Riemann sums approach a definite integral whose value we will take to be the arc length L of the curve.

To implement our idea for solving Problem 6.4.1, divide the interval $[a, b]$ into n subintervals by inserting points x_1, x_2, \dots, x_{n-1} between $a = x_0$ and $b = x_n$. As shown in Figure 6.4.3a, let P_0, P_1, \dots, P_n be the points on the curve with x -coordinates $a = x_0$,



► Figure 6.4.2



► Figure 6.4.3

$x_1, x_2, \dots, x_{n-1}, b = x_n$ and join these points with straight line segments. These line segments form a **polygonal path** that we can regard as an approximation to the curve $y = f(x)$. As indicated in Figure 6.4.3b, the length L_k of the k th line segment in the polygonal path is

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \quad (1)$$

If we now add the lengths of these line segments, we obtain the following approximation to the length L of the curve

$$L \approx \sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \quad (2)$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a point x_k^* between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \quad \text{or} \quad f(x_k) - f(x_{k-1}) = f'(x_k^*) \Delta x_k$$

and hence we can rewrite (2) as

$$L \approx \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + [f'(x_k^*)]^2 (\Delta x_k)^2} = \sum_{k=1}^n \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

Explain why the approximation in (2) cannot be greater than L .

Thus, taking the limit as n increases and the widths of all the subintervals approach zero yields the following integral that defines the arc length L :

$$L = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

In summary, we have the following definition.

6.4.2 DEFINITION If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

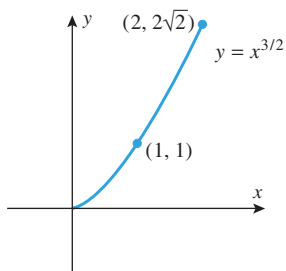
$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx \quad (3)$$

This result provides both a definition and a formula for computing arc lengths. Where convenient, (3) can also be expressed as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (4)$$

Moreover, for a curve expressed in the form $x = g(y)$, where g' is continuous on $[c, d]$, the arc length L from $y = c$ to $y = d$ can be expressed as

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \quad (5)$$



▲ Figure 6.4.4

► **Example 1** Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$ (Figure 6.4.4) in two ways: (a) using Formula (4) and (b) using Formula (5).

Solution (a).

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

and since the curve extends from $x = 1$ to $x = 2$, it follows from (4) that

$$L = \int_1^2 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2} dx = \int_1^2 \sqrt{1 + \frac{9}{4}x} dx$$

To evaluate this integral we make the u -substitution

$$u = 1 + \frac{9}{4}x, \quad du = \frac{9}{4} dx$$

and then change the x -limits of integration ($x = 1, x = 2$) to the corresponding u -limits ($u = \frac{13}{4}, u = \frac{22}{4}$):

$$\begin{aligned} L &= \frac{4}{9} \int_{13/4}^{22/4} u^{1/2} du = \frac{8}{27} u^{3/2} \Big|_{13/4}^{22/4} = \frac{8}{27} \left[\left(\frac{22}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2} \right] \\ &= \frac{22\sqrt{22} - 13\sqrt{13}}{27} \approx 2.09 \end{aligned}$$

Solution (b). To apply Formula (5) we must first rewrite the equation $y = x^{3/2}$ so that x is expressed as a function of y . This yields $x = y^{2/3}$ and

$$\frac{dx}{dy} = \frac{2}{3}y^{-1/3}$$

Since the curve extends from $y = 1$ to $y = 2\sqrt{2}$, it follows from (5) that

$$L = \int_1^{2\sqrt{2}} \sqrt{1 + \frac{4}{9}y^{-2/3}} dy = \frac{1}{3} \int_1^{2\sqrt{2}} y^{-1/3} \sqrt{9y^{2/3} + 4} dy$$

To evaluate this integral we make the u -substitution

$$u = 9y^{2/3} + 4, \quad du = 6y^{-1/3} dy$$

and change the y -limits of integration ($y = 1, y = 2\sqrt{2}$) to the corresponding u -limits ($u = 13, u = 22$). This gives

$$L = \frac{1}{18} \int_{13}^{22} u^{1/2} du = \frac{1}{27} u^{3/2} \Big|_{13}^{22} = \frac{1}{27} [(22)^{3/2} - (13)^{3/2}] = \frac{22\sqrt{22} - 13\sqrt{13}}{27}$$

The answer in part (b) agrees with that in part (a); however, the integration in part (b) is more tedious. In problems where there is a choice between using (4) or (5), it is often the case that one of the formulas leads to a simpler integral than the other. ◀

The arc from the point $(1, 1)$ to the point $(2, 2\sqrt{2})$ in Figure 6.4.4 is nearly a straight line, so the arc length should be only slightly larger than the straight-line distance between these points. Show that this is so.

FINDING ARC LENGTH BY NUMERICAL METHODS

In the next chapter we will develop some techniques of integration that will enable us to find exact values of more integrals encountered in arc length calculations; however, generally speaking, most such integrals are impossible to evaluate in terms of elementary functions. In these cases one usually approximates the integral using a numerical method such as the midpoint rule discussed in Section 5.4.

TECHNOLOGY MASTERY

If your calculating utility has a numerical integration capability, use it to confirm that the arc length L in Example 2 is approximately $L \approx 3.8202$.

► **Example 2** From (4), the arc length of $y = \sin x$ from $x = 0$ to $x = \pi$ is given by the integral

$$L = \int_0^\pi \sqrt{1 + (\cos x)^2} dx$$

This integral cannot be evaluated in terms of elementary functions; however, using a calculating utility with a numerical integration capability yields the approximation $L \approx 3.8202$. ◀

QUICK CHECK EXERCISES 6.4 (See page 443 for answers.)

- A function f is smooth on $[a, b]$ if f' is _____ on $[a, b]$.
- If a function f is smooth on $[a, b]$, then the length of the curve $y = f(x)$ over $[a, b]$ is _____.
- The distance between points $(1, 0)$ and $(e, 1)$ is _____.
- Let L be the length of the curve $y = \ln x$ from $(1, 0)$ to $(e, 1)$.
 - Integrating with respect to x , an integral expression for L is _____.
 - Integrating with respect to y , an integral expression for L is _____.

EXERCISE SET 6.4 CAS

- Use the Theorem of Pythagoras to find the length of the line segment $y = 2x$ from $(1, 2)$ to $(2, 4)$, and confirm that the value is consistent with the length computed using
 - Formula (4)
 - Formula (5).
- Use the Theorem of Pythagoras to find the length of the line segment $y = 5x$ from $(0, 0)$ and $(1, 5)$, and confirm that the value is consistent with the length computed using
 - Formula (4)
 - Formula (5).
- 3–8 Find the exact arc length of the curve over the interval.
 - $y = 3x^{3/2} - 1$ from $x = 0$ to $x = 1$
 - $x = \frac{1}{3}(y^2 + 2)^{3/2}$ from $y = 0$ to $y = 1$
 - $y = x^{2/3}$ from $x = 1$ to $x = 8$
 - $y = (x^6 + 8)/(16x^2)$ from $x = 2$ to $x = 3$
 - $24xy = y^4 + 48$ from $y = 2$ to $y = 4$
 - $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ from $y = 1$ to $y = 4$
- 9–12 **True–False** Determine whether the statement is true or false. Explain your answer.
 - The graph of $y = \sqrt{1 - x^2}$ is a smooth curve on $[-1, 1]$.

10. The approximation

$$L \approx \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

for arc length is not expressed in the form of a Riemann sum.

11. The approximation

$$L \approx \sum_{k=1}^n \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

for arc length is exact when f is a linear function of x .

12. In our definition of the arc length for the graph of
- $y = f(x)$
- , we need
- $f'(x)$
- to be a continuous function in order for
- f
- to satisfy the hypotheses of the Mean-Value Theorem (4.8.2).

- C** 13–14 Express the exact arc length of the curve over the given interval as an integral that has been simplified to eliminate the radical, and then evaluate the integral using a CAS. ■

13. $y = \ln(\sec x)$ from $x = 0$ to $x = \pi/4$

14. $y = \ln(\sin x)$ from $x = \pi/4$ to $x = \pi/2$

FOCUS ON CONCEPTS

15. Consider the curve $y = x^{2/3}$.
- Sketch the portion of the curve between $x = -1$ and $x = 8$.
 - Explain why Formula (4) cannot be used to find the arc length of the curve sketched in part (a).
 - Find the arc length of the curve sketched in part (a).
16. The curve segment $y = x^2$ from $x = 1$ to $x = 2$ may also be expressed as the graph of $x = \sqrt{y}$ from $y = 1$ to $y = 4$. Set up two integrals that give the arc length of this curve segment, one by integrating with respect to x , and the other by integrating with respect to y . Demonstrate a substitution that verifies that these two integrals are equal.
17. Consider the curve segments $y = x^2$ from $x = \frac{1}{2}$ to $x = 2$ and $y = \sqrt{x}$ from $x = \frac{1}{4}$ to $x = 4$.
- Graph the two curve segments and use your graphs to explain why the lengths of these two curve segments should be equal.
 - Set up integrals that give the arc lengths of the curve segments by integrating with respect to x . Demonstrate a substitution that verifies that these two integrals are equal.
 - Set up integrals that give the arc lengths of the curve segments by integrating with respect to y .
 - Approximate the arc length of each curve segment using Formula (2) with $n = 10$ equal subintervals.
 - Which of the two approximations in part (d) is more accurate? Explain.
 - Use the midpoint approximation with $n = 10$ subintervals to approximate each arc length integral in part (b).

- (g) Use a calculating utility with numerical integration capabilities to approximate the arc length integrals in part (b) to four decimal places.

18. Follow the directions of Exercise 17 for the curve segments $y = x^{8/3}$ from $x = 10^{-3}$ to $x = 1$ and $y = x^{3/8}$ from $x = 10^{-8}$ to $x = 1$.
19. Follow the directions of Exercise 17 for the curve segment $y = \tan x$ from $x = 0$ to $x = \pi/3$ and for the curve segment $y = \tan^{-1} x$ from $x = 0$ to $x = \sqrt{3}$.
20. Let $y = f(x)$ be a smooth curve on the closed interval $[a, b]$. Prove that if m and M are nonnegative numbers such that $m \leq |f'(x)| \leq M$ for all x in $[a, b]$, then the arc length L of $y = f(x)$ over the interval $[a, b]$ satisfies the inequalities

$$(b - a)\sqrt{1 + m^2} \leq L \leq (b - a)\sqrt{1 + M^2}$$

21. Use the result of Exercise 20 to show that the arc length L of $y = \sec x$ over the interval $0 \leq x \leq \pi/3$ satisfies

$$\frac{\pi}{3} \leq L \leq \frac{\pi}{3}\sqrt{13}$$

- C** 22. A basketball player makes a successful shot from the free throw line. Suppose that the path of the ball from the moment of release to the moment it enters the hoop is described by

$$y = 2.15 + 2.09x - 0.41x^2, \quad 0 \leq x \leq 4.6$$

where x is the horizontal distance (in meters) from the point of release, and y is the vertical distance (in meters) above the floor. Use a CAS or a scientific calculator with a numerical integration capability to approximate the distance the ball travels from the moment it is released to the moment it enters the hoop. Round your answer to two decimal places.

- C** 23. Find a positive value of k (to two decimal places) such that the curve $y = k \sin x$ has an arc length of $L = 5$ units over the interval from $x = 0$ to $x = \pi$. [Hint: Find an integral for the arc length L in terms of k , and then use a CAS or a scientific calculator with a numerical integration capability to find integer values of k at which the values of $L - 5$ have opposite signs. Complete the solution by using the Intermediate-Value Theorem (1.5.7) to approximate the value of k to two decimal places.]

- C** 24. As shown in the accompanying figure on the next page, a horizontal beam with dimensions 2 in \times 6 in \times 16 ft is fixed at both ends and is subjected to a uniformly distributed load of 120 lb/ft. As a result of the load, the centerline of the beam undergoes a deflection that is described by

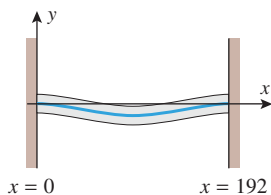
$$y = -1.67 \times 10^{-8}(x^4 - 2Lx^3 + L^2x^2)$$

($0 \leq x \leq 192$), where $L = 192$ in is the length of the unloaded beam, x is the horizontal distance along the beam measured in inches from the left end, and y is the deflection of the centerline in inches.

- (a) Graph y versus x for $0 \leq x \leq 192$.

- (b) Find the maximum deflection of the centerline. (cont.)

- (c) Use a CAS or a calculator with a numerical integration capability to find the length of the centerline of the loaded beam. Round your answer to two decimal places.



◀ Figure Ex-24

- ◻ 25. A golfer makes a successful chip shot to the green. Suppose that the path of the ball from the moment it is struck to the moment it hits the green is described by

$$y = 12.54x - 0.41x^2$$

where x is the horizontal distance (in yards) from the point where the ball is struck, and y is the vertical distance (in yards) above the fairway. Use a CAS or a calculating utility with a numerical integration capability to find the distance the ball travels from the moment it is struck to the moment it hits the green. Assume that the fairway and green are at the same level and round your answer to two decimal places.

26–34 These exercises assume familiarity with the basic concepts of parametric curves. If needed, an introduction to this material is provided in Web Appendix I. ■

- ◻ 26. Assume that no segment of the curve

$$x = x(t), \quad y = y(t), \quad (a \leq t \leq b)$$

is traced more than once as t increases from a to b . Divide the interval $[a, b]$ into n subintervals by inserting points t_1, t_2, \dots, t_{n-1} between $a = t_0$ and $b = t_n$. Let L denote the arc length of the curve. Give an informal argument for the approximation

$$L \approx \sum_{k=1}^n \sqrt{[x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2}$$

If dx/dt and dy/dt are continuous functions for $a \leq t \leq b$, then it can be shown that as $\max \Delta t_k \rightarrow 0$, this sum converges to

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

27–32 Use the arc length formula from Exercise 26 to find the arc length of the curve. ■

27. $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2 \quad (0 \leq t \leq 1)$

28. $x = (1+t)^2, \quad y = (1+t)^3 \quad (0 \leq t \leq 1)$

29. $x = \cos 2t, \quad y = \sin 2t \quad (0 \leq t \leq \pi/2)$

30. $x = \cos t + t \sin t, \quad y = \sin t - t \cos t \quad (0 \leq t \leq \pi)$

31. $x = e^t \cos t, \quad y = e^t \sin t \quad (0 \leq t \leq \pi/2)$

32. $x = e^t(\sin t + \cos t), \quad y = e^t(\cos t - \sin t) \quad (1 \leq t \leq 4)$

- ◻ 33. (a) Show that the total arc length of the ellipse

$$x = 2 \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi)$$

is given by

$$4 \int_0^{\pi/2} \sqrt{1 + 3 \sin^2 t} dt$$

- (b) Use a CAS or a scientific calculator with a numerical integration capability to approximate the arc length in part (a). Round your answer to two decimal places.

- (c) Suppose that the parametric equations in part (a) describe the path of a particle moving in the xy -plane, where t is time in seconds and x and y are in centimeters. Use a CAS or a scientific calculator with a numerical integration capability to approximate the distance traveled by the particle from $t = 1.5$ s to $t = 4.8$ s. Round your answer to two decimal places.

34. Show that the total arc length of the ellipse $x = a \cos t, \quad y = b \sin t, \quad 0 \leq t \leq 2\pi$ for $a > b > 0$ is given by

$$4a \int_0^{\pi/2} \sqrt{1 - k^2 \cos^2 t} dt$$

where $k = \sqrt{a^2 - b^2}/a$.

35. **Writing** In our discussion of Arc Length Problem 6.4.1, we derived the approximation

$$L \approx \sum_{k=1}^n \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

Discuss the geometric meaning of this approximation. (Be sure to address the appearance of the derivative f' .)

36. **Writing** Give examples in which Formula (4) for arc length cannot be applied directly, and describe how you would go about finding the arc length of the curve in each case. (Discuss both the use of alternative formulas and the use of numerical methods.)

✓ QUICK CHECK ANSWERS 6.4

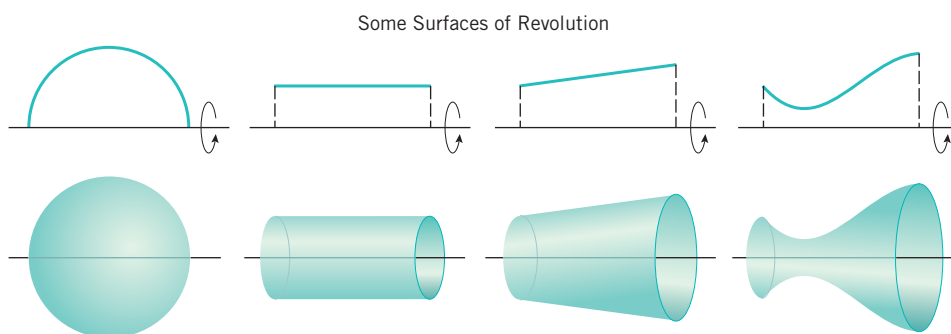
1. continuous 2. $\int_a^b \sqrt{1 + [f'(x)]^2} dx$ 3. $\sqrt{(e-1)^2 + 1}$ 4. (a) $\int_1^e \sqrt{1 + (1/x)^2} dx$ (b) $\int_0^1 \sqrt{1 + e^{2y}} dy$

6.5 AREA OF A SURFACE OF REVOLUTION

In this section we will consider the problem of finding the area of a surface that is generated by revolving a plane curve about a line.

SURFACE AREA

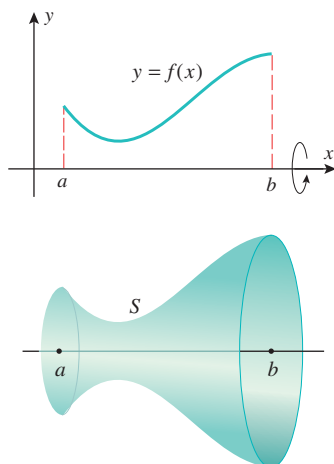
A *surface of revolution* is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter, and the lateral surface of a right circular cylinder can be generated by revolving a line segment about an axis that is parallel to it (Figure 6.5.1).



► Figure 6.5.1

In this section we will be concerned with the following problem.

6.5.1 SURFACE AREA PROBLEM Suppose that f is a smooth, nonnegative function on $[a, b]$ and that a surface of revolution is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x -axis (Figure 6.5.2). Define what is meant by the *area* S of the surface, and find a formula for computing it.



▲ Figure 6.5.2

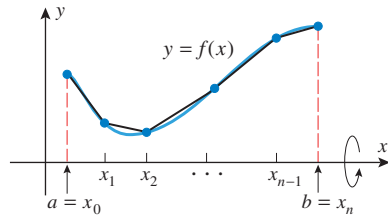
To motivate an appropriate definition for the area S of a surface of revolution, we will decompose the surface into small sections whose areas can be approximated by elementary formulas, add the approximations of the areas of the sections to form a Riemann sum that approximates S , and then take the limit of the Riemann sums to obtain an integral for the exact value of S .

To implement this idea, divide the interval $[a, b]$ into n subintervals by inserting points x_1, x_2, \dots, x_{n-1} between $a = x_0$ and $b = x_n$. As illustrated in Figure 6.5.3a, the corresponding points on the graph of f define a polygonal path that approximates the curve $y = f(x)$ over the interval $[a, b]$. As illustrated in Figure 6.5.3b, when this polygonal path is revolved about the x -axis, it generates a surface consisting of n parts, each of which is a portion of a right circular cone called a *frustum* (from the Latin meaning “bit” or “piece”). Thus, the area of each part of the approximating surface can be obtained from the formula

$$S = \pi(r_1 + r_2)l \quad (1)$$

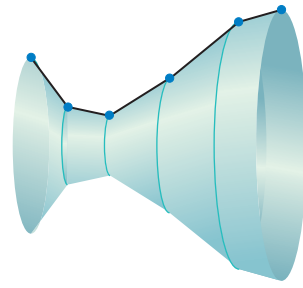
for the lateral area S of a frustum of slant height l and base radii r_1 and r_2 (Figure 6.5.4). As suggested by Figure 6.5.5, the k th frustum has radii $f(x_{k-1})$ and $f(x_k)$ and height Δx_k . Its slant height is the length L_k of the k th line segment in the polygonal path, which from Formula (1) of Section 6.4 is

$$L_k = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

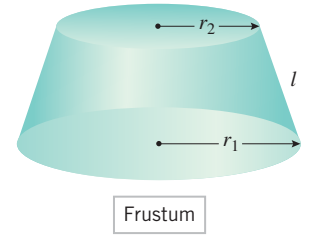


(a)

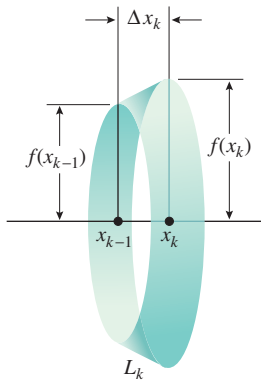
▲ Figure 6.5.3



(b)



▲ Figure 6.5.4



▲ Figure 6.5.5

This makes the lateral area S_k of the k th frustum

$$S_k = \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

If we add these areas, we obtain the following approximation to the area S of the entire surface:

$$S \approx \sum_{k=1}^n \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \quad (2)$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a point x_k^* between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \quad \text{or} \quad f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$$

and hence we can rewrite (2) as

$$\begin{aligned} S &\approx \sum_{k=1}^n \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f'(x_k^*)]^2(\Delta x_k)^2} \\ &= \sum_{k=1}^n \pi[f(x_{k-1}) + f(x_k)]\sqrt{1 + [f'(x_k^*)]^2} \Delta x_k \end{aligned} \quad (3)$$

However, this is not yet a Riemann sum because it involves the variables x_{k-1} and x_k . To eliminate these variables from the expression, observe that the average value of the numbers $f(x_{k-1})$ and $f(x_k)$ lies between these numbers, so the continuity of f and the Intermediate-Value Theorem (1.5.7) imply that there is a point x_k^{**} between x_{k-1} and x_k such that

$$\frac{1}{2}[f(x_{k-1}) + f(x_k)] = f(x_k^{**})$$

Thus, (2) can be expressed as

$$S \approx \sum_{k=1}^n 2\pi f(x_k^{**})\sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

Although this expression is close to a Riemann sum in form, it is not a true Riemann sum because it involves two variables x_k^* and x_k^{**} , rather than x_k^* alone. However, it is proved in advanced calculus courses that this has no effect on the limit because of the continuity of f . Thus, we can assume that $x_k^{**} = x_k^*$ when taking the limit, and this suggests that S can be defined as

$$S = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n 2\pi f(x_k^*)\sqrt{1 + [f'(x_k^*)]^2} \Delta x_k = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2} dx$$

In summary, we have the following definition.

