Sequences, Series, and Probability

9

Superstock

In 1998, the U.S. Postal Service handled 41 percent of the world's mail volume, approximately 630 million pieces every day. (Source: U.S. Postal Service)

The Big Picture

In this chapter you will learn how to

- use sequence, factorial, and summation notation to write the terms and sums of sequences.
- recognize, write, and use arithmetic sequences and geometric sequences.
- use mathematical induction to prove statements involving a positive integer n.
- use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and write binomial expansions.
- solve counting problems using the Fundamental Counting Principle, permutations, and combinations.
- find the probability of events and their complements.

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

- infinite sequence (p. 618)
- finite sequence (p. 618)
- recursive (p. 620)
- factorial (p. 620)
- summation or sigma notation (p. 622)
- infinite series (p. 623)
- finite series or nth partial sum (p. 623)
- arithmetic sequence (p. 629)
- geometric sequence (p. 638)
- infinite geometric series (p. 642)
- mathematical induction (p. 648)
- binomial coefficients (p. 656)
- Binomial Theorem (p. 656)
- Pascal's Triangle (p. 658)
- expanding a binomial (p. 659)
- Fundamental Counting Principle (p. 665)
- permutation (p. 666)
- permutation of n elements taken r at a time (p. 666)
- distinguishable permutation (p. 668)
- combination of n elements taken r at a time (p. 669)
- probability (p. 675)
- independent events (p. 680)
- complement of an event (p. 681)

Additional Resources Text-specific additional resources are available to help you do well in this course. See page xvi for details.

9.1 Sequences and Series

Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English. Saying that a collection is listed in *sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on.

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers. Instead of using function notation, sequences are usually written using subscript notation, as shown in the following definition.

Definition of Sequence

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

are the **terms** of the sequence. If the domain of the function consists of the first n positive integers only, the sequence is a **finite sequence.**

On occasion, it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become $a_0, a_1, a_2, a_3, \ldots$.

EXAMPLE 1 Finding the Terms of a Sequence

Find the first four terms of the sequences given by

a. $a_n = 3n - 2$ **b.** $a_n = 3 + (-1)^n$.

Solution

a. The first four terms of the sequence given by $a_n = 3n - 2$ are

$a_1 = 3(1) - 2 = 1$ 1st term

$a_2 = 3(2) - 2 = 4$ 2nd term

$a_3 = 3(3) - 2 = 7$ 3rd term

$a_4 = 3(4) - 2 = 10.$ 4th term

b. The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

$a_1 = 3 + (-1)^1 = 3 - 1 = 2$ 1st term

$a_2 = 3 + (-1)^2 = 3 + 1 = 4$ 2nd term

$a_3 = 3 + (-1)^3 = 3 - 1 = 2$ 3rd term

$a_4 = 3 + (-1)^4 = 3 + 1 = 4.$ 4th term

To graph a sequence using a graphing utility, set the mode to *dot* and *sequence* and enter the sequence. Consult your user's manual for instructions. Try graphing the sequences in Example 1 and using the *value* or *trace* feature to identify the terms.

What You Should Learn:

- How to use sequence notation to write the terms of sequences
- How to use factorial notation
- How to use summation notation to write sums
- How to find sums of infinite series
- How to use sequences and series to model and solve real-life problems

Why You Should Learn It:

Sequences and series are useful in modeling sets of values in order to identify a pattern. For instance, Exercise 115 on page 627 shows how a sequence can be used to model the average daily cost to community hospitals per patient from 1989 to 1996.

Superstock

EXAMPLE 2 Finding the Terms of a Sequence

Find the first five terms of the sequence given by $a_n = \dfrac{(-1)^n}{2n-1}$.

Algebraic Solution

The first five terms of the sequence given by

$a_n = \dfrac{(-1)^n}{2n-1}$ are as follows.

$$a_1 = \frac{(-1)^1}{2(1)-1} = \frac{-1}{2-1} = -1 \qquad \text{1st term}$$

$$a_2 = \frac{(-1)^2}{2(2)-1} = \frac{1}{4-1} = \frac{1}{3} \qquad \text{2nd term}$$

$$a_3 = \frac{(-1)^3}{2(3)-1} = \frac{-1}{6-1} = -\frac{1}{5} \qquad \text{3rd term}$$

$$a_4 = \frac{(-1)^4}{2(4)-1} = \frac{1}{8-1} = \frac{1}{7} \qquad \text{4th term}$$