6.5.2 DEFINITION If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x -axis is defined as

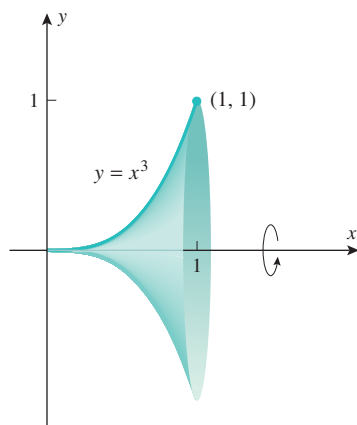
$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

This result provides both a definition and a formula for computing surface areas. Where convenient, this formula can also be expressed as

$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (4)$$

Moreover, if g is nonnegative and $x = g(y)$ is a smooth curve on the interval $[c, d]$, then the area of the surface that is generated by revolving the portion of a curve $x = g(y)$ between $y = c$ and $y = d$ about the y -axis can be expressed as

$$S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} dy = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \quad (5)$$

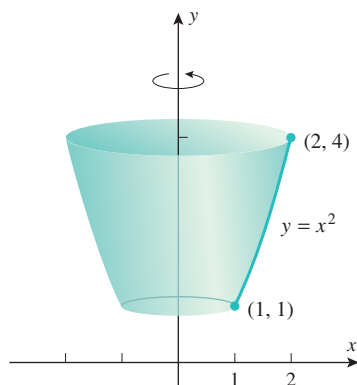


▲ Figure 6.5.6

► **Example 1** Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x -axis.

Solution. First sketch the curve; then imagine revolving it about the x -axis (Figure 6.5.6). Since $y = x^3$, we have $dy/dx = 3x^2$, and hence from (4) the surface area S is

$$\begin{aligned} S &= \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2} dx \\ &= 2\pi \int_0^1 x^3 (1 + 9x^4)^{1/2} dx \\ &= \frac{2\pi}{36} \int_1^{10} u^{1/2} du \quad \begin{matrix} u = 1 + 9x^4 \\ du = 36x^3 dx \end{matrix} \\ &= \frac{2\pi}{36} \cdot \frac{2}{3} u^{3/2} \Big|_{u=1}^{10} = \frac{\pi}{27} (10^{3/2} - 1) \approx 3.56 \quad \blacktriangleleft \end{aligned}$$



▲ Figure 6.5.7

► **Example 2** Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y -axis.

Solution. First sketch the curve; then imagine revolving it about the y -axis (Figure 6.5.7). Because the curve is revolved about the y -axis we will apply Formula (5). Toward this end, we rewrite $y = x^2$ as $x = \sqrt{y}$ and observe that the y -values corresponding to $x = 1$ and

$x = 2$ are $y = 1$ and $y = 4$. Since $x = \sqrt{y}$, we have $dx/dy = 1/(2\sqrt{y})$, and hence from (5) the surface area S is

$$\begin{aligned} S &= \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= \int_1^4 2\pi\sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2} dy \\ &= \pi \int_1^4 \sqrt{4y + 1} dy \\ &= \frac{\pi}{4} \int_5^{17} u^{1/2} du \quad \begin{array}{l} u = 4y + 1 \\ du = 4 dy \end{array} \\ &= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_{u=5}^{17} = \frac{\pi}{6} (17^{3/2} - 5^{3/2}) \approx 30.85 \quad \blacktriangleleft \end{aligned}$$

QUICK CHECK EXERCISES 6.5 (See page 449 for answers.)

- If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x -axis is _____.
- The lateral area of the frustum with slant height $\sqrt{10}$ and base radii $r_1 = 1$ and $r_2 = 2$ is _____.
- An integral expression for the area of the surface generated by rotating the line segment joining $(3, 1)$ and $(6, 2)$ about the x -axis is _____.
- An integral expression for the area of the surface generated by rotating the line segment joining $(3, 1)$ and $(6, 2)$ about the y -axis is _____.

EXERCISE SET 6.5 C CAS

1–4 Find the area of the surface generated by revolving the given curve about the x -axis. ■

- $y = 7x, 0 \leq x \leq 1$
- $y = \sqrt{x}, 1 \leq x \leq 4$
- $y = \sqrt{4 - x^2}, -1 \leq x \leq 1$
- $x = \sqrt[3]{y}, 1 \leq y \leq 8$

5–8 Find the area of the surface generated by revolving the given curve about the y -axis. ■

- $x = 9y + 1, 0 \leq y \leq 2$
- $x = y^3, 0 \leq y \leq 1$
- $x = \sqrt{9 - y^2}, -2 \leq y \leq 2$
- $x = 2\sqrt{1 - y}, -1 \leq y \leq 0$

C **9–12** Use a CAS to find the exact area of the surface generated by revolving the curve about the stated axis. ■

- $y = \sqrt{x} - \frac{1}{3}x^{3/2}, 1 \leq x \leq 3; x$ -axis
- $y = \frac{1}{3}x^3 + \frac{1}{4}x^{-1}, 1 \leq x \leq 2; x$ -axis

11. $8xy^2 = 2y^6 + 1, 1 \leq y \leq 2; y$ -axis

12. $x = \sqrt{16 - y}, 0 \leq y \leq 15; y$ -axis

C **13–16** Use a CAS or a calculating utility with a numerical integration capability to approximate the area of the surface generated by revolving the curve about the stated axis. Round your answer to two decimal places. ■

13. $y = \sin x, 0 \leq x \leq \pi; x$ -axis

14. $x = \tan y, 0 \leq y \leq \pi/4; y$ -axis

15. $y = e^x, 0 \leq x \leq 1; x$ -axis

16. $y = e^x, 1 \leq y \leq e; y$ -axis

17–20 True–False Determine whether the statement is true or false. Explain your answer. ■

17. The lateral surface area S of a right circular cone with height h and base radius r is $S = \pi r \sqrt{r^2 + h^2}$.

18. The lateral surface area of a frustum of slant height l and base radii r_1 and r_2 is equal to the lateral surface area of a right circular cylinder of height l and radius equal to the average of r_1 and r_2 .

19. The approximation

$$S \approx \sum_{k=1}^n 2\pi f(x_k^{**}) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

for surface area is exact if f is a positive-valued constant function.

20. The expression

$$\sum_{k=1}^n 2\pi f(x_k^{**}) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

is not a true Riemann sum for

$$\int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

21–22 Approximate the area of the surface using Formula (2) with $n = 20$ subintervals of equal width. Round your answer to two decimal places. ■

21. The surface of Exercise 13.

22. The surface of Exercise 16.

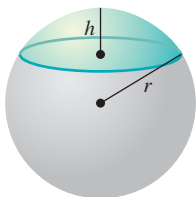
FOCUS ON CONCEPTS

23. Assume that $y = f(x)$ is a smooth curve on the interval $[a, b]$ and assume that $f(x) \geq 0$ for $a \leq x \leq b$. Derive a formula for the surface area generated when the curve $y = f(x)$, $a \leq x \leq b$, is revolved about the line $y = -k$ ($k > 0$).

24. Would it be circular reasoning to use Definition 6.5.2 to find the surface area of a frustum of a right circular cone? Explain your answer.

25. Show that the area of the surface of a sphere of radius r is $4\pi r^2$. [Hint: Revolve the semicircle $y = \sqrt{r^2 - x^2}$ about the x -axis.]

26. The accompanying figure shows a spherical cap of height h cut from a sphere of radius r . Show that the surface area S of the cap is $S = 2\pi rh$. [Hint: Revolve an appropriate portion of the circle $x^2 + y^2 = r^2$ about the y -axis.]



◀ Figure Ex-26

27. The portion of a sphere that is cut by two parallel planes is called a **zone**. Use the result of Exercise 26 to show that the surface area of a zone depends on the radius of the sphere and the distance between the planes, but not on the location of the zone.

28. Let $y = f(x)$ be a smooth curve on the interval $[a, b]$ and assume that $f(x) \geq 0$ for $a \leq x \leq b$. By the Extreme-Value

Theorem (4.4.2), the function f has a maximum value K and a minimum value k on $[a, b]$. Prove: If L is the arc length of the curve $y = f(x)$ between $x = a$ and $x = b$, and if S is the area of the surface that is generated by revolving this curve about the x -axis, then

$$2\pi kL \leq S \leq 2\pi KL$$

29. Use the results of Exercise 28 above and Exercise 21 in Section 6.4 to show that the area S of the surface generated by revolving the curve $y = \sec x$, $0 \leq x \leq \pi/3$, about the x -axis satisfies

$$\frac{2\pi^2}{3} \leq S \leq \frac{4\pi^2}{3} \sqrt{13}$$

30. Let $y = f(x)$ be a smooth curve on $[a, b]$ and assume that $f(x) \geq 0$ for $a \leq x \leq b$. Let A be the area under the curve $y = f(x)$ between $x = a$ and $x = b$, and let S be the area of the surface obtained when this section of curve is revolved about the x -axis.

(a) Prove that $2\pi A \leq S$.

(b) For what functions f is $2\pi A = S$?

31–37 These exercises assume familiarity with the basic concepts of parametric curves. If needed, an introduction to this material is provided in Web Appendix I. ■

31–32 For these exercises, divide the interval $[a, b]$ into n subintervals by inserting points t_1, t_2, \dots, t_{n-1} between $a = t_0$ and $b = t_n$, and assume that $x'(t)$ and $y'(t)$ are continuous functions and that no segment of the curve

$$x = x(t), \quad y = y(t) \quad (a \leq t \leq b)$$

is traced more than once. ■

31. Let S be the area of the surface generated by revolving the curve $x = x(t)$, $y = y(t)$ ($a \leq t \leq b$) about the x -axis. Explain how S can be approximated by

$$S \approx \sum_{k=1}^n (\pi[y(t_{k-1}) + y(t_k)] \times \sqrt{[x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2})$$

Using results from advanced calculus, it can be shown that as $\max \Delta t_k \rightarrow 0$, this sum converges to

$$S = \int_a^b 2\pi y(t) \sqrt{[x'(t)]^2 + [y'(t)]^2} dt \quad (\text{A})$$

32. Let S be the area of the surface generated by revolving the curve $x = x(t)$, $y = y(t)$ ($a \leq t \leq b$) about the y -axis. Explain how S can be approximated by

$$S \approx \sum_{k=1}^n (\pi[x(t_{k-1}) + x(t_k)] \times \sqrt{[x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2})$$

Using results from advanced calculus, it can be shown that as $\max \Delta t_k \rightarrow 0$, this sum converges to

$$S = \int_a^b 2\pi x(t) \sqrt{[x'(t)]^2 + [y'(t)]^2} dt \quad (\text{B})$$

33–37 Use Formulas (A) and (B) from Exercises 31 and 32. ■

- 33.** Find the area of the surface generated by revolving the parametric curve $x = t^2$, $y = 2t$ ($0 \leq t \leq 4$) about the x -axis.
- 34.** Use a CAS to find the area of the surface generated by revolving the parametric curve $x = \cos^2 t$, $y = 5 \sin t$ ($0 \leq t \leq \pi/2$) about the x -axis.
- 35.** Find the area of the surface generated by revolving the parametric curve $x = t$, $y = 2t^2$ ($0 \leq t \leq 1$) about the y -axis.
- 36.** Find the area of the surface generated by revolving the parametric curve $x = \cos^2 t$, $y = \sin^2 t$ ($0 \leq t \leq \pi/2$) about the y -axis.

37. By revolving the semicircle

$$x = r \cos t, \quad y = r \sin t \quad (0 \leq t \leq \pi)$$

about the x -axis, show that the surface area of a sphere of radius r is $4\pi r^2$.

- 38. Writing** Compare the derivation of Definition 6.5.2 with that of Definition 6.4.2. Discuss the geometric features that result in similarities in the two definitions.
- 39. Writing** Discuss what goes wrong if we replace the frustums of right circular cones by right circular cylinders in the derivation of Definition 6.5.2.

✓ QUICK CHECK ANSWERS 6.5

$$1. \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx \quad 2. 3\sqrt{10}\pi \quad 3. \int_3^6 (2\pi) \left(\frac{x}{3}\right) \sqrt{\frac{10}{9}} dx = \int_3^6 \frac{2\sqrt{10}\pi}{9} x dx \quad 4. \int_1^2 (2\pi)(3y) \sqrt{10} dy$$

6.6 WORK

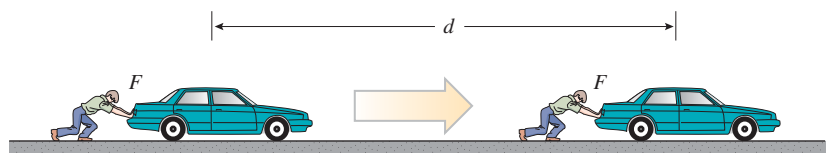
In this section we will use the integration tools developed in the preceding chapter to study some of the basic principles of “work,” which is one of the fundamental concepts in physics and engineering.

■ THE ROLE OF WORK IN PHYSICS AND ENGINEERING

In this section we will be concerned with two related concepts, *work* and *energy*. To put these ideas in a familiar setting, when you push a stalled car for a certain distance you are performing work, and the effect of your work is to make the car move. The energy of motion caused by the work is called the *kinetic energy* of the car. The exact connection between work and kinetic energy is governed by a principle of physics called the *work–energy relationship*. Although we will touch on this idea in this section, a detailed study of the relationship between work and energy will be left for courses in physics and engineering. Our primary goal here will be to explain the role of integration in the study of work.

■ WORK DONE BY A CONSTANT FORCE APPLIED IN THE DIRECTION OF MOTION

When a stalled car is pushed, the speed that the car attains depends on the force F with which it is pushed and the distance d over which that force is applied (Figure 6.6.1). Force and distance appear in the following definition of work.



► Figure 6.6.1

6.6.1 DEFINITION If a constant force of magnitude F is applied in the direction of motion of an object, and if that object moves a distance d , then we define the *work* W performed by the force on the object to be

$$W = F \cdot d \tag{1}$$

If you push against an immovable object, such as a brick wall, you may tire yourself out, but you will not perform any work. Why?

Common units for measuring force are newtons (N) in the International System of Units (SI), dynes (dyn) in the centimeter-gram-second (CGS) system, and pounds (lb) in the British Engineering (BE) system. One newton is the force required to give a mass of 1 kg an acceleration of 1 m/s^2 , one dyne is the force required to give a mass of 1 g an acceleration of 1 cm/s^2 , and one pound of force is the force required to give a mass of 1 slug an acceleration of 1 ft/s^2 .

It follows from Definition 6.6.1 that work has units of force times distance. The most common units of work are newton-meters (N·m), dyne-centimeters (dyn·cm), and foot-pounds (ft·lb). As indicated in Table 6.6.1, one newton-meter is also called a *joule* (J), and one dyne-centimeter is also called an *erg*. One foot-pound is approximately 1.36 J.

Table 6.6.1

SYSTEM	FORCE	×	DISTANCE	=	WORK
SI	newton (N)		meter (m)		joule (J)
CGS	dyne (dyn)		centimeter (cm)		erg
BE	pound (lb)		foot (ft)		foot-pound (ft·lb)

CONVERSION FACTORS:

$1 \text{ N} = 10^5 \text{ dyn} \approx 0.225 \text{ lb}$	$1 \text{ lb} \approx 4.45 \text{ N}$
$1 \text{ J} = 10^7 \text{ erg} \approx 0.738 \text{ ft}\cdot\text{lb}$	$1 \text{ ft}\cdot\text{lb} \approx 1.36 \text{ J} = 1.36 \times 10^7 \text{ erg}$

► **Example 1** An object moves 5 ft along a line while subjected to a constant force of 100 lb in its direction of motion. The work done is

$$W = F \cdot d = 100 \cdot 5 = 500 \text{ ft}\cdot\text{lb}$$

An object moves 25 m along a line while subjected to a constant force of 4 N in its direction of motion. The work done is

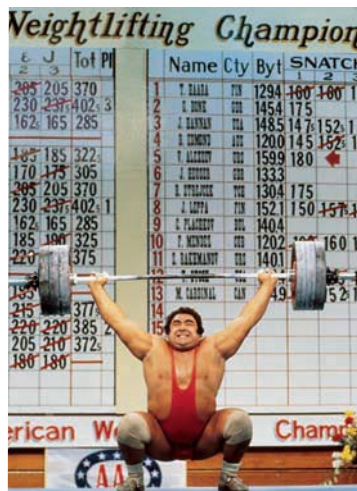
$$W = F \cdot d = 4 \cdot 25 = 100 \text{ N}\cdot\text{m} = 100 \text{ J} \blacktriangleleft$$

► **Example 2** In the 1976 Olympics, Vasili Alexeev astounded the world by lifting a record-breaking 562 lb from the floor to above his head (about 2 m). Equally astounding was the feat of strongman Paul Anderson, who in 1957 braced himself on the floor and used his back to lift 6270 lb of lead and automobile parts a distance of 1 cm. Who did more work?

Solution. To lift an object one must apply sufficient force to overcome the gravitational force that the Earth exerts on that object. The force that the Earth exerts on an object is that object’s weight; thus, in performing their feats, Alexeev applied a force of 562 lb over a distance of 2 m and Anderson applied a force of 6270 lb over a distance of 1 cm. Pounds are units in the BE system, meters are units in SI, and centimeters are units in the CGS system. We will need to decide on the measurement system we want to use and be consistent. Let us agree to use SI and express the work of the two men in joules. Using the conversion factor in Table 6.6.1 we obtain

$$562 \text{ lb} \approx 562 \text{ lb} \times 4.45 \text{ N/lb} \approx 2500 \text{ N}$$

$$6270 \text{ lb} \approx 6270 \text{ lb} \times 4.45 \text{ N/lb} \approx 27,900 \text{ N}$$



Vasili Alexeev shown lifting a record-breaking 562 lb in the 1976 Olympics. In eight successive years he won Olympic gold medals, captured six world championships, and broke 80 world records. In 1999 he was honored in Greece as the best sportsman of the 20th Century.

Using these values and the fact that $1 \text{ cm} = 0.01 \text{ m}$ we obtain

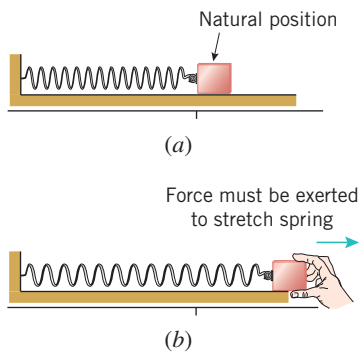
$$\text{Alexeev's work} = (2500 \text{ N}) \times (2 \text{ m}) = 5000 \text{ J}$$

$$\text{Anderson's work} = (27,900 \text{ N}) \times (0.01 \text{ m}) = 279 \text{ J}$$

Therefore, even though Anderson's lift required a tremendous upward force, it was applied over such a short distance that Alexeev did more work. ◀

WORK DONE BY A VARIABLE FORCE APPLIED IN THE DIRECTION OF MOTION

Many important problems are concerned with finding the work done by a *variable* force that is applied in the direction of motion. For example, Figure 6.6.2a shows a spring in its natural state (neither compressed nor stretched). If we want to pull the block horizontally (Figure 6.6.2b), then we would have to apply more and more force to the block to overcome the increasing force of the stretching spring. Thus, our next objective is to define what is meant by the work performed by a variable force and to find a formula for computing it. This will require calculus.



▲ Figure 6.6.2

6.6.2 PROBLEM Suppose that an object moves in the positive direction along a coordinate line while subjected to a variable force $F(x)$ that is applied in the direction of motion. Define what is meant by the *work* W performed by the force on the object as the object moves from $x = a$ to $x = b$, and find a formula for computing the work.

The basic idea for solving this problem is to break up the interval $[a, b]$ into subintervals that are sufficiently small that the force does not vary much on each subinterval. This will allow us to treat the force as constant on each subinterval and to approximate the work on each subinterval using Formula (1). By adding the approximations to the work on the subintervals, we will obtain a Riemann sum that approximates the work W over the entire interval, and by taking the limit of the Riemann sums we will obtain an integral for W .

To implement this idea, divide the interval $[a, b]$ into n subintervals by inserting points x_1, x_2, \dots, x_{n-1} between $a = x_0$ and $b = x_n$. We can use Formula (1) to approximate the work W_k done in the k th subinterval by choosing any point x_k^* in this interval and regarding the force to have a constant value $F(x_k^*)$ throughout the interval. Since the width of the k th subinterval is $x_k - x_{k-1} = \Delta x_k$, this yields the approximation

$$W_k \approx F(x_k^*) \Delta x_k$$

Adding these approximations yields the following Riemann sum that approximates the work W done over the entire interval:

$$W \approx \sum_{k=1}^n F(x_k^*) \Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$W = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n F(x_k^*) \Delta x_k = \int_a^b F(x) dx$$

In summary, we have the following result.

6.6.3 DEFINITION Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a variable force $F(x)$ that is applied in the direction of motion. Then we define the **work** W performed by the force on the object to be

$$W = \int_a^b F(x) dx \quad (2)$$

Hooke's law [Robert Hooke (1635–1703), English physicist] states that under appropriate conditions a spring that is stretched x units beyond its natural length pulls back with a force

$$F(x) = kx$$

where k is a constant (called the **spring constant** or **spring stiffness**). The value of k depends on such factors as the thickness of the spring and the material used in its composition. Since $k = F(x)/x$, the constant k has units of force per unit length.

► **Example 3** A spring exerts a force of 5 N when stretched 1 m beyond its natural length.

- (a) Find the spring constant k .
 (b) How much work is required to stretch the spring 1.8 m beyond its natural length?

Solution (a). From Hooke's law,

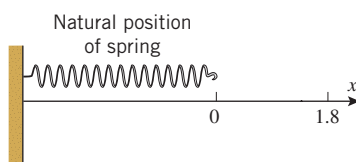
$$F(x) = kx$$

From the data, $F(x) = 5$ N when $x = 1$ m, so $5 = k \cdot 1$. Thus, the spring constant is $k = 5$ newtons per meter (N/m). This means that the force $F(x)$ required to stretch the spring x meters is

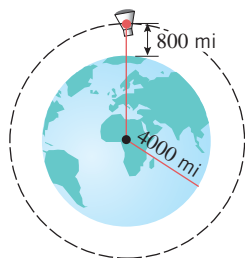
$$F(x) = 5x \quad (3)$$

Solution (b). Place the spring along a coordinate line as shown in Figure 6.6.3. We want to find the work W required to stretch the spring over the interval from $x = 0$ to $x = 1.8$. From (2) and (3) the work W required is

$$W = \int_a^b F(x) dx = \int_0^{1.8} 5x dx = \left. \frac{5x^2}{2} \right|_0^{1.8} = 8.1 \text{ J} \quad \blacktriangleleft$$



▲ Figure 6.6.3



▲ Figure 6.6.4

► **Example 4** An astronaut's *weight* (or more precisely, *Earth weight*) is the force exerted on the astronaut by the Earth's gravity. As the astronaut moves upward into space, the gravitational pull of the Earth decreases, and hence so does his or her weight. If the Earth is assumed to be a sphere of radius 4000 mi, then it can be shown using physics that an astronaut who weighs 150 lb on Earth will have a weight of

$$w(x) = \frac{2,400,000,000}{x^2} \text{ lb}, \quad x \geq 4000$$

at a distance of x mi from the Earth's center (Exercise 25). Use this formula to determine the work in foot-pounds required to lift the astronaut to a point that is 800 mi above the surface of the Earth (Figure 6.6.4).

Solution. Since the Earth has a radius of 4000 mi, the astronaut is lifted from a point that is 4000 mi from the Earth's center to a point that is 4800 mi from the Earth's center. Thus,

from (2), the work W required to lift the astronaut is

$$\begin{aligned}
 W &= \int_{4000}^{4800} \frac{2,400,000,000}{x^2} dx \\
 &= \left. -\frac{2,400,000,000}{x} \right|_{4000}^{4800} \\
 &= -500,000 + 600,000 \\
 &= 100,000 \text{ mile-pounds} \\
 &= (100,000 \text{ mi}\cdot\text{lb}) \times (5280 \text{ ft/mi}) \\
 &= 5.28 \times 10^8 \text{ ft}\cdot\text{lb} \quad \blacktriangleleft
 \end{aligned}$$

■ CALCULATING WORK FROM BASIC PRINCIPLES

Some problems cannot be solved by mechanically substituting into formulas, and one must return to basic principles to obtain solutions. This is illustrated in the next example.

► Example 5 Figure 6.6.5a shows a conical container of radius 10 ft and height 30 ft. Suppose that this container is filled with water to a depth of 15 ft. How much work is required to pump all of the water out through a hole in the top of the container?

Solution. Our strategy will be to divide the water into thin layers, approximate the work required to move each layer to the top of the container, add the approximations for the layers to obtain a Riemann sum that approximates the total work, and then take the limit of the Riemann sums to produce an integral for the total work.

To implement this idea, introduce an x -axis as shown in Figure 6.6.5a, and divide the water into n layers with Δx_k denoting the thickness of the k th layer. This division induces a partition of the interval $[15, 30]$ into n subintervals. Although the upper and lower surfaces of the k th layer are at different distances from the top, the difference will be small if the layer is thin, and we can reasonably assume that the entire layer is concentrated at a single point x_k^* (Figure 6.6.5a). Thus, the work W_k required to move the k th layer to the top of the container is approximately

$$W_k \approx F_k x_k^* \quad (4)$$

where F_k is the force required to lift the k th layer. But the force required to lift the k th layer is the force needed to overcome gravity, and this is the same as the weight of the layer. If the layer is very thin, we can approximate the volume of the k th layer with the volume of a cylinder of height Δx_k and radius r_k , where (by similar triangles)

$$\frac{r_k}{x_k^*} = \frac{10}{30} = \frac{1}{3}$$

or, equivalently, $r_k = x_k^*/3$ (Figure 6.6.5b). Therefore, the volume of the k th layer of water is approximately

$$\pi r_k^2 \Delta x_k = \pi (x_k^*/3)^2 \Delta x_k = \frac{\pi}{9} (x_k^*)^2 \Delta x_k$$

Since the weight density of water is 62.4 lb/ft^3 , it follows that

$$F_k \approx \frac{62.4\pi}{9} (x_k^*)^2 \Delta x_k$$

Thus, from (4)

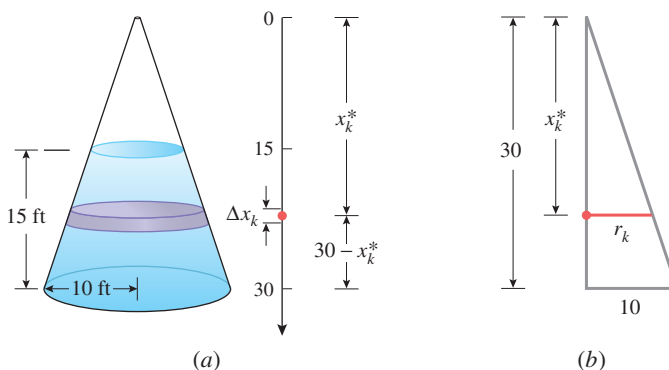
$$W_k \approx \left(\frac{62.4\pi}{9} (x_k^*)^2 \Delta x_k \right) x_k^* = \frac{62.4\pi}{9} (x_k^*)^3 \Delta x_k$$

and hence the work W required to move all n layers has the approximation

$$W = \sum_{k=1}^n W_k \approx \sum_{k=1}^n \frac{62.4\pi}{9} (x_k^*)^3 \Delta x_k$$

To find the *exact* value of the work we take the limit as $\max \Delta x_k \rightarrow 0$. This yields

$$\begin{aligned} W &= \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n \frac{62.4\pi}{9} (x_k^*)^3 \Delta x_k = \int_{15}^{30} \frac{62.4\pi}{9} x^3 dx \\ &= \frac{62.4\pi}{9} \left(\frac{x^4}{4} \right) \Big|_{15}^{30} = 1,316,250\pi \approx 4,135,000 \text{ ft}\cdot\text{lb} \quad \blacktriangleleft \end{aligned}$$



► Figure 6.6.5

THE WORK-ENERGY RELATIONSHIP



Mike Brinson/Getty Images

The work performed by the skater's stick in a brief interval of time produces the blinding speed of the hockey puck.

When you see an object in motion, you can be certain that somehow work has been expended to create that motion. For example, when you drop a stone from a building, the stone gathers speed because the force of the Earth's gravity is performing work on it, and when a hockey player strikes a puck with a hockey stick, the work performed on the puck during the brief period of contact with the stick creates the enormous speed of the puck across the ice. However, experience shows that the speed obtained by an object depends not only on the amount of work done, but also on the mass of the object. For example, the work required to throw a 5 oz baseball 50 mi/h would accelerate a 10 lb bowling ball to less than 9 mi/h.

Using the method of substitution for definite integrals, we will derive a simple equation that relates the work done on an object to the object's mass and velocity. Furthermore, this equation will allow us to motivate an appropriate definition for the "energy of motion" of an object. As in Definition 6.6.3, we will assume that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a force $F(x)$ that is applied in the direction of motion. We let m denote the mass of the object, and we let $x = x(t)$, $v = v(t) = x'(t)$, and $a = a(t) = v'(t)$ denote the respective position, velocity, and acceleration of the object at time t . We will need the following important result from physics that relates the force acting on an object with the mass and acceleration of the object.

6.6.4 NEWTON'S SECOND LAW OF MOTION If an object with mass m is subjected to a force F , then the object undergoes an acceleration a that satisfies the equation

$$F = ma \tag{5}$$

It follows from Newton's Second Law of Motion that

$$F(x(t)) = ma(t) = mv'(t)$$

Assume that

$$x(t_0) = a \quad \text{and} \quad x(t_1) = b$$

with

$$v(t_0) = v_i \quad \text{and} \quad v(t_1) = v_f$$

the initial and final velocities of the object, respectively. Then

$$\begin{aligned} W &= \int_a^b F(x) dx = \int_{x(t_0)}^{x(t_1)} F(x) dx \\ &= \int_{t_0}^{t_1} F(x(t))x'(t) dt && \text{By Theorem 5.9.1 with } x = x(t), dx = x'(t) dt \\ &= \int_{t_0}^{t_1} mv'(t)v(t) dt = \int_{t_0}^{t_1} mv(t)v'(t) dt \\ &= \int_{v(t_0)}^{v(t_1)} mv dv && \text{By Theorem 5.9.1 with } v = v(t), dv = v'(t) dt \\ &= \int_{v_i}^{v_f} mv dv = \frac{1}{2}mv^2 \Big|_{v_i}^{v_f} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \end{aligned}$$

We see from the equation

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \quad (6)$$

that the work done on the object is equal to the change in the quantity $\frac{1}{2}mv^2$ from its initial value to its final value. We will refer to Equation (6) as the **work–energy relationship**. If we define the “energy of motion” or **kinetic energy** of our object to be given by

$$K = \frac{1}{2}mv^2 \quad (7)$$

then Equation (6) tells us that the work done on an object is equal to the *change* in the object’s kinetic energy. Loosely speaking, we may think of work done on an object as being “transformed” into kinetic energy of the object. The units of kinetic energy are the same as the units of work. For example, in SI kinetic energy is measured in joules (J).

► **Example 6** A space probe of mass $m = 5.00 \times 10^4$ kg travels in deep space subjected only to the force of its own engine. Starting at a time when the speed of the probe is $v = 1.10 \times 10^4$ m/s, the engine is fired continuously over a distance of 2.50×10^6 m with a constant force of 4.00×10^5 N in the direction of motion. What is the final speed of the probe?

Solution. Since the force applied by the engine is constant and in the direction of motion, the work W expended by the engine on the probe is

$$W = \text{force} \times \text{distance} = (4.00 \times 10^5 \text{ N}) \times (2.50 \times 10^6 \text{ m}) = 1.00 \times 10^{12} \text{ J}$$

From (6), the final kinetic energy $K_f = \frac{1}{2}mv_f^2$ of the probe can be expressed in terms of the work W and the initial kinetic energy $K_i = \frac{1}{2}mv_i^2$ as

$$K_f = W + K_i$$

Thus, from the known mass and initial speed we have

$$K_f = (1.00 \times 10^{12} \text{ J}) + \frac{1}{2}(5.00 \times 10^4 \text{ kg})(1.10 \times 10^4 \text{ m/s})^2 = 4.025 \times 10^{12} \text{ J}$$

The final kinetic energy is $K_f = \frac{1}{2}mv_f^2$, so the final speed of the probe is

$$v_f = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2(4.025 \times 10^{12})}{5.00 \times 10^4}} \approx 1.27 \times 10^4 \text{ m/s} \quad \blacktriangleleft$$

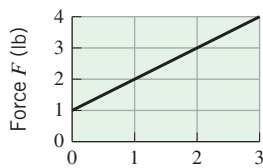
✓ QUICK CHECK EXERCISES 6.6 (See page 458 for answers.)

- If a constant force of 5 lb moves an object 10 ft, then the work done by the force on the object is _____.
- A newton-meter is also called a _____. A dyne-centimeter is also called an _____.
- Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$. The work performed on the object by a variable force $F(x)$ applied in the direction of motion is $W =$ _____.
- A force $F(x) = 10 - 2x$ N applied in the positive x -direction moves an object 3 m from $x = 2$ to $x = 5$. The work done by the force on the object is _____.

EXERCISE SET 6.6

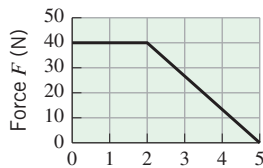
FOCUS ON CONCEPTS

- A variable force $F(x)$ in the positive x -direction is graphed in the accompanying figure. Find the work done by the force on a particle that moves from $x = 0$ to $x = 3$.



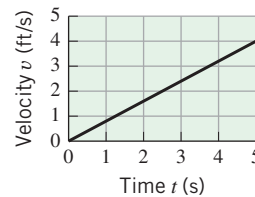
◀ Figure Ex-1

- A variable force $F(x)$ in the positive x -direction is graphed in the accompanying figure. Find the work done by the force on a particle that moves from $x = 0$ to $x = 5$.



◀ Figure Ex-2

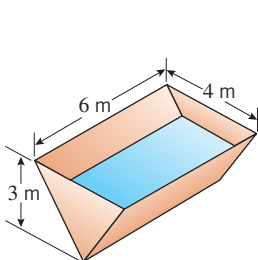
- For the variable force $F(x)$ in Exercise 2, consider the distance d for which the work done by the force on the particle when the particle moves from $x = 0$ to $x = d$ is half of the work done when the particle moves from $x = 0$ to $x = 5$. By inspecting the graph of F , is d more or less than 2.5? Explain, and then find the exact value of d .
- Suppose that a variable force $F(x)$ is applied in the positive x -direction so that an object moves from $x = a$ to $x = b$. Relate the work done by the force on the object and the average value of F over $[a, b]$, and illustrate this relationship graphically.
- A constant force of 10 lb in the positive x -direction is applied to a particle whose velocity versus time curve is shown in the accompanying figure. Find the work done by the force on the particle from time $t = 0$ to $t = 5$.



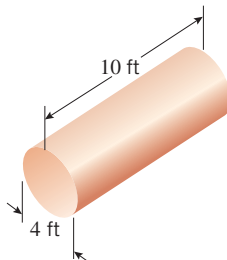
◀ Figure Ex-5

- A spring exerts a force of 6 N when it is stretched from its natural length of 4 m to a length of $4\frac{1}{2}$ m. Find the work required to stretch the spring from its natural length to a length of 6 m.
 - A spring exerts a force of 100 N when it is stretched 0.2 m beyond its natural length. How much work is required to stretch the spring 0.8 m beyond its natural length?
 - A spring whose natural length is 15 cm exerts a force of 45 N when stretched to a length of 20 cm.
 - Find the spring constant (in newtons/meter).
 - Find the work that is done in stretching the spring 3 cm beyond its natural length.
 - Find the work done in stretching the spring from a length of 20 cm to a length of 25 cm.
 - Assume that 10 ft-lb of work is required to stretch a spring 1 ft beyond its natural length. What is the spring constant?
- 10–13 True–False** Determine whether the statement is true or false. Explain your answer. ■
- In order to support the weight of a parked automobile, the surface of a driveway must do work against the force of gravity on the vehicle.
 - A force of 10 lb in the direction of motion of an object that moves 5 ft in 2 s does six times the work of a force of 10 lb in the direction of motion of an object that moves 5 ft in 12 s.
 - It follows from Hooke's law that in order to double the distance a spring is stretched beyond its natural length, four times as much work is required.
 - In the International System of Units, work and kinetic energy have the same units.

14. A cylindrical tank of radius 5 ft and height 9 ft is two-thirds filled with water. Find the work required to pump all the water over the upper rim.
15. Solve Exercise 14 assuming that the tank is half-filled with water.
16. A cone-shaped water reservoir is 20 ft in diameter across the top and 15 ft deep. If the reservoir is filled to a depth of 10 ft, how much work is required to pump all the water to the top of the reservoir?
17. The vat shown in the accompanying figure contains water to a depth of 2 m. Find the work required to pump all the water to the top of the vat. [Use 9810 N/m^3 as the weight density of water.]
18. The cylindrical tank shown in the accompanying figure is filled with a liquid weighing 50 lb/ft^3 . Find the work required to pump all the liquid to a level 1 ft above the top of the tank.



▲ Figure Ex-17



▲ Figure Ex-18

19. A swimming pool is built in the shape of a rectangular parallelepiped 10 ft deep, 15 ft wide, and 20 ft long.
- If the pool is filled to 1 ft below the top, how much work is required to pump all the water into a drain at the top edge of the pool?
 - A one-horsepower motor can do 550 ft·lb of work per second. What size motor is required to empty the pool in 1 hour?
20. How much work is required to fill the swimming pool in Exercise 19 to 1 ft below the top if the water is pumped in through an opening located at the bottom of the pool?
21. A 100 ft length of steel chain weighing 15 lb/ft is dangling from a pulley. How much work is required to wind the chain onto the pulley?
22. A 3 lb bucket containing 20 lb of water is hanging at the end of a 20 ft rope that weighs 4 oz/ft . The other end of the rope is attached to a pulley. How much work is required to wind the length of rope onto the pulley, assuming that the rope is wound onto the pulley at a rate of 2 ft/s and that as the bucket is being lifted, water leaks from the bucket at a rate of 0.5 lb/s ?
23. A rocket weighing 3 tons is filled with 40 tons of liquid fuel. In the initial part of the flight, fuel is burned off at a constant rate of 2 tons per 1000 ft of vertical height. How much work in foot-tons (ft·ton) is done lifting the rocket 3000 ft?
24. It follows from Coulomb's law in physics that two like electrostatic charges repel each other with a force inversely proportional to the square of the distance between them. Suppose that two charges A and B repel with a force of k newtons when they are positioned at points $A(-a, 0)$ and $B(a, 0)$, where a is measured in meters. Find the work W required to move charge A along the x -axis to the origin if charge B remains stationary.
25. It is a law of physics that the gravitational force exerted by the Earth on an object above the Earth's surface varies inversely as the square of its distance from the Earth's center. Thus, an object's weight $w(x)$ is related to its distance x from the Earth's center by a formula of the form
- $$w(x) = \frac{k}{x^2}$$
- where k is a constant of proportionality that depends on the mass of the object.
- Use this fact and the assumption that the Earth is a sphere of radius 4000 mi to obtain the formula for $w(x)$ in Example 4.
 - Find a formula for the weight $w(x)$ of a satellite that is x mi from the Earth's surface if its weight on Earth is 6000 lb.
 - How much work is required to lift the satellite from the surface of the Earth to an orbital position that is 1000 mi high?
26. (a) The formula $w(x) = k/x^2$ in Exercise 25 is applicable to all celestial bodies. Assuming that the Moon is a sphere of radius 1080 mi, find the force that the Moon exerts on an astronaut who is x mi from the surface of the Moon if her weight on the Moon's surface is 20 lb.
- (b) How much work is required to lift the astronaut to a point that is 10.8 mi above the Moon's surface?
27. The world's first commercial high-speed magnetic levitation (MAGLEV) train, a 30 km double-track project connecting Shanghai, China, to Pudong International Airport, began full revenue service in 2003. Suppose that a MAGLEV train has a mass $m = 4.00 \times 10^5 \text{ kg}$ and that starting at a time when the train has a speed of 20 m/s the engine applies a force of $6.40 \times 10^5 \text{ N}$ in the direction of motion over a distance of $3.00 \times 10^3 \text{ m}$. Use the work–energy relationship (6) to find the final speed of the train.
28. Assume that a Mars probe of mass $m = 2.00 \times 10^3 \text{ kg}$ is subjected only to the force of its own engine. Starting at a time when the speed of the probe is $v = 1.00 \times 10^4 \text{ m/s}$, the engine is fired continuously over a distance of $1.50 \times 10^5 \text{ m}$ with a constant force of $2.00 \times 10^5 \text{ N}$ in the direction of motion. Use the work–energy relationship (6) to find the final speed of the probe.
29. On August 10, 1972 a meteorite with an estimated mass of $4 \times 10^6 \text{ kg}$ and an estimated speed of 15 km/s skipped across the atmosphere above the western United States and Canada but fortunately did not hit the Earth. (cont.)

- (a) Assuming that the meteorite had hit the Earth with a speed of 15 km/s, what would have been its change in kinetic energy in joules (J)?
- (b) Express the energy as a multiple of the explosive energy of 1 megaton of TNT, which is 4.2×10^{15} J.
- (c) The energy associated with the Hiroshima atomic bomb was 13 kilotons of TNT. To how many such bombs would the meteorite impact have been equivalent?
30. **Writing** After reading Examples 3–5, a student classifies work problems as either “pushing/pulling” or “pumping.”

Describe these categories in your own words and discuss the methods used to solve each type. Give examples to illustrate that these categories are not mutually exclusive.

31. **Writing** How might you recognize that a problem can be solved by means of the work–energy relationship? That is, what sort of “givens” and “unknowns” would suggest such a solution? Discuss two or three examples.

✓ QUICK CHECK ANSWERS 6.6

1. 50 ft·lb 2. joule; erg 3. $\int_a^b F(x) dx$ 4. 9 J

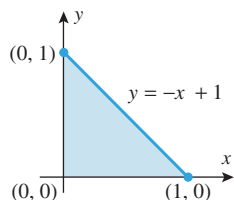
6.7 MOMENTS, CENTERS OF GRAVITY, AND CENTROIDS

Suppose that a rigid physical body is acted on by a constant gravitational field. Because the body is composed of many particles, each of which is affected by gravity, the action of the gravitational field on the body consists of a large number of forces distributed over the entire body. However, it is a fact of physics that these individual forces can be replaced by a single force acting at a point called the **center of gravity** of the body. In this section we will show how integrals can be used to locate centers of gravity.