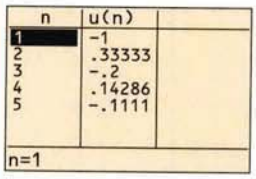
$$a_5 = \frac{(-1)^5}{2(5)-1} = \frac{-1}{10-1} = -\frac{1}{9} \qquad \text{5th term}$$

Numerical Solution

Use the *table* feature of a graphing utility to create a table showing the terms of the sequence $u_n = (-1)^n/(2n-1)$ for $n = 1, 2, 3, 4$, and 5. From the table in Figure 9.1, you can estimate the first five terms of the sequence as follows.

$$u_1 = -1, \qquad u_2 = 0.33333 \approx \frac{1}{3}, \qquad u_3 = -0.2 = -\frac{1}{5},$$

$$u_4 = 0.14286 \approx \frac{1}{7} \qquad \text{and} \qquad u_5 = -0.1111 \approx -\frac{1}{9}$$



n	u(n)
1	-1
2	.33333
3	-.2
4	.14286
5	-.1111

n=1

Figure 9.1

Simply listing the first few terms is not sufficient to define a unique sequence—the nth term *must be given*. To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2-n+6)}, \ldots$$

EXAMPLE 3 Finding the nth Term of a Sequence

Write an expression for the apparent nth term (a_n) of each sequence.

a. 1, 3, 5, 7, . . . **b.** 2, 5, 10, 17, . . .

Solution

a. n: 1 2 3 4 . . . n

Terms: 1 3 5 7 . . . a_n

Apparent Pattern: Each term is 1 less than twice n, which implies that

$$a_n = 2n - 1.$$

b. n: 1 2 3 4 . . . n

Terms: 2 5 10 17 . . . a_n

Apparent Pattern: Each term is 1 more than the square of n, which implies that

$$a_n = n^2 + 1.$$

Some sequences are defined **recursively.** To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms. A well-known example is the Fibonacci sequence shown in Example 4.

EXAMPLE 4 The Fibonacci Sequence: A Recursive Sequence

The Fibonacci sequence is defined recursively as follows.

$$a_0 = 1,\ a_1 = 1,\ a_k = a_{k-2} + a_{k-1}, \qquad \text{where } k \geq 2$$

Write the first six terms of this sequence.

Solution

$a_0 = 1$	0th term is given.
$a_1 = 1$	1st term is given.
$a_2 = a_0 + a_1 = 1 + 1 = 2$	Use recursive formula.
$a_3 = a_1 + a_2 = 1 + 2 = 3$	Use recursive formula.
$a_4 = a_2 + a_3 = 2 + 3 = 5$	Use recursive formula.
$a_5 = a_3 + a_4 = 3 + 5 = 8$	Use recursive formula.

The *Interactive* CD-ROM and *Internet* versions if this text show every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

Factorial Notation

Some very important sequences in mathematics involve terms that are defined with special types of products called **factorials.**

Definition of Factorial

If n is a positive integer, ***n* factorial** is defined by

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots (n-1) \cdot n.$$

As a special case, zero factorial is defined as $0! = 1$.

Here are some values of $n!$ for the first several nonnegative integers. Notice that $0! = 1$ by definition.

$$0! = 1$$

$$1! = 1$$

$$2! = 1 \cdot 2 = 2$$

$$3! = 1 \cdot 2 \cdot 3 = 6$$

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

The value of n does not have to be very large before the value of $n!$ becomes huge. For instance, $10! = 3{,}628{,}800$.

STUDY TIP

Most graphing utilities have the capability to compute $n!$. Use your utility to compare $3 \cdot 5!$ and $(3 \cdot 5)!$. How do they differ? How large a value of $n!$ will your graphing utility allow you to compute?

Factorials follow the same conventions for order of operations as do exponents. For instance,

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots n)$$

whereas $(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots 2n$.

EXAMPLE 5 Finding the Terms of a Sequence Involving Factorials

List the first five terms of the sequence given by $a_n = \dfrac{2^n}{n!}$. Begin with $n = 0$.

Algebraic Solution

$$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1 \quad \text{0th term}$$

$$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2 \quad \text{1st term}$$

$$a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2 \quad \text{2nd term}$$

$$a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3} \quad \text{3rd term}$$

$$a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3} \quad \text{4th term}$$

Graphical Solution

Using a graphing utility set to *dot* and *sequence* modes, enter the sequence $u_n = 2^n/n!$.