▲ Figure 6.7.1

The units in Equation (1) are consistent since mass = (mass/area) \times area.



▲ Figure 6.7.2

■ DENSITY AND MASS OF A LAMINA

Let us consider an idealized flat object that is thin enough to be viewed as a two-dimensional plane region (Figure 6.7.1). Such an object is called a **lamina**. A lamina is called **homogeneous** if its composition is uniform throughout and **inhomogeneous** otherwise. We will consider homogeneous laminas in this section. Inhomogeneous laminas will be discussed in Chapter 14. The **density** of a **homogeneous** lamina is defined to be its mass per unit area. Thus, the density δ of a homogeneous lamina of mass M and area A is given by $\delta = M/A$. Notice that the mass M of a homogeneous lamina can be expressed as

$$M = \delta A \quad (1)$$

► **Example 1** A triangular lamina with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$ has density $\delta = 3$. Find its total mass.

Solution. Referring to (1) and Figure 6.7.2, the mass M of the lamina is

$$M = \delta A = 3 \cdot \frac{1}{2} = \frac{3}{2} \text{ (unit of mass) } \blacktriangleleft$$

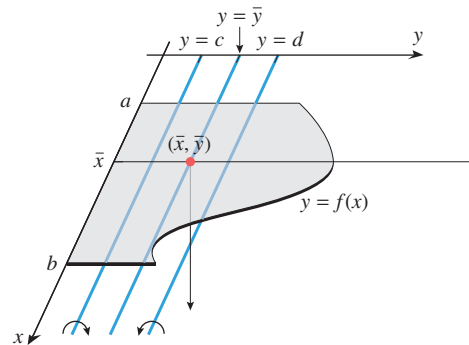
■ CENTER OF GRAVITY OF A LAMINA

Assume that the acceleration due to the force of gravity is constant and acts downward, and suppose that a lamina occupies a region R in a horizontal xy -plane. It can be shown that there exists a unique point (\bar{x}, \bar{y}) (which may or may not belong to R) such that the effect

of gravity on the lamina is “equivalent” to that of a single force acting at the point (\bar{x}, \bar{y}) . This point is called the **center of gravity** of the lamina, and if it is in R , then the lamina will balance horizontally on the point of a support placed at (\bar{x}, \bar{y}) . For example, the center of gravity of a homogeneous disk is at the center of the disk, and the center of gravity of a homogeneous rectangular region is at the center of the rectangle. For an irregularly shaped homogeneous lamina, locating the center of gravity requires calculus.

6.7.1 PROBLEM Let f be a positive continuous function on the interval $[a, b]$. Suppose that a homogeneous lamina with constant density δ occupies a region R in a horizontal xy -plane bounded by the graphs of $y = f(x)$, $y = 0$, $x = a$, and $x = b$. Find the coordinates (\bar{x}, \bar{y}) of the center of gravity of the lamina.

To motivate the solution, consider what happens if we try to balance the lamina on a knife-edge parallel to the x -axis. Suppose the lamina in Figure 6.7.3 is placed on a knife-edge along a line $y = c$ that does not pass through the center of gravity. Because the lamina behaves as if its entire mass is concentrated at the center of gravity (\bar{x}, \bar{y}) , the lamina will be rotationally unstable and the force of gravity will cause a rotation about $y = c$. Similarly, the lamina will undergo a rotation if placed on a knife-edge along $y = d$. However, if the knife-edge runs along the line $y = \bar{y}$ through the center of gravity, the lamina will be in perfect balance. Similarly, the lamina will be in perfect balance on a knife-edge along the line $x = \bar{x}$ through the center of gravity. This suggests that the center of gravity of a lamina can be determined as the intersection of two lines of balance, one parallel to the x -axis and the other parallel to the y -axis. In order to find these lines of balance, we will need some preliminary results about rotations.

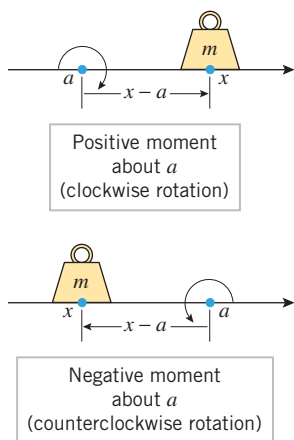


Force of gravity acting on the center of gravity of the lamina

► Figure 6.7.3

Children on a seesaw learn by experience that a lighter child can balance a heavier one by sitting farther from the fulcrum or pivot point. This is because the tendency for an object to produce rotation is proportional not only to its mass but also to the distance between the object and the fulcrum. To make this more precise, consider an x -axis, which we view as a weightless beam. If a mass m is located on the axis at x , then the tendency for that mass to produce a rotation of the beam about a point a on the axis is measured by the following quantity, called the **moment of m about $x = a$** :

$$\left[\begin{array}{l} \text{moment of } m \\ \text{about } a \end{array} \right] = m(x - a)$$



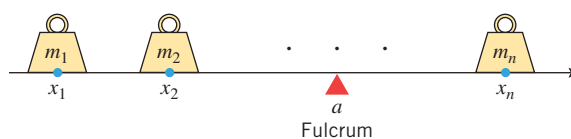
▲ Figure 6.7.4

The number $x - a$ is called the **lever arm**. Depending on whether the mass is to the right or left of a , the lever arm is either the distance between x and a or the negative of this distance (Figure 6.7.4). Positive lever arms result in positive moments and clockwise rotations, and negative lever arms result in negative moments and counterclockwise rotations.

Suppose that masses m_1, m_2, \dots, m_n are located at x_1, x_2, \dots, x_n on a coordinate axis and a fulcrum is positioned at the point a (Figure 6.7.5). Depending on whether the sum of the moments about a ,

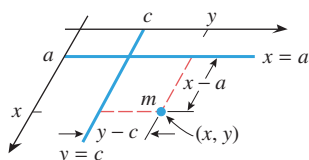
$$\sum_{k=1}^n m_k(x_k - a) = m_1(x_1 - a) + m_2(x_2 - a) + \dots + m_n(x_n - a)$$

is positive, negative, or zero, a weightless beam along the axis will rotate clockwise about a , rotate counterclockwise about a , or balance perfectly. In the last case, the system of masses is said to be in **equilibrium**.



► Figure 6.7.5

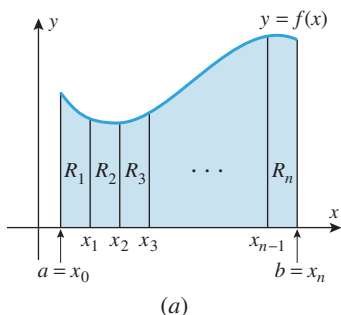
The preceding ideas can be extended to masses distributed in two-dimensional space. If we imagine the xy -plane to be a weightless sheet supporting a mass m located at a point (x, y) , then the tendency for the mass to produce a rotation of the sheet about the line $x = a$ is $m(x - a)$, called the **moment of m about $x = a$** , and the tendency for the mass to produce a rotation about the line $y = c$ is $m(y - c)$, called the **moment of m about $y = c$** (Figure 6.7.6). In summary,



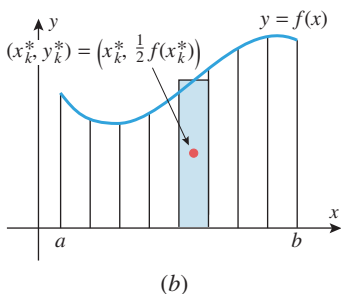
$$\left[\begin{array}{l} \text{moment of } m \\ \text{about the} \\ \text{line } x = a \end{array} \right] = m(x - a) \quad \text{and} \quad \left[\begin{array}{l} \text{moment of } m \\ \text{about the} \\ \text{line } y = c \end{array} \right] = m(y - c) \quad (2-3)$$

▲ Figure 6.7.6

If a number of masses are distributed throughout the xy -plane, then the plane (viewed as a weightless sheet) will balance on a knife-edge along the line $x = a$ if the sum of the moments about the line is zero. Similarly, the plane will balance on a knife-edge along the line $y = c$ if the sum of the moments about that line is zero.



We are now ready to solve Problem 6.7.1. The basic idea for solving this problem is to divide the lamina into strips whose areas may be approximated by the areas of rectangles. These area approximations, along with Formulas (2) and (3), will allow us to create a Riemann sum that approximates the moment of the lamina about a horizontal or vertical line. By taking the limit of Riemann sums we will then obtain an integral for the moment of a lamina about a horizontal or vertical line. We observe that since the lamina balances on the lines $x = \bar{x}$ and $y = \bar{y}$, the moment of the lamina about those lines should be zero. This observation will enable us to calculate \bar{x} and \bar{y} .



To implement this idea, we divide the interval $[a, b]$ into n subintervals by inserting the points x_1, x_2, \dots, x_{n-1} between $a = x_0$ and $b = x_n$. This has the effect of dividing the lamina R into n strips R_1, R_2, \dots, R_n (Figure 6.7.7a). Suppose that the k th strip extends from x_{k-1} to x_k and that the width of this strip is

$$\Delta x_k = x_k - x_{k-1}$$

We will let x_k^* be the midpoint of the k th subinterval and we will approximate R_k by a rectangle of width Δx_k and height $f(x_k^*)$. From (1), the mass ΔM_k of this rectangle is $\Delta M_k = \delta f(x_k^*) \Delta x_k$, and we will assume that the rectangle behaves as if its entire mass is concentrated at its center $(x_k^*, y_k^*) = (x_k^*, \frac{1}{2} f(x_k^*))$ (Figure 6.7.7b). It then follows from (2) and (3) that the moments of R_k about the lines $x = \bar{x}$ and $y = \bar{y}$ may be approximated

▲ Figure 6.7.7

by $(x_k^* - \bar{x})\Delta M_k$ and $(y_k^* - \bar{y})\Delta M_k$, respectively. Adding these approximations yields the following Riemann sums that approximate the moment of the entire lamina about the lines $x = \bar{x}$ and $y = \bar{y}$:

$$\begin{aligned} \sum_{k=1}^n (x_k^* - \bar{x})\Delta M_k &= \sum_{k=1}^n (x_k^* - \bar{x})\delta f(x_k^*)\Delta x_k \\ \sum_{k=1}^n (y_k^* - \bar{y})\Delta M_k &= \sum_{k=1}^n \left(\frac{f(x_k^*)}{2} - \bar{y} \right) \delta f(x_k^*)\Delta x_k \end{aligned}$$

Taking the limits as n increases and the widths of all the rectangles approach zero yields the definite integrals

$$\int_a^b (x - \bar{x})\delta f(x) dx \quad \text{and} \quad \int_a^b \left(\frac{f(x)}{2} - \bar{y} \right) \delta f(x) dx$$

that represent the moments of the lamina about the lines $x = \bar{x}$ and $y = \bar{y}$. Since the lamina balances on those lines, the moments of the lamina about those lines should be zero:

$$\int_a^b (x - \bar{x})\delta f(x) dx = \int_a^b \left(\frac{f(x)}{2} - \bar{y} \right) \delta f(x) dx = 0$$

Since \bar{x} and \bar{y} are constant, these equations can be rewritten as

$$\begin{aligned} \int_a^b \delta x f(x) dx &= \bar{x} \int_a^b \delta f(x) dx \\ \int_a^b \frac{1}{2} \delta (f(x))^2 dx &= \bar{y} \int_a^b \delta f(x) dx \end{aligned}$$

from which we obtain the following formulas for the center of gravity of the lamina:

Center of Gravity (\bar{x}, \bar{y}) of a Lamina

$$\bar{x} = \frac{\int_a^b \delta x f(x) dx}{\int_a^b \delta f(x) dx}, \quad \bar{y} = \frac{\int_a^b \frac{1}{2} \delta (f(x))^2 dx}{\int_a^b \delta f(x) dx} \tag{4-5}$$

Observe that in both formulas the denominator is the mass M of the lamina. The numerator in the formula for \bar{x} is denoted by M_y and is called the **first moment of the lamina about the y-axis**; the numerator of the formula for \bar{y} is denoted by M_x and is called the **first moment of the lamina about the x-axis**. Thus, we can write (4) and (5) as

Alternative Formulas for Center of Gravity (\bar{x}, \bar{y}) of a Lamina

$$\bar{x} = \frac{M_y}{M} = \frac{1}{\text{mass of } R} \int_a^b \delta x f(x) dx \tag{6}$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{\text{mass of } R} \int_a^b \frac{1}{2} \delta (f(x))^2 dx \tag{7}$$

► **Example 2** Find the center of gravity of the triangular lamina with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$ and density $\delta = 3$.

Solution. The lamina is shown in Figure 6.7.2. In Example 1 we found the mass of the lamina to be

$$M = \frac{3}{2}$$

The moment of the lamina about the y -axis is

$$\begin{aligned} M_y &= \int_0^1 \delta x f(x) dx = \int_0^1 3x(-x+1) dx \\ &= \int_0^1 (-3x^2 + 3x) dx = \left(-x^3 + \frac{3}{2}x^2 \right) \Big|_0^1 = -1 + \frac{3}{2} = \frac{1}{2} \end{aligned}$$

and the moment about the x -axis is

$$\begin{aligned} M_x &= \int_0^1 \frac{1}{2} \delta (f(x))^2 dx = \int_0^1 \frac{3}{2} (-x+1)^2 dx \\ &= \int_0^1 \frac{3}{2} (x^2 - 2x + 1) dx = \frac{3}{2} \left(\frac{1}{3}x^3 - x^2 + x \right) \Big|_0^1 = \frac{3}{2} \left(\frac{1}{3} \right) = \frac{1}{2} \end{aligned}$$

From (6) and (7),

$$\bar{x} = \frac{M_y}{M} = \frac{1/2}{3/2} = \frac{1}{3}, \quad \bar{y} = \frac{M_x}{M} = \frac{1/2}{3/2} = \frac{1}{3}$$

so the center of gravity is $(\frac{1}{3}, \frac{1}{3})$. ◀

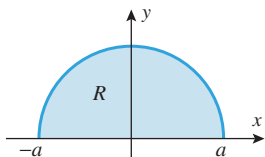
In the case of a *homogeneous* lamina, the center of gravity of a lamina occupying the region R is called the **centroid of the region R** . Since the lamina is homogeneous, δ is constant. The factor δ in (4) and (5) may thus be moved through the integral signs and canceled, and (4) and (5) can be expressed as

Since the density factor has canceled, we may interpret the centroid as a **geometric property** of the region, and distinguish it from the center of gravity, which is a **physical property** of an idealized object that occupies the region.

Centroid of a Region R

$$\bar{x} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} = \frac{1}{\text{area of } R} \int_a^b x f(x) dx \quad (8)$$

$$\bar{y} = \frac{\int_a^b \frac{1}{2} (f(x))^2 dx}{\int_a^b f(x) dx} = \frac{1}{\text{area of } R} \int_a^b \frac{1}{2} (f(x))^2 dx \quad (9)$$



▲ Figure 6.7.8

► **Example 3** Find the centroid of the semicircular region in Figure 6.7.8.

Solution. By symmetry, $\bar{x} = 0$ since the y -axis is obviously a line of balance. To find \bar{y} , first note that the equation of the semicircle is $y = f(x) = \sqrt{a^2 - x^2}$. From (9),

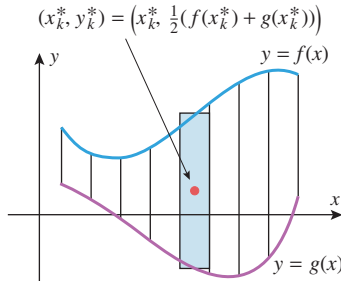
$$\begin{aligned} \bar{y} &= \frac{1}{\text{area of } R} \int_{-a}^a \frac{1}{2} (f(x))^2 dx = \frac{1}{\frac{1}{2}\pi a^2} \int_{-a}^a \frac{1}{2} (a^2 - x^2) dx \\ &= \frac{1}{\pi a^2} \left(a^2 x - \frac{1}{3} x^3 \right) \Big|_{-a}^a \\ &= \frac{1}{\pi a^2} \left[\left(a^3 - \frac{1}{3} a^3 \right) - \left(-a^3 + \frac{1}{3} a^3 \right) \right] \\ &= \frac{1}{\pi a^2} \left(\frac{4a^3}{3} \right) = \frac{4a}{3\pi} \end{aligned}$$

so the centroid is $(0, 4a/3\pi)$. ◀

■ OTHER TYPES OF REGIONS

The strategy used to find the center of gravity of the region in Problem 6.7.1 can be used to find the center of gravity of regions that are not of that form.

Consider a homogeneous lamina that occupies the region R between two continuous functions $f(x)$ and $g(x)$ over the interval $[a, b]$, where $f(x) \geq g(x)$ for $a \leq x \leq b$. To find the center of gravity of this lamina we can subdivide it into n strips using lines parallel to the y -axis. If x_k^* is the midpoint of the k th strip, the strip can be approximated by a rectangle of width Δx_k and height $f(x_k^*) - g(x_k^*)$. We assume that the entire mass of the k th rectangle is concentrated at its center $(x_k^*, y_k^*) = (x_k^*, \frac{1}{2}(f(x_k^*) + g(x_k^*)))$ (Figure 6.7.9). Continuing the argument as in the solution of Problem 6.7.1, we find that the center of gravity of the lamina is



▲ Figure 6.7.9

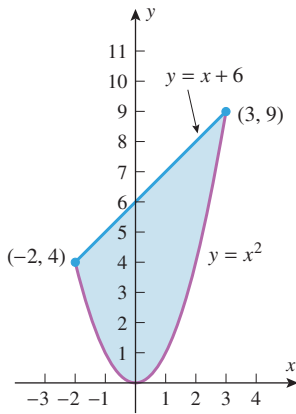
$$\bar{x} = \frac{\int_a^b x(f(x) - g(x)) dx}{\int_a^b (f(x) - g(x)) dx} = \frac{1}{\text{area of } R} \int_a^b x(f(x) - g(x)) dx \quad (10)$$

$$\bar{y} = \frac{\int_a^b \frac{1}{2} ([f(x)]^2 - [g(x)]^2) dx}{\int_a^b (f(x) - g(x)) dx} = \frac{1}{\text{area of } R} \int_a^b \frac{1}{2} ([f(x)]^2 - [g(x)]^2) dx \quad (11)$$

Note that the density of the lamina does not appear in Equations (10) and (11). This reflects the fact that the centroid is a geometric property of R .

► **Example 4** Find the centroid of the region R enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution. To begin, we note that the two curves intersect when $x = -2$ and $x = 3$ and that $x + 6 \geq x^2$ over that interval (Figure 6.7.10). The area of R is



▲ Figure 6.7.10

$$\int_{-2}^3 [(x + 6) - x^2] dx = \frac{125}{6}$$

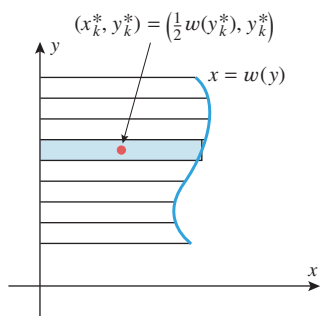
From (10) and (11),

$$\begin{aligned} \bar{x} &= \frac{1}{\text{area of } R} \int_{-2}^3 x[(x + 6) - x^2] dx \\ &= \frac{6}{125} \left(\frac{1}{3}x^3 + 3x^2 - \frac{1}{4}x^4 \right) \Big|_{-2}^3 \\ &= \frac{6}{125} \cdot \frac{125}{12} = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \bar{y} &= \frac{1}{\text{area of } R} \int_{-2}^3 \frac{1}{2} ((x + 6)^2 - (x^2)^2) dx \\ &= \frac{6}{125} \int_{-2}^3 \frac{1}{2} (x^2 + 12x + 36 - x^4) dx \\ &= \frac{6}{125} \cdot \frac{1}{2} \left(\frac{1}{3}x^3 + 6x^2 + 36x - \frac{1}{5}x^5 \right) \Big|_{-2}^3 \\ &= \frac{6}{125} \cdot \frac{250}{3} = 4 \end{aligned}$$

so the centroid of R is $(\frac{1}{2}, 4)$. ◀



▲ Figure 6.7.11

Suppose that w is a continuous function of y on an interval $[c, d]$ with $w(y) \geq 0$ for $c \leq y \leq d$. Consider a lamina that occupies a region R bounded above by $y = d$, below by $y = c$, on the left by the y -axis, and on the right by $x = w(y)$ (Figure 6.7.11). To find the center of gravity of this lamina, we note that the roles of x and y in Problem 6.7.1 have been reversed. We now imagine the lamina to be subdivided into n strips using lines parallel to the x -axis. We let y_k^* be the midpoint of the k th subinterval and approximate the strip by a rectangle of width Δy_k and height $w(y_k^*)$. We assume that the entire mass of the k th rectangle is concentrated at its center $(x_k^*, y_k^*) = (\frac{1}{2}w(y_k^*), y_k^*)$ (Figure 6.7.11). Continuing the argument as in the solution of Problem 6.7.1, we find that the center of gravity of the lamina is

$$\bar{x} = \frac{\int_c^d \frac{1}{2} (w(y))^2 dy}{\int_c^d w(y) dy} = \frac{1}{\text{area of } R} \int_c^d \frac{1}{2} (w(y))^2 dy \quad (12)$$

$$\bar{y} = \frac{\int_c^d y w(y) dy}{\int_c^d w(y) dy} = \frac{1}{\text{area of } R} \int_c^d y w(y) dy \quad (13)$$

Once again, the absence of the density in Equations (12) and (13) reflects the geometric nature of the centroid.

► **Example 5** Find the centroid of the region R enclosed between the curves $y = \sqrt{x}$, $y = 1$, $y = 2$, and the y -axis (Figure 6.7.12).

Solution. Note that $x = w(y) = y^2$ and that the area of R is

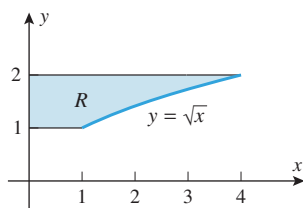
$$\int_1^2 y^2 dy = \frac{7}{3}$$

From (12) and (13),

$$\bar{x} = \frac{1}{\text{area of } R} \int_1^2 \frac{1}{2} (y^2)^2 dy = \frac{3}{7} \cdot \left. \frac{1}{10} y^5 \right|_1^2 = \frac{3}{7} \cdot \frac{31}{10} = \frac{93}{70}$$

$$\bar{y} = \frac{1}{\text{area of } R} \int_1^2 y (y^2) dy = \frac{3}{7} \cdot \left. \frac{1}{4} y^4 \right|_1^2 = \frac{3}{7} \cdot \frac{15}{4} = \frac{45}{28}$$

so the centroid of R is $(93/70, 45/28) \approx (1.329, 1.607)$. ◀



▲ Figure 6.7.12

THEOREM OF PAPPUS

The following theorem, due to the Greek mathematician Pappus, gives an important relationship between the centroid of a plane region R and the volume of the solid generated when the region is revolved about a line.

6.7.2 THEOREM (Theorem of Pappus) If R is a bounded plane region and L is a line that lies in the plane of R such that R is entirely on one side of L , then the volume of the solid formed by revolving R about L is given by

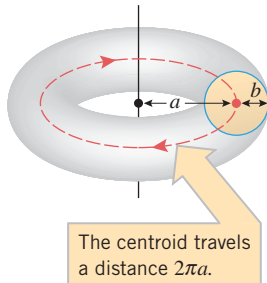
$$\text{volume} = (\text{area of } R) \cdot \left(\begin{array}{l} \text{distance traveled} \\ \text{by the centroid} \end{array} \right)$$

PROOF We prove this theorem in the special case where L is the y -axis, the region R is in the first quadrant, and the region R is of the form given in Problem 6.7.1. (A more general proof will be outlined in the Exercises of Section 14.8.) In this case, the volume V of the solid formed by revolving R about L can be found by the method of cylindrical shells (Section 6.3) to be

$$V = 2\pi \int_a^b x f(x) dx$$

Thus, it follows from (8) that $V = 2\pi\bar{x}[\text{area of } R]$

This completes the proof since $2\pi\bar{x}$ is the distance traveled by the centroid when R is revolved about the y -axis. ■



▲ Figure 6.7.13

► **Example 6** Use Pappus' Theorem to find the volume V of the torus generated by revolving a circular region of radius b about a line at a distance a (greater than b) from the center of the circle (Figure 6.7.13).

Solution. By symmetry, the centroid of a circular region is its center. Thus, the distance traveled by the centroid is $2\pi a$. Since the area of a circle of radius b is πb^2 , it follows from Pappus' Theorem that the volume of the torus is

$$V = (2\pi a)(\pi b^2) = 2\pi^2 ab^2 \quad \blacktriangleleft$$

✓ QUICK CHECK EXERCISES 6.7 (See page 467 for answers.)

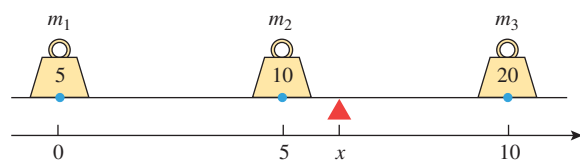
- The total mass of a homogeneous lamina of area A and density δ is _____.
- A homogeneous lamina of mass M and density δ occupies a region in the xy -plane bounded by the graphs of $y = f(x)$, $y = 0$, $x = a$, and $x = b$, where f is a nonnegative continuous function defined on an interval $[a, b]$. The x -coordinate of the center of gravity of the lamina is M_y/M , where M_y is called the _____ and is given by the integral _____.
- Let R be the region between the graphs of $y = x^2$ and $y = 2 - x$ for $0 \leq x \leq 1$. The area of R is $\frac{7}{6}$ and the centroid of R is _____.
- If the region R in Quick Check Exercise 3 is used to generate a solid G by rotating R about a horizontal line 6 units above its centroid, then the volume of G is _____.

EXERCISE SET 6.7 C CAS

FOCUS ON CONCEPTS

- Masses $m_1 = 5$, $m_2 = 10$, and $m_3 = 20$ are positioned on a weightless beam as shown in the accompanying figure.
 - Suppose that the fulcrum is positioned at $x = 5$. Without computing the sum of moments about 5, determine whether the sum is positive, zero, or negative. Explain.

(b) Where should the fulcrum be placed so that the beam is in equilibrium?



▲ Figure Ex-1

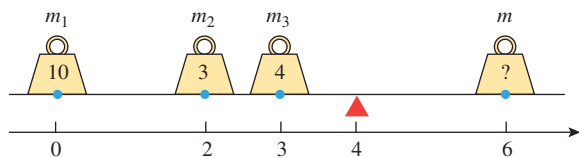
Pappus of Alexandria (4th century A.D.) Greek mathematician. Pappus lived during the early Christian era when mathematical activity was in a period of decline. His main contributions to mathematics appeared in a series of eight books called *The Collection* (written about 340 A.D.). This work, which survives only partially, contained some original results but was devoted mostly to state-

ments, refinements, and proofs of results by earlier mathematicians. Pappus' Theorem, stated without proof in Book VII of *The Collection*, was probably known and proved in earlier times. This result is sometimes called *Guldin's Theorem* in recognition of the Swiss mathematician, Paul Guldin (1577–1643), who rediscovered it independently.

2. Masses $m_1 = 10$, $m_2 = 3$, $m_3 = 4$, and m are positioned on a weightless beam, with the fulcrum positioned at point 4, as shown in the accompanying figure.

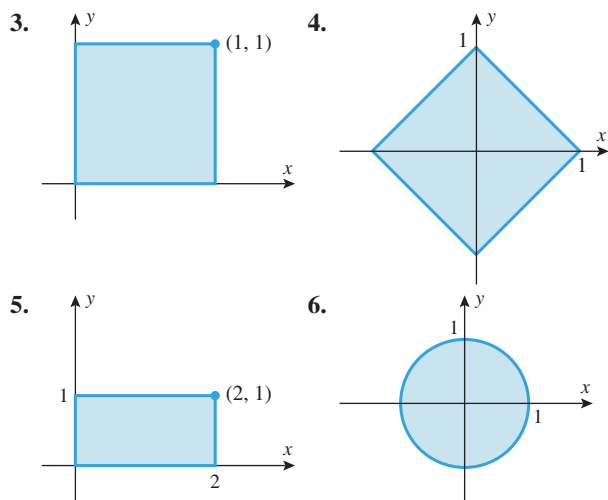
(a) Suppose that $m = 14$. Without computing the sum of the moments about 4, determine whether the sum is positive, zero, or negative. Explain.

(b) For what value of m is the beam in equilibrium?

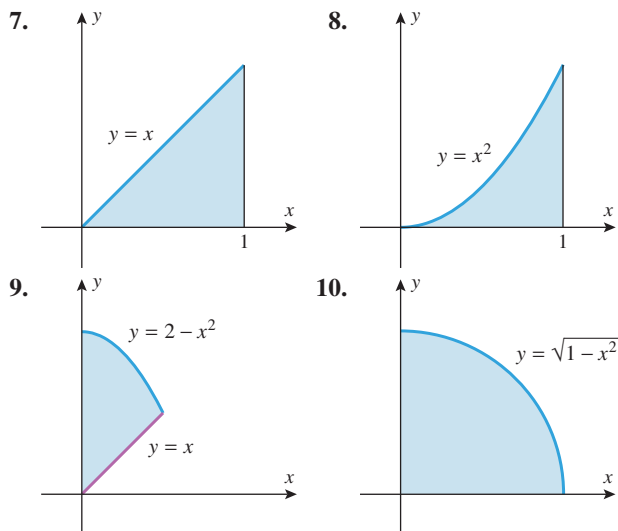


▲ Figure Ex-2

3–6 Find the centroid of the region by inspection and confirm your answer by integrating. ■



7–20 Find the centroid of the region. ■



11. The triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 1)$.

12. The triangle with vertices $(0, 0)$, $(1, 1)$, and $(2, 0)$.

13. The region bounded by the graphs of $y = x^2$ and $x + y = 6$.

14. The region bounded on the left by the y -axis, on the right by the line $x = 2$, below by the parabola $y = x^2$, and above by the line $y = x + 6$.

15. The region bounded by the graphs of $y = x^2$ and $y = x + 2$.

16. The region bounded by the graphs of $y = x^2$ and $y = 1$.

17. The region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$.

18. The region bounded by the graphs of $x = 1/y$, $x = 0$, $y = 1$, and $y = 2$.

19. The region bounded by the graphs of $y = x$, $x = 1/y^2$, and $y = 2$.

20. The region bounded by the graphs of $xy = 4$ and $x + y = 5$.

FOCUS ON CONCEPTS

21. Use symmetry considerations to argue that the centroid of an isosceles triangle lies on the median to the base of the triangle.
22. Use symmetry considerations to argue that the centroid of an ellipse lies at the intersection of the major and minor axes of the ellipse.

23–26 Find the mass and center of gravity of the lamina with density δ . ■

23. A lamina bounded by the x -axis, the line $x = 1$, and the curve $y = \sqrt{x}$; $\delta = 2$.

24. A lamina bounded by the graph of $x = y^4$ and the line $x = 1$; $\delta = 15$.

25. A lamina bounded by the graph of $y = |x|$ and the line $y = 1$; $\delta = 3$.

26. A lamina bounded by the x -axis and the graph of the equation $y = 1 - x^2$; $\delta = 3$.

© 27–30 Use a CAS to find the mass and center of gravity of the lamina with density δ . ■

27. A lamina bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$; $\delta = 4$.

28. A lamina bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$; $\delta = 1/(e - 1)$.

29. A lamina bounded by the graph of $y = \ln x$, the x -axis, and the line $x = 2$; $\delta = 1$.

30. A lamina bounded by the graphs of $y = \cos x$, $y = \sin x$, $x = 0$, and $x = \pi/4$; $\delta = 1 + \sqrt{2}$.

31–34 True-False Determine whether the statement is true or false. Explain your answer. [In Exercise 34, assume that the (rotated) square lies in the xy -plane to the right of the y -axis.] ■

31. The centroid of a rectangle is the intersection of the diagonals of the rectangle.
32. The centroid of a rhombus is the intersection of the diagonals of the rhombus.
33. The centroid of an equilateral triangle is the intersection of the medians of the triangle.
34. By rotating a square about its center, it is possible to change the volume of the solid of revolution generated by revolving the square about the y -axis.
35. Find the centroid of the triangle with vertices $(0, 0)$, (a, b) , and $(a, -b)$.
36. Prove that the centroid of a triangle is the point of intersection of the three medians of the triangle. [Hint: Choose coordinates so that the vertices of the triangle are located at $(0, -a)$, $(0, a)$, and (b, c) .]
37. Find the centroid of the isosceles trapezoid with vertices $(-a, 0)$, $(a, 0)$, $(-b, c)$, and (b, c) .
38. Prove that the centroid of a parallelogram is the point of intersection of the diagonals of the parallelogram. [Hint: Choose coordinates so that the vertices of the parallelogram are located at $(0, 0)$, $(0, a)$, (b, c) , and $(b, a + c)$.]
39. Use the Theorem of Pappus and the fact that the volume of a sphere of radius a is $V = \frac{4}{3}\pi a^3$ to show that the centroid of the lamina that is bounded by the x -axis and the semicircle $y = \sqrt{a^2 - x^2}$ is $(0, 4a/(3\pi))$. (This problem was solved directly in Example 3.)
40. Use the Theorem of Pappus and the result of Exercise 39 to find the volume of the solid generated when the region bounded by the x -axis and the semicircle $y = \sqrt{a^2 - x^2}$ is revolved about
 (a) the line $y = -a$ (b) the line $y = x - a$.
41. Use the Theorem of Pappus and the fact that the area of an ellipse with semiaxes a and b is πab to find the volume of the elliptical torus generated by revolving the ellipse
$$\frac{(x - k)^2}{a^2} + \frac{y^2}{b^2} = 1$$
 about the y -axis. Assume that $k > a$.
42. Use the Theorem of Pappus to find the volume of the solid that is generated when the region enclosed by $y = x^2$ and $y = 8 - x^2$ is revolved about the x -axis.
43. Use the Theorem of Pappus to find the centroid of the triangular region with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$, where $a > 0$ and $b > 0$. [Hint: Revolve the region about the x -axis to obtain \bar{y} and about the y -axis to obtain \bar{x} .]
44. **Writing** Suppose that a region R in the plane is decomposed into two regions R_1 and R_2 whose areas are A_1 and A_2 , respectively, and whose centroids are (\bar{x}_1, \bar{y}_1) and (\bar{x}_2, \bar{y}_2) , respectively. Investigate the problem of expressing the centroid of R in terms of A_1 , A_2 , (\bar{x}_1, \bar{y}_1) , and (\bar{x}_2, \bar{y}_2) . Write a short report on your investigations, supporting your reasoning with plausible arguments. Can you extend your results to decompositions of R into more than two regions?
45. **Writing** How might you recognize that a problem can be solved by means of the Theorem of Pappus? That is, what sort of “givens” and “unknowns” would suggest such a solution? Discuss two or three examples.

✓ QUICK CHECK ANSWERS 6.7

1. δA 2. first moment about the y -axis; $\int_a^b \delta x f(x) dx$ 3. $\left(\frac{5}{14}, \frac{32}{35}\right)$ 4. 14π

6.8 FLUID PRESSURE AND FORCE

In this section we will use the integration tools developed in the preceding chapter to study the pressures and forces exerted by fluids on submerged objects.

■ WHAT IS A FLUID?

A **fluid** is a substance that flows to conform to the boundaries of any container in which it is placed. Fluids include *liquids*, such as water, oil, and mercury, as well as *gases*, such as helium, oxygen, and air. The study of fluids falls into two categories: *fluid statics* (the study of fluids at rest) and *fluid dynamics* (the study of fluids in motion). In this section we will be concerned only with fluid statics; toward the end of this text we will investigate problems in fluid dynamics.

THE CONCEPT OF PRESSURE



Jupiter Images Corp.

Snowshoes prevent the woman from sinking by spreading her weight over a large area to reduce her pressure on the snow.

The effect that a force has on an object depends on how that force is spread over the surface of the object. For example, when you walk on soft snow with boots, the weight of your body crushes the snow and you sink into it. However, if you put on a pair of snowshoes to spread the weight of your body over a greater surface area, then the weight of your body has less of a crushing effect on the snow. The concept that accounts for both the magnitude of a force and the area over which it is applied is called *pressure*.

6.8.1 DEFINITION If a force of magnitude F is applied to a surface of area A , then we define the *pressure* P exerted by the force on the surface to be

$$P = \frac{F}{A} \quad (1)$$

It follows from this definition that pressure has units of force per unit area. The most common units of pressure are newtons per square meter (N/m^2) in SI and pounds per square inch (lb/in^2) or pounds per square foot (lb/ft^2) in the BE system. As indicated in Table 6.8.1, one newton per square meter is called a *pascal* (Pa). A pressure of 1 Pa is quite small ($1 \text{ Pa} = 1.45 \times 10^{-4} \text{ lb}/\text{in}^2$), so in countries using SI, tire pressure gauges are usually calibrated in kilopascals (kPa), which is 1000 pascals.

Table 6.8.1

UNITS OF FORCE AND PRESSURE

SYSTEM	FORCE	÷	AREA	=	PRESSURE
SI	newton (N)		square meter (m^2)		pascal (Pa)
BE	pound (lb)		square foot (ft^2)		lb/ft^2
BE	pound (lb)		square inch (in^2)		lb/in^2 (psi)

CONVERSION FACTORS:

$$1 \text{ Pa} \approx 1.45 \times 10^{-4} \text{ lb}/\text{in}^2 \approx 2.09 \times 10^{-2} \text{ lb}/\text{ft}^2$$

$$1 \text{ lb}/\text{in}^2 \approx 6.89 \times 10^3 \text{ Pa} \quad 1 \text{ lb}/\text{ft}^2 \approx 47.9 \text{ Pa}$$



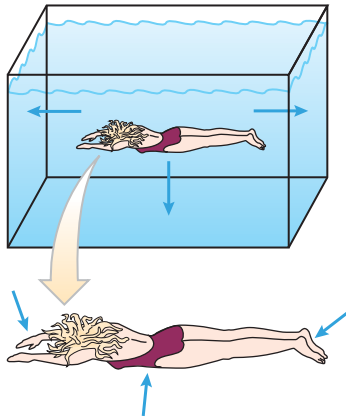
Blaise Pascal (1623–1662) French mathematician and scientist. Pascal's mother died when he was three years old and his father, a highly educated magistrate, personally provided the boy's early education. Although Pascal showed an inclination for science and mathematics, his father refused to tutor him in those subjects until he mastered

Latin and Greek. Pascal's sister and primary biographer claimed that he independently discovered the first thirty-two propositions of Euclid without ever reading a book on geometry. (However, it is generally agreed that the story is apocryphal.) Nevertheless, the precocious Pascal published a highly respected essay on conic sections by the time he was sixteen years old. Descartes, who read the essay, thought it so brilliant that he could not believe that it was written by such a young man. By age 18 his health began to fail and

until his death he was in frequent pain. However, his creativity was unimpaired.

Pascal's contributions to physics include the discovery that air pressure decreases with altitude and the principle of fluid pressure that bears his name. However, the originality of his work is questioned by some historians. Pascal made major contributions to a branch of mathematics called "projective geometry," and he helped to develop probability theory through a series of letters with Fermat.

In 1646, Pascal's health problems resulted in a deep emotional crisis that led him to become increasingly concerned with religious matters. Although born a Catholic, he converted to a religious doctrine called Jansenism and spent most of his final years writing on religion and philosophy.



Fluid forces always act perpendicular to the surface of a submerged object.

▲ Figure 6.8.1

In this section we will be interested in pressures and forces on objects submerged in fluids. Pressures themselves have no directional characteristics, but the forces that they create always act perpendicular to the face of the submerged object. Thus, in Figure 6.8.1 the water pressure creates horizontal forces on the sides of the tank, vertical forces on the bottom of the tank, and forces that vary in direction, so as to be perpendicular to the different parts of the swimmer’s body.

► **Example 1** Referring to Figure 6.8.1, suppose that the back of the swimmer’s hand has a surface area of $8.4 \times 10^{-3} \text{ m}^2$ and that the pressure acting on it is $5.1 \times 10^4 \text{ Pa}$ (a realistic value near the bottom of a deep diving pool). Find the force that acts on the swimmer’s hand.

Solution. From (1), the force F is

$$F = PA = (5.1 \times 10^4 \text{ N/m}^2)(8.4 \times 10^{-3} \text{ m}^2) \approx 4.3 \times 10^2 \text{ N}$$

This is quite a large force (nearly 100 lb in the BE system). ◀

Table 6.8.2

WEIGHT DENSITIES

SI	N/m ³
Machine oil	4708
Gasoline	6602
Fresh water	9810
Seawater	10,045
Mercury	133,416
BE SYSTEM	lb/ft ³
Machine oil	30.0
Gasoline	42.0
Fresh water	62.4
Seawater	64.0
Mercury	849.0

All densities are affected by variations in temperature and pressure. Weight densities are also affected by variations in g .

■ **FLUID DENSITY**

Scuba divers know that the pressure and forces on their bodies increase with the depth they dive. This is caused by the weight of the water and air above—the deeper the diver goes, the greater the weight above and so the greater the pressure and force exerted on the diver.

To calculate pressures and forces on submerged objects, we need to know something about the characteristics of the fluids in which they are submerged. For simplicity, we will assume that the fluids under consideration are *homogeneous*, by which we mean that any two samples of the fluid with the same volume have the same mass. It follows from this assumption that the mass per unit volume is a constant δ that depends on the physical characteristics of the fluid but not on the size or location of the sample; we call

$$\delta = \frac{m}{V} \tag{2}$$

the *mass density* of the fluid. Sometimes it is more convenient to work with weight per unit volume than with mass per unit volume. Thus, we define the *weight density* ρ of a fluid to be

$$\rho = \frac{w}{V} \tag{3}$$

where w is the weight of a fluid sample of volume V . Thus, if the weight density of a fluid is known, then the weight w of a fluid sample of volume V can be computed from the formula $w = \rho V$. Table 6.8.2 shows some typical weight densities.

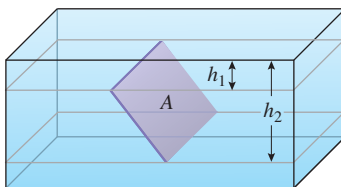
■ **FLUID PRESSURE**

To calculate fluid pressures and forces we will need to make use of an experimental observation. Suppose that a flat surface of area A is submerged in a homogeneous fluid of weight density ρ such that the entire surface lies between depths h_1 and h_2 , where $h_1 \leq h_2$ (Figure 6.8.2). Experiments show that on both sides of the surface, the fluid exerts a force that is perpendicular to the surface and whose magnitude F satisfies the inequalities

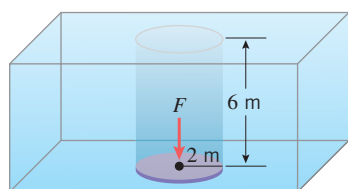
$$\rho h_1 A \leq F \leq \rho h_2 A \tag{4}$$

Thus, it follows from (1) that the pressure $P = F/A$ on a given side of the surface satisfies the inequalities

$$\rho h_1 \leq P \leq \rho h_2 \tag{5}$$



▲ Figure 6.8.2



The fluid force is the fluid pressure times the area.