Set the viewing window to $0 \le n \le 4$, $0 \le x \le 6$, and $0 \le y \le 4$. Then graph the sequence as shown in Figure 9.2. Use the *value* or *trace* feature to approximate the first five terms as follows.

$$u_0 = 1, \quad u_1 = 2, \quad u_2 = 2, \quad u_3 \approx 1.333 = \frac{4}{3}, \quad u_4 \approx 0.666 = \frac{2}{3}$$

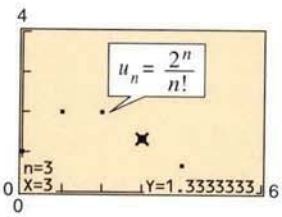


Figure 9.2

When working with fractions involving factorials, you will often find that the fractions can be reduced to simplify the computations.

EXAMPLE 6 Evaluating Factorial Expressions

Evaluate each factorial expression.

a. $\dfrac{8!}{2! \cdot 6!}$ **b.** $\dfrac{2! \cdot 6!}{3! \cdot 5!}$ **c.** $\dfrac{n!}{(n-1)!}$

Solution

a. $$\frac{8!}{2! \cdot 6!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 7 \cdot 8}{1 \cdot 2 \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}} = \frac{7 \cdot 8}{2} = 28$$

b. $$\frac{2! \cdot 6!}{3! \cdot 5!} = \frac{\cancel{1 \cdot 2} \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot 6}{\cancel{1 \cdot 2} \cdot 3 \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}} = \frac{6}{3} = 2$$

c. $$\frac{n!}{(n-1)!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdots (n-1)} \cdot n}{\cancel{1 \cdot 2 \cdot 3 \cdots (n-1)}} = n$$

Note in Example 6(a) that you can simplify the computation as follows.

$$\frac{8!}{2!\,6!} = \frac{8 \cdot 7 \cdot \cancel{6!}}{2!\,\cancel{6!}} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

Summation Notation

There is a convenient notation for the sum of the terms of a finite sequence. It is called **summation notation** or **sigma notation** because it involves the use of the uppercase Greek letter sigma, written as Σ.

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

where i is called the **index of summation,** n is the **upper limit of summation,** and 1 is the **lower limit of summation.**

EXAMPLE 7 Sigma Notation for Sums

a. $\displaystyle\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$

$$= 3(1 + 2 + 3 + 4 + 5)$$
$$= 3(15) = 45$$

b. $\displaystyle\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2)$

$$= 10 + 17 + 26 + 37 = 90$$

c. $\displaystyle\sum_{n=0}^{8} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320}$$
$$\approx 2.71828$$

For this summation, note that the sum is very close to the irrational number $e \approx 2.718281828$. It can be shown that as more terms of the sequence whose nth term is $1/n!$ are added, the sum becomes closer and closer to e.

In Example 7, note that the lower limit of a summation does not have to be 1. Also note that the index of summation does not have to be the letter i. For instance, in part (b) the letter k is the index of summation.

Most graphing utilities are able to sum the first n terms of a sequence. Check your user's manual for a *sum sequence* feature or a *series* feature. Figure 9.3 is an example of how one graphing utility displays the sum of the terms of the sequence

$$a_n = \frac{1}{n!} \quad \text{from} \quad n = 0 \quad \text{to} \quad n = 8.$$

In Example 7(a), note that $\displaystyle\sum_{i=1}^{5} 3i = 3(1 + 2 + 3 + 4 + 5) = 3\sum_{i=1}^{5} i.$

This is an example of one of the *properties of sums* listed on page 623.

```
sum(seq(1/n!,n,0,8)
          2.71827877
```

Figure 9.3

Properties of Sums

1. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, c is any constant.
2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
3. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

A proof of Property 1 is given in Appendix A.

STUDY TIP

Variations in the upper and lower limits of summation can produce quite different-looking summation notations for *the same sum.* For example, the following two sums have identical terms.

$$\sum_{i=1}^{5} 3(2^i) \quad \text{and} \quad \sum_{i=0}^{4} 3(2^{i+1})$$

Series

Many applications involve the sum of the terms of an infinite sequence. Such a sum is called an **infinite series** or simply a **series.**

Definition of a Series

Consider the infinite sequence $a_1, a_2, a_3, \ldots, a_i, \ldots$.

1. The sum of all terms of the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i.$$

2. The sum of the first n terms of the sequence is called a **finite series** or the ***nth* partial sum** of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

EXAMPLE 8 Finding the Sum of a Series

For the series $\sum_{i=1}^{\infty} \frac{3}{10^i}$, find (a) the 3rd partial sum and (b) the sum.