▲ Figure 6.8.3

Note that it is now a straightforward matter to calculate fluid force and pressure on a flat surface that is submerged *horizontally* at depth h , for then $h = h_1 = h_2$ and inequalities (4) and (5) become the *equalities*

$$F = \rho h A \tag{6}$$

and

$$P = \rho h \tag{7}$$

► **Example 2** Find the fluid pressure and force on the top of a flat circular plate of radius 2 m that is submerged horizontally in water at a depth of 6 m (Figure 6.8.3).

Solution. Since the weight density of water is $\rho = 9810 \text{ N/m}^3$, it follows from (7) that the fluid pressure is

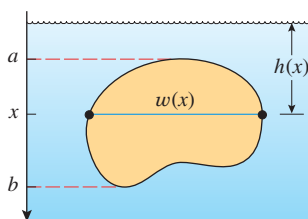
$$P = \rho h = (9810)(6) = 58,860 \text{ Pa}$$

and it follows from (6) that the fluid force is

$$F = \rho h A = \rho h(\pi r^2) = (9810)(6)(4\pi) = 235,440\pi \approx 739,700 \text{ N} \blacktriangleleft$$

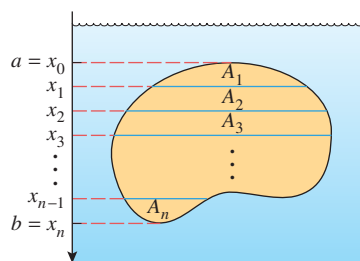
■ FLUID FORCE ON A VERTICAL SURFACE

It was easy to calculate the fluid force on the horizontal plate in Example 2 because each point on the plate was at the same depth. The problem of finding the fluid force on a vertical surface is more complicated because the depth, and hence the pressure, is not constant over the surface. To find the fluid force on a vertical surface we will need calculus.



(a)

6.8.2 PROBLEM Suppose that a flat surface is immersed vertically in a fluid of weight density ρ and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an x -axis whose positive direction is down (Figure 6.8.4a). For $a \leq x \leq b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point x . Define what is meant by the *fluid force* F on the surface, and find a formula for computing it.



(b)

The basic idea for solving this problem is to divide the surface into horizontal strips whose areas may be approximated by areas of rectangles. These area approximations, along with inequalities (4), will allow us to create a Riemann sum that approximates the total force on the surface. By taking a limit of Riemann sums we will then obtain an integral for F .

To implement this idea, we divide the interval $[a, b]$ into n subintervals by inserting the points x_1, x_2, \dots, x_{n-1} between $a = x_0$ and $b = x_n$. This has the effect of dividing the surface into n strips of area $A_k, k = 1, 2, \dots, n$ (Figure 6.8.4b). It follows from (4) that the force F_k on the k th strip satisfies the inequalities

$$\rho h(x_{k-1})A_k \leq F_k \leq \rho h(x_k)A_k$$

or, equivalently,

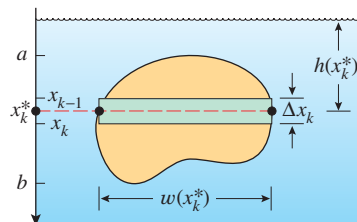
$$h(x_{k-1}) \leq \frac{F_k}{\rho A_k} \leq h(x_k)$$

Since the depth function $h(x)$ increases linearly, there must exist a point x_k^* between x_{k-1} and x_k such that

$$h(x_k^*) = \frac{F_k}{\rho A_k}$$

or, equivalently,

$$F_k = \rho h(x_k^*)A_k$$



(c)

▲ Figure 6.8.4

We now approximate the area A_k of the k th strip of the surface by the area of a rectangle of width $w(x_k^*)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.8.4c). It follows that F_k may be approximated as

$$F_k = \rho h(x_k^*) A_k \approx \rho h(x_k^*) \cdot \underbrace{w(x_k^*) \Delta x_k}_{\text{Area of rectangle}}$$

Adding these approximations yields the following Riemann sum that approximates the total force F on the surface:

$$F = \sum_{k=1}^n F_k \approx \sum_{k=1}^n \rho h(x_k^*) w(x_k^*) \Delta x_k$$

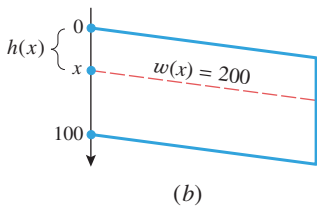
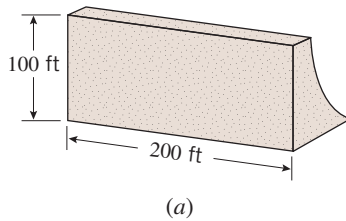
Taking the limit as n increases and the widths of all the subintervals approach zero yields the definite integral

$$F = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n \rho h(x_k^*) w(x_k^*) \Delta x_k = \int_a^b \rho h(x) w(x) dx$$

In summary, we have the following result.

6.8.3 DEFINITION Suppose that a flat surface is immersed vertically in a fluid of weight density ρ and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an x -axis whose positive direction is down (Figure 6.8.4a). For $a \leq x \leq b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point x . Then we define the **fluid force** F on the surface to be

$$F = \int_a^b \rho h(x) w(x) dx \tag{8}$$

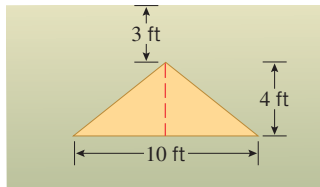


▲ Figure 6.8.5

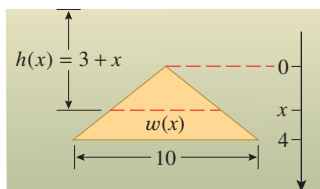
► **Example 3** The face of a dam is a vertical rectangle of height 100 ft and width 200 ft (Figure 6.8.5a). Find the total fluid force exerted on the face when the water surface is level with the top of the dam.

Solution. Introduce an x -axis with its origin at the water surface as shown in Figure 6.8.5b. At a point x on this axis, the width of the dam in feet is $w(x) = 200$ and the depth in feet is $h(x) = x$. Thus, from (8) with $\rho = 62.4 \text{ lb/ft}^3$ (the weight density of water) we obtain as the total force on the face

$$\begin{aligned} F &= \int_0^{100} (62.4)(x)(200) dx = 12,480 \int_0^{100} x dx \\ &= 12,480 \left. \frac{x^2}{2} \right|_0^{100} = 62,400,000 \text{ lb} \quad \blacktriangleleft \end{aligned}$$



(a)



(b)

▲ Figure 6.8.6

► **Example 4** A plate in the form of an isosceles triangle with base 10 ft and altitude 4 ft is submerged vertically in machine oil as shown in Figure 6.8.6a. Find the fluid force F against the plate surface if the oil has weight density $\rho = 30 \text{ lb/ft}^3$.

Solution. Introduce an x -axis as shown in Figure 6.8.6b. By similar triangles, the width of the plate, in feet, at a depth of $h(x) = (3 + x)$ ft satisfies

$$\frac{w(x)}{10} = \frac{x}{4}, \quad \text{so} \quad w(x) = \frac{5}{2}x$$

Thus, it follows from (8) that the force on the plate is

$$\begin{aligned} F &= \int_a^b \rho h(x)w(x) dx = \int_0^4 (30)(3+x) \left(\frac{5}{2}x\right) dx \\ &= 75 \int_0^4 (3x+x^2) dx = 75 \left[\frac{3x^2}{2} + \frac{x^3}{3} \right]_0^4 = 3400 \text{ lb} \quad \blacktriangleleft \end{aligned}$$

✓ QUICK CHECK EXERCISES 6.8 (See page 473 for answers.)

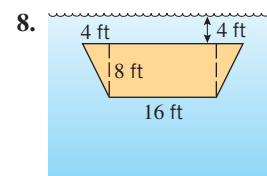
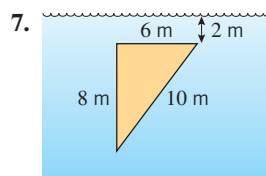
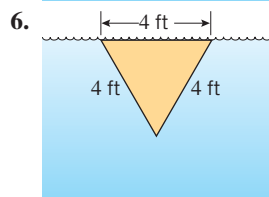
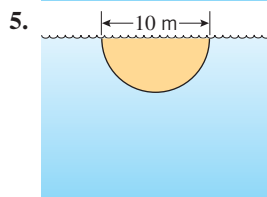
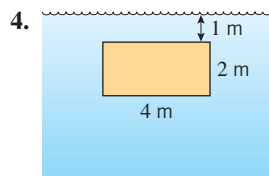
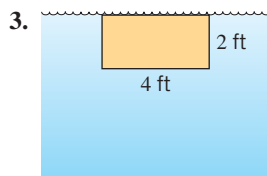
- The pressure unit equivalent to a newton per square meter (N/m^2) is called a _____. The pressure unit psi stands for _____.
- Given that the weight density of water is $9810 \text{ N}/\text{m}^3$, the fluid pressure on a rectangular $2 \text{ m} \times 3 \text{ m}$ flat plate submerged horizontally in water at a depth of 10 m is _____. The fluid force on the plate is _____.
- Suppose that a flat surface is immersed vertically in a fluid of weight density ρ and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an x -axis whose positive direction is down. If, for $a \leq x \leq b$, the surface has width $w(x)$ and depth $h(x)$, then the fluid force on the surface is $F =$ _____.
- A rectangular plate 2 m wide and 3 m high is submerged vertically in water so that the top of the plate is 5 m below the water surface. An integral expression for the force of the water on the plate surface is $F =$ _____.

EXERCISE SET 6.8

In this exercise set, refer to Table 6.8.2 for weight densities of fluids, where needed. ■

- A flat rectangular plate is submerged horizontally in water.
 - Find the force (in lb) and the pressure (in lb/ft^2) on the top surface of the plate if its area is 100 ft^2 and the surface is at a depth of 5 ft .
 - Find the force (in N) and the pressure (in Pa) on the top surface of the plate if its area is 25 m^2 and the surface is at a depth of 10 m .
- (a) Find the force (in N) on the deck of a sunken ship if its area is 160 m^2 and the pressure acting on it is $6.0 \times 10^5 \text{ Pa}$.
(b) Find the force (in lb) on a diver's face mask if its area is 60 in^2 and the pressure acting on it is $100 \text{ lb}/\text{in}^2$.

3–8 The flat surfaces shown are submerged vertically in water. Find the fluid force against each surface. ■



- Suppose that a flat surface is immersed vertically in a fluid of weight density ρ . If ρ is doubled, is the force on the plate also doubled? Explain your reasoning.
 - An oil tank is shaped like a right circular cylinder of diameter 4 ft . Find the total fluid force against one end when the axis is horizontal and the tank is half filled with oil of weight density $50 \text{ lb}/\text{ft}^3$.
 - A square plate of side a feet is dipped in a liquid of weight density $\rho \text{ lb}/\text{ft}^3$. Find the fluid force on the plate if a vertex is at the surface and a diagonal is perpendicular to the surface.
- 12–15 True–False** Determine whether the statement is true or false. Explain your answer. ■
- In the International System of Units, pressure and force have the same units.
 - In a cylindrical water tank (with vertical axis), the fluid force on the base of the tank is equal to the weight of water in the tank.
 - In a rectangular water tank, the fluid force on any side of the tank must be less than the fluid force on the base of the tank.

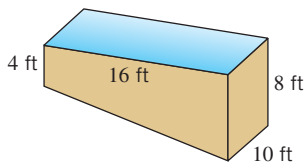
15. In any water tank with a flat base, no matter what the shape of the tank, the fluid force on the base is at most equal to the weight of water in the tank.

16–19 Formula (8) gives the fluid force on a flat surface immersed vertically in a fluid. More generally, if a flat surface is immersed so that it makes an angle of $0 \leq \theta < \pi/2$ with the vertical, then the fluid force on the surface is given by

$$F = \int_a^b \rho h(x)w(x) \sec \theta \, dx$$

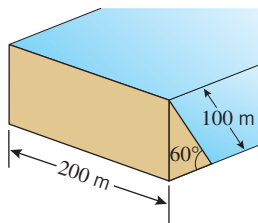
Use this formula in these exercises. ■

16. Derive the formula given above for the fluid force on a flat surface immersed at an angle in a fluid.
17. The accompanying figure shows a rectangular swimming pool whose bottom is an inclined plane. Find the fluid force on the bottom when the pool is filled to the top.



◀ Figure Ex-17

18. By how many feet should the water in the pool of Exercise 17 be lowered in order for the force on the bottom to be reduced by a factor of $\frac{1}{2}$?
19. The accompanying figure shows a dam whose face is an inclined rectangle. Find the fluid force on the face when the water is level with the top of this dam.



◀ Figure Ex-19

20. An observation window on a submarine is a square with 2 ft sides. Using ρ_0 for the weight density of seawater, find

the fluid force on the window when the submarine has descended so that the window is vertical and its top is at a depth of h feet.

FOCUS ON CONCEPTS

21. (a) Show: If the submarine in Exercise 20 descends vertically at a constant rate, then the fluid force on the window increases at a constant rate.
 (b) At what rate is the force on the window increasing if the submarine is descending vertically at 20 ft/min?
22. (a) Let $D = D_a$ denote a disk of radius a submerged in a fluid of weight density ρ such that the center of D is h units below the surface of the fluid. For each value of r in the interval $(0, a]$, let D_r denote the disk of radius r that is concentric with D . Select a side of the disk D and define $P(r)$ to be the fluid pressure on the chosen side of D_r . Use (5) to prove that

$$\lim_{r \rightarrow 0^+} P(r) = \rho h$$

- (b) Explain why the result in part (a) may be interpreted to mean that *fluid pressure at a given depth is the same in all directions*. (This statement is one version of a result known as **Pascal's Principle**.)

23. **Writing** Suppose that we model the Earth's atmosphere as a "fluid." Atmospheric pressure at sea level is $P = 14.7$ lb/in² and the weight density of air at sea level is about $\rho = 4.66 \times 10^{-5}$ lb/in³. With these numbers, what would Formula (7) yield as the height of the atmosphere above the Earth? Do you think this answer is reasonable? If not, explain how we might modify our assumptions to yield a more plausible answer.
24. **Writing** Suppose that the weight density ρ of a fluid is a function $\rho = \rho(x)$ of the depth x within the fluid. How do you think that Formula (7) for fluid pressure will need to be modified? Support your answer with plausible arguments.

✓ QUICK CHECK ANSWERS 6.8

1. pascal; pounds per square inch 2. 98,100 Pa; 588,600 N 3. $\int_a^b \rho h(x)w(x) \, dx$ 4. $\int_0^3 9810[(5+x)2] \, dx$

6.9 HYPERBOLIC FUNCTIONS AND HANGING CABLES

In this section we will study certain combinations of e^x and e^{-x} , called “hyperbolic functions.” These functions, which arise in various engineering applications, have many properties in common with the trigonometric functions. This similarity is somewhat surprising, since there is little on the surface to suggest that there should be any relationship between exponential and trigonometric functions. This is because the relationship occurs within the context of complex numbers, a topic which we will leave for more advanced courses.

DEFINITIONS OF HYPERBOLIC FUNCTIONS

To introduce the hyperbolic functions, observe from Exercise 61 in Section 0.2 that the function e^x can be expressed in the following way as the sum of an even function and an odd function:

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{Even}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{Odd}}$$

These functions are sufficiently important that there are names and notation associated with them: the odd function is called the *hyperbolic sine* of x and the even function is called the *hyperbolic cosine* of x . They are denoted by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

where \sinh is pronounced “cinch” and \cosh rhymes with “gosh.” From these two building blocks we can create four more functions to produce the following set of six *hyperbolic functions*.

6.9.1 DEFINITION

Hyperbolic sine	$\sinh x = \frac{e^x - e^{-x}}{2}$
Hyperbolic cosine	$\cosh x = \frac{e^x + e^{-x}}{2}$
Hyperbolic tangent	$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
Hyperbolic cotangent	$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
Hyperbolic secant	$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
Hyperbolic cosecant	$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

The terms “tanh,” “sech,” and “csch” are pronounced “tanch,” “seech,” and “coseech,” respectively.

TECHNOLOGY MASTERY

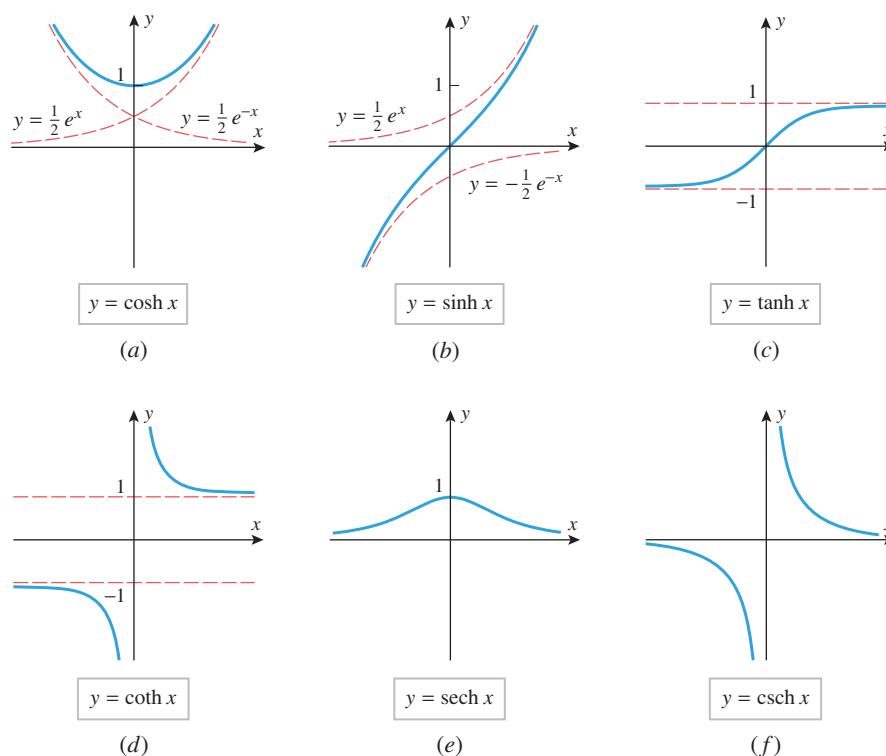
Computer algebra systems have built-in capabilities for evaluating hyperbolic functions directly, but some calculators do not. However, if you need to evaluate a hyperbolic function on a calculator, you can do so by expressing it in terms of exponential functions, as in Example 1.

► Example 1

$$\begin{aligned} \sinh 0 &= \frac{e^0 - e^{-0}}{2} = \frac{1 - 1}{2} = 0 \\ \cosh 0 &= \frac{e^0 + e^{-0}}{2} = \frac{1 + 1}{2} = 1 \\ \sinh 2 &= \frac{e^2 - e^{-2}}{2} \approx 3.6269 \quad \blacktriangleleft \end{aligned}$$

■ GRAPHS OF THE HYPERBOLIC FUNCTIONS

The graphs of the hyperbolic functions, which are shown in Figure 6.9.1, can be generated with a graphing utility, but it is worthwhile to observe that the general shape of the graph of $y = \cosh x$ can be obtained by sketching the graphs of $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ separately and adding the corresponding y -coordinates [see part (a) of the figure]. Similarly, the general shape of the graph of $y = \sinh x$ can be obtained by sketching the graphs of $y = \frac{1}{2}e^x$ and $y = -\frac{1}{2}e^{-x}$ separately and adding corresponding y -coordinates [see part (b) of the figure].



▲ Figure 6.9.1

Observe that $\sinh x$ has a domain of $(-\infty, +\infty)$ and a range of $(-\infty, +\infty)$, whereas $\cosh x$ has a domain of $(-\infty, +\infty)$ and a range of $[1, +\infty)$. Observe also that $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ are **curvilinear asymptotes** for $y = \cosh x$ in the sense that the graph of $y = \cosh x$ gets closer and closer to the graph of $y = \frac{1}{2}e^x$ as $x \rightarrow +\infty$ and gets closer and closer to the graph of $y = \frac{1}{2}e^{-x}$ as $x \rightarrow -\infty$. (See Section 4.3.) Similarly, $y = \frac{1}{2}e^x$ is a curvilinear asymptote for $y = \sinh x$ as $x \rightarrow +\infty$ and $y = -\frac{1}{2}e^{-x}$ is a curvilinear asymptote as $x \rightarrow -\infty$. Other properties of the hyperbolic functions are explored in the exercises.

■ HANGING CABLES AND OTHER APPLICATIONS

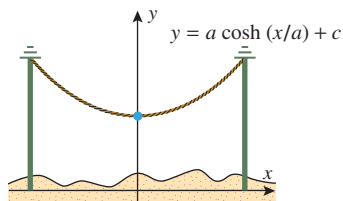
Hyperbolic functions arise in vibratory motions inside elastic solids and more generally in many problems where mechanical energy is gradually absorbed by a surrounding medium. They also occur when a homogeneous, flexible cable is suspended between two points, as with a telephone line hanging between two poles. Such a cable forms a curve, called a **catenary** (from the Latin *catena*, meaning “chain”). If, as in Figure 6.9.2, a coordinate system is introduced so that the low point of the cable lies on the y -axis, then it can be shown using principles of physics that the cable has an equation of the form

$$y = a \cosh\left(\frac{x}{a}\right) + c$$



Glen Allison/Stone/Getty Images

The design of the Gateway Arch near St. Louis is based on an inverted hyperbolic cosine curve (Exercise 73).



▲ Figure 6.9.2

where the parameters a and c are determined by the distance between the poles and the composition of the cable.