Solution

a. The 3rd partial sum is

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} = 0.3 + 0.03 + 0.003 = 0.333.$$

b. The sum of the series is

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{3}{10^i} &= \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots \\ &= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots \\ &= 0.33333\ldots \approx \frac{1}{3}.\end{aligned}$$

Notice in Example 8(b) that the sum of an infinite series can be a finite number.

Application

Sequences have many applications in situations that involve a recognizable pattern. One is illustrated in Example 9.

EXAMPLE 9 Population of the United States

From 1960 to 1997, the resident population of the United States can be approximated by the model

$$a_n = \sqrt{33{,}282 + 801.3n + 6.12n^2}, \qquad n = 0, 1, \ldots, 37$$

where a_n is the population in millions and n represents the calendar year, with $n = 0$ corresponding to 1960. Find the last five terms of this finite sequence. (Source: U.S. Bureau of the Census)

Algebraic Solution

The last five terms of this finite sequence are as follows.

$$a_{33} = \sqrt{33{,}282 + 801.3(33) + 6.12(33)^2}$$
$$\approx 257.7 \qquad \text{1993 population}$$
$$a_{34} = \sqrt{33{,}282 + 801.3(34) + 6.12(34)^2}$$
$$\approx 260.0 \qquad \text{1994 population}$$
$$a_{35} = \sqrt{33{,}282 + 801.3(35) + 6.12(35)^2}$$
$$\approx 262.3 \qquad \text{1995 population}$$
$$a_{36} = \sqrt{33{,}282 + 801.3(36) + 6.12(36)^2}$$
$$\approx 264.7 \qquad \text{1996 population}$$
$$a_{37} = \sqrt{33{,}282 + 801.3(37) + 6.12(37)^2}$$
$$\approx 267.0 \qquad \text{1997 population}$$

Graphical Solution

Using a graphing utility set to *dot* and *sequence* modes, enter the sequence

$$u_n = \sqrt{33{,}282 + 801.3n + 6.12n^2}.$$

Set the viewing window to $0 \le n \le 40$, $0 \le x \le 40$, and $140 \le y \le 280$. Then graph the sequence, as shown in Figure 9.4. Use the *value* or *trace* feature to approximate the last five terms.

$u_{33} \approx 257.7$ $\quad u_{34} \approx 260.0$ $\quad u_{35} \approx 262.3$

$u_{36} \approx 264.7$ $\quad u_{37} \approx 267.0$

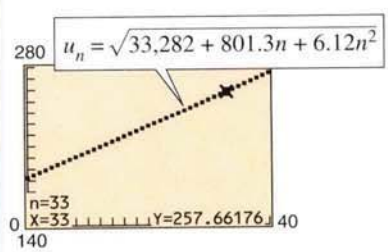
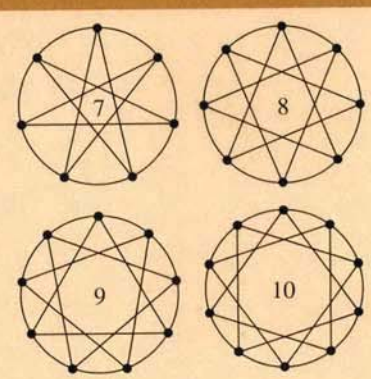


Figure 9.4

Writing About Math

The stars in the figure at the right are formed by placing n equally spaced points on a circle and connecting each point with the third point from it. For these stars, the measure of the angle (in degrees) of each point is

$$d_n = \frac{180(n-6)}{n}, \qquad n \ge 7.$$



a. Write the first four terms of the sequence.

b. If you form the stars by connecting each point with the fourth point from it, you obtain stars with the following number of points and angle measures: 9 points (20°), 10 points (36°), 11 points $\left(49\frac{1}{11}°\right)$, 12 points (60°). Find a formula for the measure of the angle (in degrees) of each point of an n-pointed star. Explain how you found the formula.

9.1 Exercises

In Exercises 1–22, write the first five terms of the sequence. (Assume n begins with 1.) Use the *table* feature of a graphing utility to verify your results.