■ **HYPERBOLIC IDENTITIES**

The hyperbolic functions satisfy various identities that are similar to identities for trigonometric functions. The most fundamental of these is

$$\cosh^2 x - \sinh^2 x = 1 \tag{1}$$

which can be proved by writing

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= (\cosh x + \sinh x)(\cosh x - \sinh x) \\ &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \right) \\ &= e^x \cdot e^{-x} = 1 \end{aligned}$$

Other hyperbolic identities can be derived in a similar manner or, alternatively, by performing algebraic operations on known identities. For example, if we divide (1) by $\cosh^2 x$, we obtain

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

and if we divide (1) by $\sinh^2 x$, we obtain

$$\coth^2 x - 1 = \operatorname{csch}^2 x$$

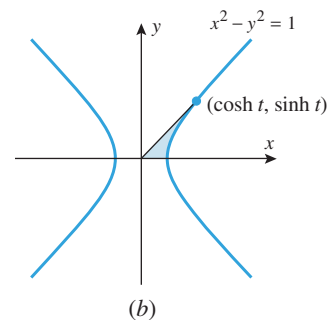
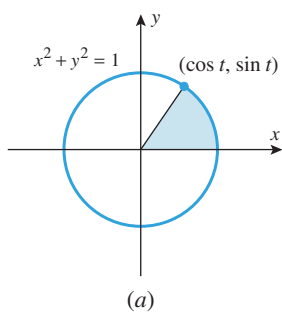
The following theorem summarizes some of the more useful hyperbolic identities. The proofs of those not already obtained are left as exercises.



Larry Auippy/Mira.com/Digital Railroad, Inc.
A flexible cable suspended between two poles forms a catenary.

6.9.2 THEOREM

$\cosh x + \sinh x = e^x$	$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
$\cosh x - \sinh x = e^{-x}$	$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
$\cosh^2 x - \sinh^2 x = 1$	$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\sinh 2x = 2 \sinh x \cosh x$
$\cosh(-x) = \cosh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\sinh(-x) = -\sinh x$	$\cosh 2x = 2 \sinh^2 x + 1 = 2 \cosh^2 x - 1$



▲ Figure 6.9.3

■ **WHY THEY ARE CALLED HYPERBOLIC FUNCTIONS**

Recall that the parametric equations

$$x = \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi)$$

represent the unit circle $x^2 + y^2 = 1$ (Figure 6.9.3a), as may be seen by writing

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

If $0 \leq t \leq 2\pi$, then the parameter t can be interpreted as the angle in radians from the positive x -axis to the point $(\cos t, \sin t)$ or, alternatively, as twice the shaded area of the sector in Figure 6.9.3a (verify). Analogously, the parametric equations

$$x = \cosh t, \quad y = \sinh t \quad (-\infty < t < +\infty)$$

represent a portion of the curve $x^2 - y^2 = 1$, as may be seen by writing

$$x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$$

and observing that $x = \cosh t > 0$. This curve, which is shown in Figure 6.9.3b, is the right half of a larger curve called the **unit hyperbola**; this is the reason why the functions in this section are called *hyperbolic* functions. It can be shown that if $t \geq 0$, then the parameter t can be interpreted as twice the shaded area in Figure 6.9.3b. (We omit the details.)

DERIVATIVE AND INTEGRAL FORMULAS

Derivative formulas for $\sinh x$ and $\cosh x$ can be obtained by expressing these functions in terms of e^x and e^{-x} :

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx} \left[\frac{e^x - e^{-x}}{2} \right] = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \frac{d}{dx} \left[\frac{e^x + e^{-x}}{2} \right] = \frac{e^x - e^{-x}}{2} = \sinh x$$

Derivatives of the remaining hyperbolic functions can be obtained by expressing them in terms of \sinh and \cosh and applying appropriate identities. For example,

$$\begin{aligned} \frac{d}{dx}[\tanh x] &= \frac{d}{dx} \left[\frac{\sinh x}{\cosh x} \right] = \frac{\cosh x \frac{d}{dx}[\sinh x] - \sinh x \frac{d}{dx}[\cosh x]}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \end{aligned}$$

The following theorem provides a complete list of the generalized derivative formulas and corresponding integration formulas for the hyperbolic functions.

6.9.3 THEOREM

$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx}$	$\int \cosh u \, du = \sinh u + C$
$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx}$	$\int \sinh u \, du = \cosh u + C$
$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx}$	$\int \operatorname{sech}^2 u \, du = \tanh u + C$
$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx}$	$\int \operatorname{csch}^2 u \, du = -\coth u + C$
$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx}$	$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$
$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx}$	$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$

► Example 2

$$\frac{d}{dx}[\cosh(x^3)] = \sinh(x^3) \cdot \frac{d}{dx}[x^3] = 3x^2 \sinh(x^3)$$

$$\frac{d}{dx}[\ln(\tanh x)] = \frac{1}{\tanh x} \cdot \frac{d}{dx}[\tanh x] = \frac{\operatorname{sech}^2 x}{\tanh x} \blacktriangleleft$$

► **Example 3**

$$\int \sinh^5 x \cosh x \, dx = \frac{1}{6} \sinh^6 x + C$$

$$\begin{array}{l} u = \sinh x \\ du = \cosh x \, dx \end{array}$$

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx$$

$$= \ln |\cosh x| + C$$

$$\begin{array}{l} u = \cosh x \\ du = \sinh x \, dx \end{array}$$

$$= \ln(\cosh x) + C$$

We were justified in dropping the absolute value signs since $\cosh x > 0$ for all x . ◀

► **Example 4** A 100 ft wire is attached at its ends to the tops of two 50 ft poles that are positioned 90 ft apart. How high above the ground is the middle of the wire?

Solution. From above, the wire forms a catenary curve with equation

$$y = a \cosh\left(\frac{x}{a}\right) + c$$

where the origin is on the ground midway between the poles. Using Formula (4) of Section 6.4 for the length of the catenary, we have

$$100 = \int_{-45}^{45} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= 2 \int_0^{45} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

By symmetry
about the y -axis

$$= 2 \int_0^{45} \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)} \, dx$$

$$= 2 \int_0^{45} \cosh\left(\frac{x}{a}\right) \, dx$$

By (1) and the fact
that $\cosh x > 0$

$$= 2a \sinh\left(\frac{x}{a}\right) \Big|_0^{45} = 2a \sinh\left(\frac{45}{a}\right)$$

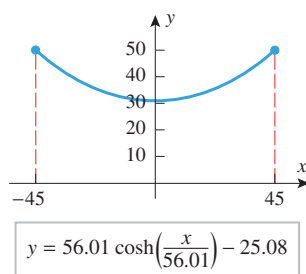
Using a calculating utility's numeric solver to solve

$$100 = 2a \sinh\left(\frac{45}{a}\right)$$

for a gives $a \approx 56.01$. Then

$$50 = y(45) = 56.01 \cosh\left(\frac{45}{56.01}\right) + c \approx 75.08 + c$$

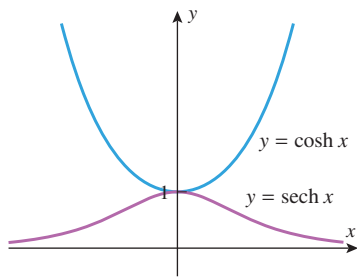
so $c \approx -25.08$. Thus, the middle of the wire is $y(0) \approx 56.01 - 25.08 = 30.93$ ft above the ground (Figure 6.9.4). ◀



▲ Figure 6.9.4

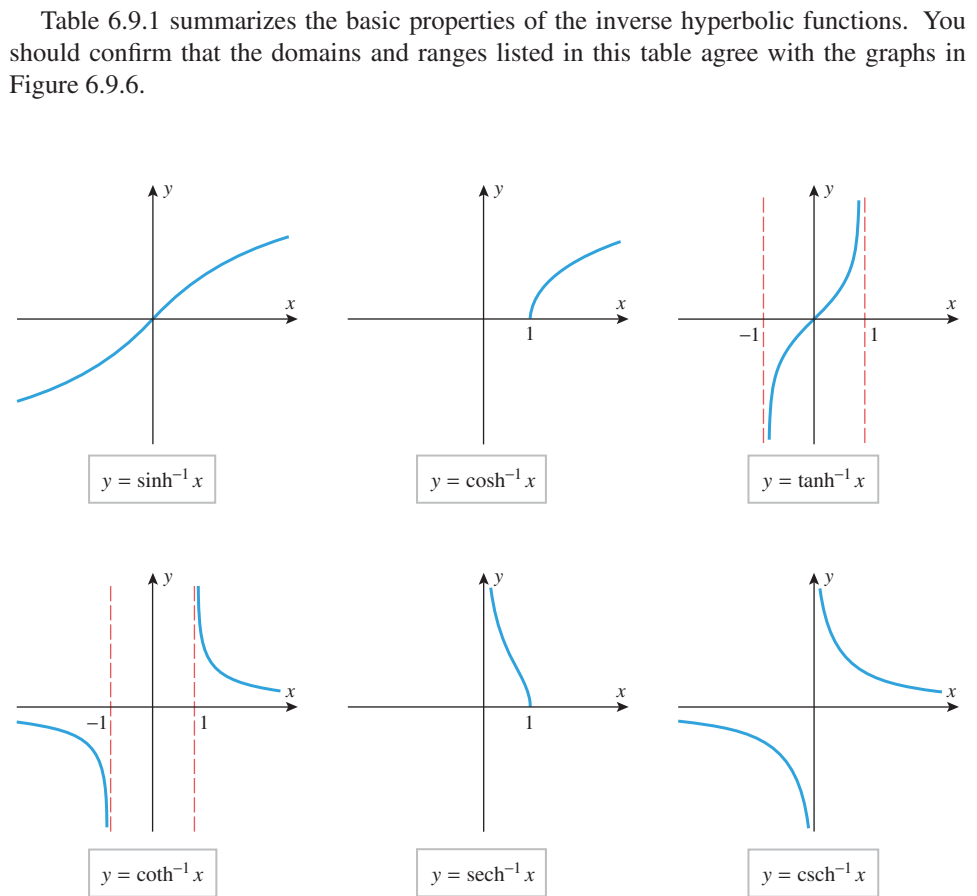
INVERSES OF HYPERBOLIC FUNCTIONS

Referring to Figure 6.9.1, it is evident that the graphs of $\sinh x$, $\tanh x$, $\coth x$, and $\operatorname{csch} x$ pass the horizontal line test, but the graphs of $\cosh x$ and $\operatorname{sech} x$ do not. In the latter case, restricting x to be nonnegative makes the functions invertible (Figure 6.9.5). The graphs of the six inverse hyperbolic functions in Figure 6.9.6 were obtained by reflecting the graphs of the hyperbolic functions (with the appropriate restrictions) about the line $y = x$.



With the restriction that $x \geq 0$, the curves $y = \cosh x$ and $y = \operatorname{sech} x$ pass the horizontal line test.

▲ Figure 6.9.5



► Figure 6.9.6

Table 6.9.1

PROPERTIES OF INVERSE HYPERBOLIC FUNCTIONS

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sinh^{-1} x$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$\sinh^{-1}(\sinh x) = x$ if $-\infty < x < +\infty$ $\sinh(\sinh^{-1} x) = x$ if $-\infty < x < +\infty$
$\cosh^{-1} x$	$[1, +\infty)$	$[0, +\infty)$	$\cosh^{-1}(\cosh x) = x$ if $x \geq 0$ $\cosh(\cosh^{-1} x) = x$ if $x \geq 1$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, +\infty)$	$\tanh^{-1}(\tanh x) = x$ if $-\infty < x < +\infty$ $\tanh(\tanh^{-1} x) = x$ if $-1 < x < 1$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\coth^{-1}(\coth x) = x$ if $x < 0$ or $x > 0$ $\coth(\coth^{-1} x) = x$ if $x < -1$ or $x > 1$
$\operatorname{sech}^{-1} x$	$(0, 1]$	$[0, +\infty)$	$\operatorname{sech}^{-1}(\operatorname{sech} x) = x$ if $x \geq 0$ $\operatorname{sech}(\operatorname{sech}^{-1} x) = x$ if $0 < x \leq 1$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\operatorname{csch}^{-1}(\operatorname{csch} x) = x$ if $x < 0$ or $x > 0$ $\operatorname{csch}(\operatorname{csch}^{-1} x) = x$ if $x < 0$ or $x > 0$

LOGARITHMIC FORMS OF INVERSE HYPERBOLIC FUNCTIONS

Because the hyperbolic functions are expressible in terms of e^x , it should not be surprising that the inverse hyperbolic functions are expressible in terms of natural logarithms; the next theorem shows that this is so.

6.9.4 THEOREM *The following relationships hold for all x in the domains of the stated inverse hyperbolic functions:*

$$\begin{aligned} \sinh^{-1} x &= \ln(x + \sqrt{x^2 + 1}) & \cosh^{-1} x &= \ln(x + \sqrt{x^2 - 1}) \\ \tanh^{-1} x &= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) & \coth^{-1} x &= \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right) \\ \operatorname{sech}^{-1} x &= \ln\left(\frac{1 + \sqrt{1-x^2}}{x}\right) & \operatorname{csch}^{-1} x &= \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right) \end{aligned}$$

We will show how to derive the first formula in this theorem and leave the rest as exercises. The basic idea is to write the equation $x = \sinh y$ in terms of exponential functions and solve this equation for y as a function of x . This will produce the equation $y = \sinh^{-1} x$ with $\sinh^{-1} x$ expressed in terms of natural logarithms. Expressing $x = \sinh y$ in terms of exponentials yields

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

which can be rewritten as

$$e^y - 2x - e^{-y} = 0$$

Multiplying this equation through by e^y we obtain

$$e^{2y} - 2xe^y - 1 = 0$$

and applying the quadratic formula yields

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Since $e^y > 0$, the solution involving the minus sign is extraneous and must be discarded. Thus,

$$e^y = x + \sqrt{x^2 + 1}$$

Taking natural logarithms yields

$$y = \ln(x + \sqrt{x^2 + 1}) \quad \text{or} \quad \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

► Example 5

$$\sinh^{-1} 1 = \ln(1 + \sqrt{1^2 + 1}) = \ln(1 + \sqrt{2}) \approx 0.8814$$

$$\tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2} \ln\left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}\right) = \frac{1}{2} \ln 3 \approx 0.5493 \quad \blacktriangleleft$$

DERIVATIVES AND INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

Show that the derivative of the function $\sinh^{-1} x$ can also be obtained by letting $y = \sinh^{-1} x$ and then differentiating $x = \sinh y$ implicitly.

Formulas for the derivatives of the inverse hyperbolic functions can be obtained from Theorem 6.9.4. For example,

$$\begin{aligned} \frac{d}{dx}[\sinh^{-1} x] &= \frac{d}{dx}[\ln(x + \sqrt{x^2 + 1})] = \frac{1}{x + \sqrt{x^2 + 1}} \left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \\ &= \frac{\sqrt{x^2 + 1} + x}{(x + \sqrt{x^2 + 1})(\sqrt{x^2 + 1})} = \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$

This computation leads to two integral formulas, a formula that involves $\sinh^{-1} x$ and an equivalent formula that involves logarithms:

$$\int \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1} x + C = \ln(x + \sqrt{x^2 + 1}) + C$$

The following two theorems list the generalized derivative formulas and corresponding integration formulas for the inverse hyperbolic functions. Some of the proofs appear as exercises.

6.9.5 THEOREM

$$\begin{aligned} \frac{d}{dx}(\sinh^{-1} u) &= \frac{1}{\sqrt{1+u^2}} \frac{du}{dx} & \frac{d}{dx}(\coth^{-1} u) &= \frac{1}{1-u^2} \frac{du}{dx}, \quad |u| > 1 \\ \frac{d}{dx}(\cosh^{-1} u) &= \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}, \quad u > 1 & \frac{d}{dx}(\operatorname{sech}^{-1} u) &= -\frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad 0 < u < 1 \\ \frac{d}{dx}(\tanh^{-1} u) &= \frac{1}{1-u^2} \frac{du}{dx}, \quad |u| < 1 & \frac{d}{dx}(\operatorname{csch}^{-1} u) &= -\frac{1}{|u|\sqrt{1+u^2}} \frac{du}{dx}, \quad u \neq 0 \end{aligned}$$

6.9.6 THEOREM If $a > 0$, then

$$\begin{aligned} \int \frac{du}{\sqrt{a^2 + u^2}} &= \sinh^{-1} \left(\frac{u}{a}\right) + C \quad \text{or} \quad \ln(u + \sqrt{u^2 + a^2}) + C \\ \int \frac{du}{\sqrt{u^2 - a^2}} &= \cosh^{-1} \left(\frac{u}{a}\right) + C \quad \text{or} \quad \ln(u + \sqrt{u^2 - a^2}) + C, \quad u > a \\ \int \frac{du}{a^2 - u^2} &= \begin{cases} \frac{1}{a} \tanh^{-1} \left(\frac{u}{a}\right) + C, & |u| < a \\ \frac{1}{a} \coth^{-1} \left(\frac{u}{a}\right) + C, & |u| > a \end{cases} \quad \text{or} \quad \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + C, \quad |u| \neq a \\ \int \frac{du}{u\sqrt{a^2 - u^2}} &= -\frac{1}{a} \operatorname{sech}^{-1} \left| \frac{u}{a} \right| + C \quad \text{or} \quad -\frac{1}{a} \ln \left(\frac{a + \sqrt{a^2 - u^2}}{|u|} \right) + C, \quad 0 < |u| < a \\ \int \frac{du}{u\sqrt{a^2 + u^2}} &= -\frac{1}{a} \operatorname{csch}^{-1} \left| \frac{u}{a} \right| + C \quad \text{or} \quad -\frac{1}{a} \ln \left(\frac{a + \sqrt{a^2 + u^2}}{|u|} \right) + C, \quad u \neq 0 \end{aligned}$$

► **Example 6** Evaluate $\int \frac{dx}{\sqrt{4x^2 - 9}}$, $x > \frac{3}{2}$.

Solution. Let $u = 2x$. Thus, $du = 2 dx$ and

$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2 - 9}} &= \frac{1}{2} \int \frac{2 dx}{\sqrt{4x^2 - 9}} = \frac{1}{2} \int \frac{du}{\sqrt{u^2 - 3^2}} \\ &= \frac{1}{2} \cosh^{-1} \left(\frac{u}{3} \right) + C = \frac{1}{2} \cosh^{-1} \left(\frac{2x}{3} \right) + C\end{aligned}$$

Alternatively, we can use the logarithmic equivalent of $\cosh^{-1}(2x/3)$,

$$\cosh^{-1} \left(\frac{2x}{3} \right) = \ln(2x + \sqrt{4x^2 - 9}) - \ln 3$$

(verify), and express the answer as

$$\int \frac{dx}{\sqrt{4x^2 - 9}} = \frac{1}{2} \ln(2x + \sqrt{4x^2 - 9}) + C \blacktriangleleft$$

 **QUICK CHECK EXERCISES 6.9** (See page 485 for answers.)

1. $\cosh x = \underline{\hspace{2cm}}$ $\sinh x = \underline{\hspace{2cm}}$
 $\tanh x = \underline{\hspace{2cm}}$

2. Complete the table.

	$\cosh x$	$\sinh x$	$\tanh x$	$\coth x$	$\operatorname{sech} x$	$\operatorname{csch} x$
DOMAIN						
RANGE						

3. The parametric equations

$$x = \cosh t, \quad y = \sinh t \quad (-\infty < t < +\infty)$$

represent the right half of the curve called a . Eliminating the parameter, the equation of this curve is .

4. $\frac{d}{dx}[\cosh x] = \underline{\hspace{2cm}}$ $\frac{d}{dx}[\sinh x] = \underline{\hspace{2cm}}$

$\frac{d}{dx}[\tanh x] = \underline{\hspace{2cm}}$

5. $\int \cosh x dx = \underline{\hspace{2cm}}$ $\int \sinh x dx = \underline{\hspace{2cm}}$

$\int \tanh x dx = \underline{\hspace{2cm}}$

6. $\frac{d}{dx}[\cosh^{-1} x] = \underline{\hspace{2cm}}$ $\frac{d}{dx}[\sinh^{-1} x] = \underline{\hspace{2cm}}$

$\frac{d}{dx}[\tanh^{-1} x] = \underline{\hspace{2cm}}$

EXERCISE SET 6.9  Graphing Utility

1–2 Approximate the expression to four decimal places. ■

1. (a) $\sinh 3$ (b) $\cosh(-2)$ (c) $\tanh(\ln 4)$
 (d) $\sinh^{-1}(-2)$ (e) $\cosh^{-1} 3$ (f) $\tanh^{-1} \frac{3}{4}$

2. (a) $\operatorname{csch}(-1)$ (b) $\operatorname{sech}(\ln 2)$ (c) $\coth 1$
 (d) $\operatorname{sech}^{-1} \frac{1}{2}$ (e) $\coth^{-1} 3$ (f) $\operatorname{csch}^{-1}(-\sqrt{3})$

3. Find the exact numerical value of each expression.

(a) $\sinh(\ln 3)$ (b) $\cosh(-\ln 2)$
 (c) $\tanh(2 \ln 5)$ (d) $\sinh(-3 \ln 2)$

4. In each part, rewrite the expression as a ratio of polynomials.

(a) $\cosh(\ln x)$ (b) $\sinh(\ln x)$
 (c) $\tanh(2 \ln x)$ (d) $\cosh(-\ln x)$

5. In each part, a value for one of the hyperbolic functions is given at an unspecified positive number x_0 . Use appropriate

identities to find the exact values of the remaining five hyperbolic functions at x_0 .

(a) $\sinh x_0 = 2$ (b) $\cosh x_0 = \frac{5}{4}$ (c) $\tanh x_0 = \frac{4}{5}$

6. Obtain the derivative formulas for $\operatorname{csch} x$, $\operatorname{sech} x$, and $\coth x$ from the derivative formulas for $\sinh x$, $\cosh x$, and $\tanh x$.

7. Find the derivatives of $\cosh^{-1} x$ and $\tanh^{-1} x$ by differentiating the formulas in Theorem 6.9.4.

8. Find the derivatives of $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$ by differentiating the equations $x = \sinh y$, $x = \cosh y$, and $x = \tanh y$ implicitly.