1. $a_n = 2n + 5$

2. $a_n = 4n - 7$

3. $a_n = 2^n$

4. $a_n = \left(\frac{1}{2}\right)^n$

5. $a_n = (-2)^n$

6. $a_n = \left(-\frac{1}{2}\right)^n$

7. $a_n = \dfrac{n+1}{n}$

8. $a_n = \dfrac{n}{n+1}$

9. $a_n = \dfrac{6n}{3n^2 - 1}$

10. $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$

11. $a_n = \dfrac{1 + (-1)^n}{n}$

12. $a_n = \dfrac{1 + (-1)^n}{2n}$

13. $a_n = 3 - \dfrac{1}{2^n}$

14. $a_n = \dfrac{3^n}{4^n}$

15. $a_n = \dfrac{1}{n^{3/2}}$

16. $a_n = \dfrac{10}{n^{2/3}}$

17. $a_n = \dfrac{3^n}{n!}$

18. $a_n = \dfrac{n!}{2^n}$

19. $a_n = \dfrac{(-1)^n}{n^2}$

20. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$

21. $a_n = (2n - 1)(2n + 1)$

22. $a_n = n(n - 1)(n - 2)$

In Exercises 23–28, find the indicated term of the sequence.

23. $a_n = (-1)^n(3n - 2)$

$a_{25} =$

24. $a_n = (-1)^{n-1}[n(n - 1)]$

$a_{16} =$

25. $a_n = \dfrac{2^n}{n!}$

$a_{10} =$

26. $a_n = \dfrac{n!}{2n}$

$a_8 =$

27. $a_n = \dfrac{4n}{2n^2 - 3}$

$a_{12} =$

28. $a_n = \dfrac{4n^2 - n + 3}{n(n - 1)(n + 2)}$

$a_{15} =$

In Exercises 29–34, write the first five terms of the sequence defined recursively.

29. $a_1 = 28, \quad a_{k+1} = a_k - 4$

30. $a_1 = 15, \quad a_{k+1} = a_k + 3$

31. $a_1 = 3, \quad a_{k+1} = 2(a_k - 1)$

32. $a_1 = 32, \quad a_{k+1} = \frac{1}{2}a_k$

33. $a_1 = 2, a_2 = 6, \quad a_{k+2} = a_{k+1} + 2a_k$

34. $a_1 = 52, a_2 = 40, \quad a_{k+2} = \frac{1}{2}a_{k+1} - a_k$

In Exercises 35–40, use a graphing utility to graph the first ten terms of the sequence. (Assume n begins with 1.)

35. $a_n = \dfrac{2}{3}n$

36. $a_n = 2 - \dfrac{4}{n}$

37. $a_n = 16(-0.5)^{n-1}$

38. $a_n = 8(0.75)^{n-1}$

39. $a_n = \dfrac{2n}{n+1}$

40. $a_n = \dfrac{3n^2}{n^2 + 1}$

In Exercises 41–46, use the *table* feature of a graphing utility to find the first ten terms of the sequence. (Assume n begins with 1.)

41. $a_n = 2(3n - 1) + 5$

42. $a_n = 2n(n + 1)(n + 2)$

43. $a_n = \dfrac{6^n}{n!}$

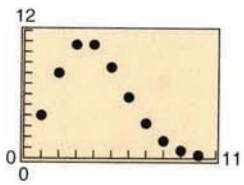
44. $a_n = \dfrac{n!}{(n^2 - 10)}$

45. $a_n = 1 + \dfrac{n+1}{n}$

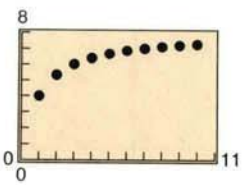
46. $a_n = \dfrac{4n^2}{n+2}$

In Exercises 47–50, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)



(b)



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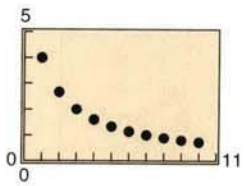
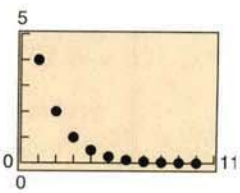


(d)



The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered Section and Review Exercises. They also provide Tutorial Exercises, which link to Guided Examples for additional help.

47. $a_n = \dfrac{8}{n+1}$

48. $a_n = \dfrac{8n}{n+1}$

49. $a_n = 4(0.5)^{n-1}$

50. $a_n = \dfrac{4^n}{n!}$

In Exercises 51–64, write an expression for the *apparent* nth term of the sequence. (Assume n begins with 1.)

51. $1, 4, 7, 10, 13, \ldots$

52. $3, 7, 11, 15, 19, \ldots$

53. $0, 3, 8, 15, 24, \ldots$

54. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

55. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$

56. $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

57. $\frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \frac{-1}{16}, \ldots$

58. $\frac{1}{3}, -\frac{2}{9}, \frac{4}{27}, -\frac{8}{81}, \ldots$

59. $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

60. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

61. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

62. $1, 2, \dfrac{2^2}{2}, \dfrac{2^3}{6}, \dfrac{2^4}{24}, \dfrac{2^5}{120}, \ldots$

63. $1, 3, 1, 3, 1, \ldots$

64. $1, -1, 1, -1, 1, \ldots$

In Exercises 65–68, write the first five terms of the sequence defined recursively. Use the pattern to write the nth term of the sequence as a function of n. (Assume n begins with 1.)

65. $a_1 = 6, \quad a_{k+1} = a_k + 2$

66. $a_1 = 25, \quad a_{k+1} = a_k - 5$

67. $a_1 = 81, \quad a_{k+1} = \frac{1}{3}a_k$

68. $a_1 = 14, \quad a_{k+1} = -2a_k$

In Exercises 69–78, simplify the ratio of factorials.

69. $\dfrac{3!}{6!}$

70. $\dfrac{4!}{7!}$

71. $\dfrac{10!}{8!}$

72. $\dfrac{25!}{23!}$

73. $\dfrac{12!}{4! \cdot 8!}$

74. $\dfrac{10! \cdot 3!}{4! \cdot 6!}$

75. $\dfrac{(n+1)!}{n!}$

76. $\dfrac{(n+2)!}{n!}$

77. $\dfrac{(2n-1)!}{(2n+1)!}$

78. $\dfrac{(2n+2)!}{(2n)!}$

In Exercises 79–90, find the sum.

79. $\displaystyle\sum_{i=1}^{5}(2i+1)$

80. $\displaystyle\sum_{i=1}^{6}(3i-1)$

81. $\displaystyle\sum_{k=1}^{4}10$

82. $\displaystyle\sum_{k=1}^{5}6$

83. $\displaystyle\sum_{i=0}^{4}i^2$

84. $\displaystyle\sum_{i=0}^{5}3i^2$

85. $\displaystyle\sum_{k=0}^{3}\frac{1}{k^2+1}$

86. $\displaystyle\sum_{j=3}^{5}\frac{1}{j}$

87. $\displaystyle\sum_{i=1}^{4}[(i-1)^2 + (i+1)^3]$

88. $\displaystyle\sum_{k=2}^{5}(k+1)(k-3)$

89. $\displaystyle\sum_{i=1}^{4}2^i$

90. $\displaystyle\sum_{j=0}^{4}(-2)^j$

In Exercises 91–94, use a graphing utility to find the sum.

91. $\displaystyle\sum_{j=1}^{6}(24-3j)$

92. $\displaystyle\sum_{j=1}^{10}\frac{3}{j+1}$

93. $\displaystyle\sum_{k=0}^{4}\frac{(-1)^k}{k+1}$

94. $\displaystyle\sum_{k=0}^{4}\frac{(-1)^k}{k!}$

In Exercises 95–104, use sigma notation to write the sum. Then use a graphing utility to find the sum.

95. $\dfrac{1}{3(1)} + \dfrac{1}{3(2)} + \dfrac{1}{3(3)} + \cdots + \dfrac{1}{3(9)}$

96. $\dfrac{5}{1+1} + \dfrac{5}{1+2} + \dfrac{5}{1+3} + \cdots + \dfrac{5}{1+15}$

97. $\left[2\left(\frac{1}{8}\right) + 3\right] + \left[2\left(\frac{2}{8}\right) + 3\right] + \cdots + \left[2\left(\frac{8}{8}\right) + 3\right]$

98. $\left[1 - \left(\frac{1}{6}\right)^2\right] + \left[1 - \left(\frac{2}{6}\right)^2\right] + \cdots + \left[1 - \left(\frac{6}{6}\right)^2\right]$

99. $3 - 9 + 27 - 81 + 243 - 729$

100. $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \cdots - \dfrac{1}{128}$

101. $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots - \dfrac{1}{20^2}$

102. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{2 \cdot 4} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{10 \cdot 12}$

103. $\dfrac{1}{4} + \dfrac{3}{8} + \dfrac{7}{16} + \dfrac{15}{32} + \dfrac{31}{64}$

104. $\dfrac{1}{2} + \dfrac{2}{4} + \dfrac{6}{8} + \dfrac{24}{16} + \dfrac{120}{32} + \dfrac{720}{64}$

In Exercises 105–108, find the indicated partial sum of the series.

105. $\sum_{i=1}^{\infty} 5\left(\frac{1}{2}\right)^i$, 4th partial sum

106. $\sum_{i=1}^{\infty} 2\left(\frac{1}{3}\right)^i$, 5th partial sum

107. $\sum_{n=1}^{\infty} 4\left(-\frac{1}{2}\right)^n$, 3rd partial sum

108. $\sum_{n=1}^{\infty} 8\left(-\frac{1}{4}\right)^n$, 4th partial sum

In Exercises 109–112, find the sum of the infinite series.