9–28 Find dy/dx . ■

9. $y = \sinh(4x - 8)$

10. $y = \cosh(x^4)$

11. $y = \coth(\ln x)$ 12. $y = \ln(\tanh 2x)$
 13. $y = \operatorname{csch}(1/x)$ 14. $y = \operatorname{sech}(e^{2x})$
 15. $y = \sqrt{4x + \cosh^2(5x)}$ 16. $y = \sinh^3(2x)$
 17. $y = x^3 \tanh^2(\sqrt{x})$ 18. $y = \sinh(\cos 3x)$
 19. $y = \sinh^{-1}(\frac{1}{3}x)$ 20. $y = \sinh^{-1}(1/x)$
 21. $y = \ln(\cosh^{-1} x)$ 22. $y = \cosh^{-1}(\sinh^{-1} x)$
 23. $y = \frac{1}{\tanh^{-1} x}$ 24. $y = (\coth^{-1} x)^2$
 25. $y = \cosh^{-1}(\cosh x)$ 26. $y = \sinh^{-1}(\tanh x)$
 27. $y = e^x \operatorname{sech}^{-1} \sqrt{x}$ 28. $y = (1 + x \operatorname{csch}^{-1} x)^{10}$

29–44 Evaluate the integrals. ■

29. $\int \sinh^6 x \cosh x \, dx$ 30. $\int \cosh(2x - 3) \, dx$
 31. $\int \sqrt{\tanh x} \operatorname{sech}^2 x \, dx$ 32. $\int \operatorname{csch}^2(3x) \, dx$
 33. $\int \tanh 2x \, dx$ 34. $\int \coth^2 x \operatorname{csch}^2 x \, dx$
 35. $\int_{\ln 2}^{\ln 3} \tanh x \operatorname{sech}^3 x \, dx$ 36. $\int_0^{\ln 3} \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$
 37. $\int \frac{dx}{\sqrt{1 + 9x^2}}$ 38. $\int \frac{dx}{\sqrt{x^2 - 2}} \quad (x > \sqrt{2})$
 39. $\int \frac{dx}{\sqrt{1 - e^{2x}}} \quad (x < 0)$ 40. $\int \frac{\sin \theta \, d\theta}{\sqrt{1 + \cos^2 \theta}}$
 41. $\int \frac{dx}{x\sqrt{1 + 4x^2}}$ 42. $\int \frac{dx}{\sqrt{9x^2 - 25}} \quad (x > 5/3)$
 43. $\int_0^{1/2} \frac{dx}{1 - x^2}$ 44. $\int_0^{\sqrt{3}} \frac{dt}{\sqrt{t^2 + 1}}$

45–48 True–False Determine whether the statement is true or false. Explain your answer. ■

45. The equation $\cosh x = \sinh x$ has no solutions.
 46. Exactly two of the hyperbolic functions are bounded.
 47. There is exactly one hyperbolic function $f(x)$ such that for all real numbers a , the equation $f(x) = a$ has a unique solution x .
 48. The identities in Theorem 6.9.2 may be obtained from the corresponding trigonometric identities by replacing each trigonometric function with its hyperbolic analogue.
 49. Find the area enclosed by $y = \sinh 2x$, $y = 0$, and $x = \ln 3$.
 50. Find the volume of the solid that is generated when the region enclosed by $y = \operatorname{sech} x$, $y = 0$, $x = 0$, and $x = \ln 2$ is revolved about the x -axis.
 51. Find the volume of the solid that is generated when the region enclosed by $y = \cosh 2x$, $y = \sinh 2x$, $x = 0$, and $x = 5$ is revolved about the x -axis.
 52. Approximate the positive value of the constant a such that the area enclosed by $y = \cosh ax$, $y = 0$, $x = 0$, and $x = 1$

is 2 square units. Express your answer to at least five decimal places.

53. Find the arc length of the catenary $y = \cosh x$ between $x = 0$ and $x = \ln 2$.
 54. Find the arc length of the catenary $y = a \cosh(x/a)$ between $x = 0$ and $x = x_1$ ($x_1 > 0$).
 55. In parts (a)–(f) find the limits, and confirm that they are consistent with the graphs in Figures 6.9.1 and 6.9.6.
 (a) $\lim_{x \rightarrow +\infty} \sinh x$ (b) $\lim_{x \rightarrow -\infty} \sinh x$
 (c) $\lim_{x \rightarrow +\infty} \tanh x$ (d) $\lim_{x \rightarrow -\infty} \tanh x$
 (e) $\lim_{x \rightarrow +\infty} \sinh^{-1} x$ (f) $\lim_{x \rightarrow 1^-} \tanh^{-1} x$

FOCUS ON CONCEPTS

56. Explain how to obtain the asymptotes for $y = \tanh x$ from the curvilinear asymptotes for $y = \cosh x$ and $y = \sinh x$.
 57. Prove that $\sinh x$ is an odd function of x and that $\cosh x$ is an even function of x , and check that this is consistent with the graphs in Figure 6.9.1.

58–59 Prove the identities. ■

58. (a) $\cosh x + \sinh x = e^x$
 (b) $\cosh x - \sinh x = e^{-x}$
 (c) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
 (d) $\sinh 2x = 2 \sinh x \cosh x$
 (e) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
 (f) $\cosh 2x = \cosh^2 x + \sinh^2 x$
 (g) $\cosh 2x = 2 \sinh^2 x + 1$
 (h) $\cosh 2x = 2 \cosh^2 x - 1$
 59. (a) $1 - \tanh^2 x = \operatorname{sech}^2 x$
 (b) $\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
 (c) $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$
 60. Prove:
 (a) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$
 (b) $\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$, $-1 < x < 1$.
 61. Use Exercise 60 to obtain the derivative formulas for $\cosh^{-1} x$ and $\tanh^{-1} x$.
 62. Prove:
 $\operatorname{sech}^{-1} x = \cosh^{-1}(1/x)$, $0 < x \leq 1$
 $\coth^{-1} x = \tanh^{-1}(1/x)$, $|x| > 1$
 $\operatorname{csch}^{-1} x = \sinh^{-1}(1/x)$, $x \neq 0$

63. Use Exercise 62 to express the integral

$$\int \frac{du}{1 - u^2}$$

entirely in terms of \tanh^{-1} .

64. Show that

$$(a) \frac{d}{dx}[\operatorname{sech}^{-1}|x|] = -\frac{1}{x\sqrt{1-x^2}}$$

$$(b) \frac{d}{dx}[\operatorname{csch}^{-1}|x|] = -\frac{1}{x\sqrt{1+x^2}}$$

65. In each part, find the limit.

$$(a) \lim_{x \rightarrow +\infty} (\cosh^{-1} x - \ln x) \quad (b) \lim_{x \rightarrow +\infty} \frac{\cosh x}{e^x}$$

66. Use the first and second derivatives to show that the graph of $y = \tanh^{-1} x$ is always increasing and has an inflection point at the origin.

67. The integration formulas for $1/\sqrt{u^2 - a^2}$ in Theorem 6.9.6 are valid for $u > a$. Show that the following formula is valid for $u < -a$:

$$\int \frac{du}{\sqrt{u^2 - a^2}} = -\cosh^{-1}\left(-\frac{u}{a}\right) + C \quad \text{or} \quad \ln\left|u + \sqrt{u^2 - a^2}\right| + C$$

68. Show that $(\sinh x + \cosh x)^n = \sinh nx + \cosh nx$.

69. Show that

$$\int_{-a}^a e^{tx} dx = \frac{2 \sinh at}{t}$$

70. A cable is suspended between two poles as shown in Figure 6.9.2. Assume that the equation of the curve formed by the cable is $y = a \cosh(x/a)$, where a is a positive constant. Suppose that the x -coordinates of the points of support are $x = -b$ and $x = b$, where $b > 0$.

(a) Show that the length L of the cable is given by

$$L = 2a \sinh \frac{b}{a}$$

(b) Show that the sag S (the vertical distance between the highest and lowest points on the cable) is given by

$$S = a \cosh \frac{b}{a} - a$$

71–72 These exercises refer to the hanging cable described in Exercise 70. ■

71. Assuming that the poles are 400 ft apart and the sag in the cable is 30 ft, approximate the length of the cable by approximating a . Express your final answer to the nearest tenth of a foot. [Hint: First let $u = 200/a$.]

72. Assuming that the cable is 120 ft long and the poles are 100 ft apart, approximate the sag in the cable by approximating a . Express your final answer to the nearest tenth of a foot. [Hint: First let $u = 50/a$.]

73. The design of the Gateway Arch in St. Louis, Missouri, by architect Eero Saarinen was implemented using equations provided by Dr. Hannskarl Badel. The equation used for the centerline of the arch was

$$y = 693.8597 - 68.7672 \cosh(0.0100333x) \text{ ft}$$

for x between -299.2239 and 299.2239 .

(a) Use a graphing utility to graph the centerline of the arch.

(b) Find the length of the centerline to four decimal places.

(c) For what values of x is the height of the arch 100 ft? Round your answers to four decimal places.

(d) Approximate, to the nearest degree, the acute angle that the tangent line to the centerline makes with the ground at the ends of the arch.

74. Suppose that a hollow tube rotates with a constant angular velocity of ω rad/s about a horizontal axis at one end of the tube, as shown in the accompanying figure. Assume that an object is free to slide without friction in the tube while the tube is rotating. Let r be the distance from the object to the pivot point at time $t \geq 0$, and assume that the object is at rest and $r = 0$ when $t = 0$. It can be shown that if the tube is horizontal at time $t = 0$ and rotating as shown in the figure, then

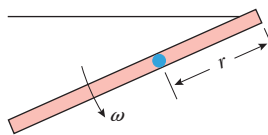
$$r = \frac{g}{2\omega^2} [\sinh(\omega t) - \sin(\omega t)]$$

during the period that the object is in the tube. Assume that t is in seconds and r is in meters, and use $g = 9.8 \text{ m/s}^2$ and $\omega = 2 \text{ rad/s}$.

(a) Graph r versus t for $0 \leq t \leq 1$.

(b) Assuming that the tube has a length of 1 m, approximately how long does it take for the object to reach the end of the tube?

(c) Use the result of part (b) to approximate dr/dt at the instant that the object reaches the end of the tube.



◀ Figure Ex-74

75. The accompanying figure (on the next page) shows a person pulling a boat by holding a rope of length a attached to the bow and walking along the edge of a dock. If we assume that the rope is always tangent to the curve traced by the bow of the boat, then this curve, which is called a **tractrix**, has the property that the segment of the tangent line between the curve and the y -axis has a constant length a . It can be proved that the equation of this tractrix is

$$y = a \operatorname{sech}^{-1} \frac{x}{a} - \sqrt{a^2 - x^2}$$

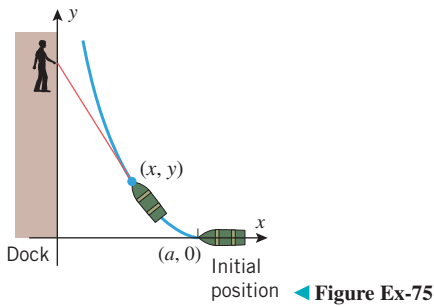
(a) Show that to move the bow of the boat to a point (x, y) , the person must walk a distance

$$D = a \operatorname{sech}^{-1} \frac{x}{a}$$

from the origin.

(b) If the rope has a length of 15 m, how far must the person walk from the origin to bring the boat 10 m from the dock? Round your answer to two decimal places.

(c) Find the distance traveled by the bow along the tractrix as it moves from its initial position to the point where it is 5 m from the dock.



76. **Writing** Suppose that, by analogy with the trigonometric functions, we *define* $\cosh t$ and $\sinh t$ geometrically using Figure 6.9.3b:
 “For any real number t , define $x = \cosh t$ and $y = \sinh t$ to be the unique values of x and y such that

- (i) $P(x, y)$ is on the right branch of the unit hyperbola $x^2 - y^2 = 1$;
- (ii) t and y have the same sign (or are both 0);
- (iii) the area of the region bounded by the x -axis, the right branch of the unit hyperbola, and the segment from the origin to P is $|t|/2$.”

Discuss what properties would first need to be verified in order for this to be a legitimate definition.

77. **Writing** Investigate what properties of $\cosh t$ and $\sinh t$ can be proved directly from the geometric definition in Exercise 76. Write a short description of the results of your investigation.

QUICK CHECK ANSWERS 6.9

1. $\frac{e^x + e^{-x}}{2}$; $\frac{e^x - e^{-x}}{2}$; $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

2.

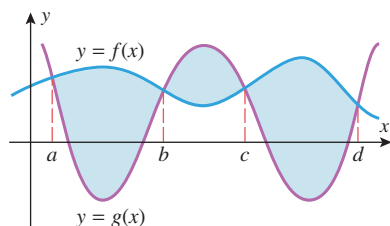
	$\cosh x$	$\sinh x$	$\tanh x$	$\coth x$	$\operatorname{sech} x$	$\operatorname{csch} x$
DOMAIN	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$
RANGE	$[1, +\infty)$	$(-\infty, +\infty)$	$(-1, 1)$	$(-\infty, -1) \cup (1, +\infty)$	$(0, 1]$	$(-\infty, 0) \cup (0, +\infty)$

3. unit hyperbola; $x^2 - y^2 = 1$ 4. $\sinh x$; $\cosh x$; $\operatorname{sech}^2 x$ 5. $\sinh x + C$; $\cosh x + C$; $\ln(\cosh x) + C$
 6. $\frac{1}{\sqrt{x^2 - 1}}$; $\frac{1}{\sqrt{1 + x^2}}$; $\frac{1}{1 - x^2}$

CHAPTER 6 REVIEW EXERCISES

1. Describe the method of slicing for finding volumes, and use that method to derive an integral formula for finding volumes by the method of disks.
2. State an integral formula for finding a volume by the method of cylindrical shells, and use Riemann sums to derive the formula.
3. State an integral formula for finding the arc length of a smooth curve $y = f(x)$ over an interval $[a, b]$, and use Riemann sums to derive the formula.
4. State an integral formula for the work W done by a variable force $F(x)$ applied in the direction of motion to an object moving from $x = a$ to $x = b$, and use Riemann sums to derive the formula.
5. State an integral formula for the fluid force F exerted on a vertical flat surface immersed in a fluid of weight density ρ , and use Riemann sums to derive the formula.
6. Let R be the region in the first quadrant enclosed by $y = x^2$, $y = 2 + x$, and $x = 0$. In each part, set up, but *do not evaluate*, an integral or a sum of integrals that will solve the problem.
 - (a) Find the area of R by integrating with respect to x .
 - (b) Find the area of R by integrating with respect to y .
 - (c) Find the volume of the solid generated by revolving R about the x -axis by integrating with respect to x .
 - (d) Find the volume of the solid generated by revolving R about the x -axis by integrating with respect to y .
 - (e) Find the volume of the solid generated by revolving R about the y -axis by integrating with respect to x .
 - (f) Find the volume of the solid generated by revolving R about the y -axis by integrating with respect to y .
 - (g) Find the volume of the solid generated by revolving R about the line $y = -3$ by integrating with respect to x .
 - (h) Find the volume of the solid generated by revolving R about the line $x = 5$ by integrating with respect to x .
7. (a) Set up a sum of definite integrals that represents the total shaded area between the curves $y = f(x)$ and $y = g(x)$ in the accompanying figure on the next page. (cont.)

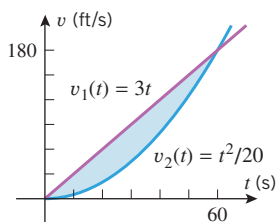
- (b) Find the total area enclosed between $y = x^3$ and $y = x$ over the interval $[-1, 2]$.



◀ Figure Ex-7

8. The accompanying figure shows velocity versus time curves for two cars that move along a straight track, accelerating from rest at a common starting line.

- (a) How far apart are the cars after 60 seconds?
 (b) How far apart are the cars after T seconds, where $0 \leq T \leq 60$?



◀ Figure Ex-8

9. Let R be the region enclosed by the curves $y = x^2 + 4$, $y = x^3$, and the y -axis. Find and evaluate a definite integral that represents the volume of the solid generated by revolving R about the x -axis.
10. A football has the shape of the solid generated by revolving the region bounded between the x -axis and the parabola $y = 4R(x^2 - \frac{1}{4}L^2)/L^2$ about the x -axis. Find its volume.
11. Find the volume of the solid whose base is the region bounded between the curves $y = \sqrt{x}$ and $y = 1/\sqrt{x}$ for $1 \leq x \leq 4$ and whose cross sections perpendicular to the x -axis are squares.
12. Consider the region enclosed by $y = \sin^{-1} x$, $y = 0$, and $x = 1$. Set up, but *do not evaluate*, an integral that represents the volume of the solid generated by revolving the region about the x -axis using
 (a) disks (b) cylindrical shells.
13. Find the arc length in the second quadrant of the curve $x^{2/3} + y^{2/3} = 4$ from $x = -8$ to $x = -1$.
14. Let C be the curve $y = e^x$ between $x = 0$ and $x = \ln 10$. In each part, set up, but *do not evaluate*, an integral that solves the problem.
 (a) Find the arc length of C by integrating with respect to x .
 (b) Find the arc length of C by integrating with respect to y .
15. Find the area of the surface generated by revolving the curve $y = \sqrt{25 - x}$, $9 \leq x \leq 16$, about the x -axis.

16. Let C be the curve $27x - y^3 = 0$ between $y = 0$ and $y = 2$. In each part, set up, but *do not evaluate*, an integral or a sum of integrals that solves the problem.

- (a) Find the area of the surface generated by revolving C about the x -axis by integrating with respect to x .
 (b) Find the area of the surface generated by revolving C about the y -axis by integrating with respect to y .
 (c) Find the area of the surface generated by revolving C about the line $y = -2$ by integrating with respect to y .

17. (a) A spring exerts a force of 0.5 N when stretched 0.25 m beyond its natural length. Assuming that Hooke's law applies, how much work was performed in stretching the spring to this length?

- (b) How far beyond its natural length can the spring be stretched with 25 J of work?

18. A boat is anchored so that the anchor is 150 ft below the surface of the water. In the water, the anchor weighs 2000 lb and the chain weighs 30 lb/ft. How much work is required to raise the anchor to the surface?

19–20 Find the centroid of the region. ■

19. The region bounded by $y^2 = 4x$ and $y^2 = 8(x - 2)$.

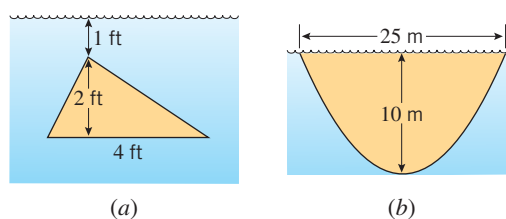
20. The upper half of the ellipse $(x/a)^2 + (y/b)^2 = 1$.

21. In each part, set up, but *do not evaluate*, an integral that solves the problem.

- (a) Find the fluid force exerted on a side of a box that has a 3 m square base and is filled to a depth of 1 m with a liquid of weight density ρ N/m³.

- (b) Find the fluid force exerted by a liquid of weight density ρ lb/ft³ on a face of the vertical plate shown in part (a) of the accompanying figure.

- (c) Find the fluid force exerted on the parabolic dam in part (b) of the accompanying figure by water that extends to the top of the dam.



▲ Figure Ex-21

22. Show that for any constant a , the function $y = \sinh(ax)$ satisfies the equation $y'' = a^2y$.

23. In each part, prove the identity.

(a) $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$

(b) $\cosh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x + 1)}$

(c) $\sinh \frac{1}{2}x = \pm \sqrt{\frac{1}{2}(\cosh x - 1)}$

CHAPTER 6 MAKING CONNECTIONS

- Suppose that f is a nonnegative function defined on $[0, 1]$ such that the area between the graph of f and the interval $[0, 1]$ is A_1 and such that the area of the region R between the graph of $g(x) = f(x^2)$ and the interval $[0, 1]$ is A_2 . In each part, express your answer in terms of A_1 and A_2 .
 - What is the volume of the solid of revolution generated by revolving R about the y -axis?
 - Find a value of a such that if the xy -plane were horizontal, the region R would balance on the line $x = a$.
- A water tank has the shape of a conical frustum with radius of the base 5 ft, radius of the top 10 ft and (vertical) height 15 ft. Suppose the tank is filled with water and consider the problem of finding the work required to pump all the water out through a hole in the top of the tank.
 - Solve this problem using the method of Example 5 in Section 6.6.
 - Solve this problem using Definition 6.6.3. [*Hint*: Think of the base as the head of a piston that expands to a water-tight fit against the sides of the tank as the piston is pushed upward. What important result about water pressure do you need to use?]
- A disk of radius a is an inhomogeneous lamina whose density is a function $f(r)$ of the distance r to the center of the lamina.

Modify the argument used to derive the method of cylindrical shells to find a formula for the mass of the lamina.

 - Compare Formula (10) in Section 6.7 with Formula (8) in Section 6.8. Then give a plausible argument that the force on a flat surface immersed vertically in a fluid of constant weight density is equal to the product of the area of the surface and the pressure at the centroid of the surface. Conclude that the force on the surface is the same as if the surface were immersed horizontally at the depth of the centroid.
- Archimedes' Principle** states that a solid immersed in a fluid experiences a buoyant force equal to the weight of the fluid displaced by the solid.
 - Use the results of Section 6.8 to verify Archimedes' Principle in the case of (i) a box-shaped solid with a pair of faces parallel to the surface of the fluid, (ii) a solid cylinder with vertical axis, and (iii) a cylindrical shell with vertical axis.
 - Give a plausible argument for Archimedes' Principle in the case of a solid of revolution immersed in fluid such that the axis of revolution of the solid is vertical. [*Hint*: Approximate the solid by a union of cylindrical shells and use the result from part (a).]