109. $\sum_{i=1}^{\infty} 6\left(\frac{1}{10}\right)^i$

110. $\sum_{k=1}^{\infty} 4\left(\frac{1}{10}\right)^k$

111. $\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k$

112. $\sum_{i=1}^{\infty} 2\left(\frac{1}{10}\right)^i$

113. ***Compound Interest*** A deposit of \$5000 is made in an account that earns 8% interest compounded quarterly. The balance in the account after n quarters is

$$A_n = 5000\left(1 + \frac{0.08}{4}\right)^n, \quad n = 1, 2, 3, \ldots.$$

(a) Compute the first eight terms of this sequence.

(b) Find the balance in this account after 10 years by computing the 40th term of the sequence.

114. ***Compound Interest*** A deposit of \$100 is made *each month* in an account that earns 12% interest compounded monthly. The balance in the account after n months is

$$A_n = 100(101)[(1.01)^n - 1], \quad n = 1, 2, 3, \ldots.$$

(a) Compute the first six terms of this sequence.

(b) Find the balance in this account after 5 years by computing the 60th term of the sequence.

(c) Find the balance in this account after 20 years by computing the 240th term of the sequence.

115. ***Per Capita Hospital Care*** The average cost to community hospitals per patient per day from 1989 to 1996 can be approximated by the model $a_n = 696.39 + 66.44n - 2.37n^2$, $n = -1, \ldots, 6$ where a_n is the cost (in dollars) and n is the year, with $n = 0$ corresponding to 1990. Find the terms of this finite sequence and use a graphing utility to construct a bar graph that represents the sequence. What does the pattern of the bar graph say about the future of hospital costs? (Source: American Hospital Association)

116. ***Federal Debt*** From 1987 to 1998, the federal debt rose from just over \$2 trillion dollars to over \$5 trillion dollars. The federal debt from 1987 to 1998 is approximated by the model

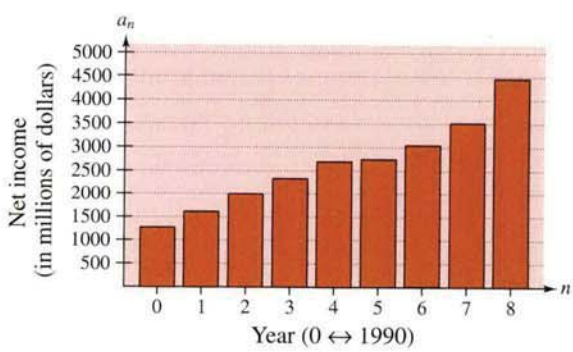
$$a_n = \sqrt{11.7 + 2.4n}, \quad n = -3, \ldots, 8$$

where a_n is the debt (in trillions of dollars) and n is the year, with $n = 0$ corresponding to 1990. Find the terms of this finite sequence and use a graphing utility to construct a bar graph that represents the sequence. What does the pattern in the bar graph say about the future of the federal debt? (Source: Bureau of the Public Debt)

117. ***Corporate Income*** The net income a_n (in millions of dollars) of Wal-Mart for the years 1990 through 1998 are shown in the graph. The income can be approximated by the model

$$a_n = 1215.16 + 608.19n - 114.83n^2 + 11n^3,$$
$$n = 0, \ldots, 8$$

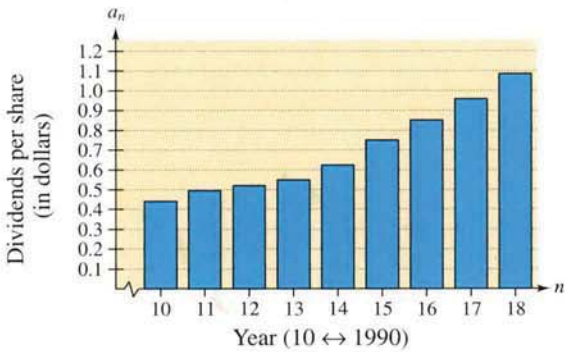
where $n = 0$ represents 1990. Use this model to approximate the total net income from 1990 through 1998. Compare this sum with the result of adding the incomes shown in the graph. (Source: Wal-Mart Stores, Inc.)



118. ***Corporate Dividends*** The dividends a_n (in dollars) declared per share of common stock of Procter & Gamble Company for the years 1990 through 1998 are shown in the graph. These dividends can be approximated by the model

$$a_n = 4.27 + 0.29n - 2.93 \ln n,$$
$$n = 10, \ldots, 18$$

where $n = 10$ represents 1990. Use this model to approximate the total dividends per share of common stock from 1990 through 1998. Compare this sum with the result of adding the dividends shown in the graph. (Source: Procter & Gamble Company)



Synthesis

True or False? **In Exercises 119 and 120, determine whether the statement is true or false. Justify your answer.**

119. $\sum_{i=1}^{4}(i^2 + 2i) = \sum_{i=1}^{4} i^2 + 2\sum_{i=1}^{4} i$

120. $\sum_{j=1}^{4} 2^j = \sum_{j=3}^{6} 2^{j-2}$

Fibonacci Sequence **In Exercises 121 and 122, use the Fibonacci sequence. (See Example 4.)**

121. Write the first 12 terms of the Fibonacci sequence a_n and the first 10 terms of the sequence given by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n > 1.$$

122. Using the definition for b_n in Exercise 121, show that b_n can be defined recursively by

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

123. Find the first few terms of $a_n = n^2 - n + 11$. Describe any pattern or make an observation about the terms of the sequence.

In Exercises 124–127, find the first five terms of the sequence.

124. $a_n = \dfrac{x^n}{n!}$

125. $a_n = \dfrac{(-1)^n x^{2n+1}}{2n + 1}$

126. $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$

127. $a_n = \dfrac{(-1)^n x^{2n+1}}{(2n + 1)!}$

Review

In Exercises 128 and 129, write the augmented matrix for the system of linear equations.

128. $\begin{cases} -4x + y = -7 \\ 6x - 9y = 3 \end{cases}$

129. $\begin{cases} 2x + y + 3z = -3 \\ -x + 5y = 14 \\ -3x - 6y - 7z = -7 \end{cases}$

In Exercises 130–133, find (a) $A - B$, (b) $2B - 3A$, (c) AB, (d) BA.

130. $A = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 6 & -3 \end{bmatrix}$

131. $A = \begin{bmatrix} 10 & 7 \\ -4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -12 \\ 8 & 11 \end{bmatrix}$

132. $A = \begin{bmatrix} -2 & -3 & 6 \\ 4 & 5 & 7 \\ 1 & 7 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 6 \\ 0 & 3 & 1 \end{bmatrix}$

133. $A = \begin{bmatrix} -1 & 4 & 0 \\ 5 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$

In Exercises 134–137, find the determinant of the matrix.

134. $A = \begin{bmatrix} 3 & 7 \\ -2 & 9 \end{bmatrix}$

135. $A = \begin{bmatrix} -4 & 11 \\ 13 & 20 \end{bmatrix}$

136. $A = \begin{bmatrix} 4 & 0 & 5 \\ 0 & -7 & 2 \\ 9 & 1 & -1 \end{bmatrix}$

137. $A = \begin{bmatrix} 10 & 9 & 12 & 2 \\ -2 & 5 & 8 & 7 \\ -2 & -1 & 0 & 3 \\ -4 & 6 & 2 & 1 \end{bmatrix}$

9.2 Arithmetic Sequences and Partial Sums

Arithmetic Sequences

A sequence whose consecutive terms have a common difference is called an **arithmetic sequence.**

Definition of Arithmetic Sequence

A sequence is **arithmetic** if the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

is arithmetic if there is a number d such that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = d.$$

The number d is the **common difference** of the arithmetic sequence.

EXAMPLE 1 Examples of Arithmetic Sequences

a. The sequence whose nth term is $4n + 3$ is arithmetic. For this sequence, the common difference between consecutive terms is 4.

$$\underbrace{7, 11}_{11 - 7 = 4}, 15, 19, \ldots, 4n + 3, \ldots$$

b. The sequence whose nth term is $7 - 5n$ is arithmetic. For this sequence, the common difference between consecutive terms is -5.

$$\underbrace{2, -3}_{-3 - 2 = -5}, -8, -13, \ldots, 7 - 5n, \ldots$$

c. The sequence whose nth term is $\frac{1}{4}(n + 3)$ is arithmetic. For this sequence, the common difference between consecutive terms is $\frac{1}{4}$.

$$\underbrace{1, \frac{5}{4}}_{\frac{5}{4} - 1 = \frac{1}{4}}, \frac{3}{2}, \frac{7}{4}, \ldots, \frac{n + 3}{4}, \ldots$$

d. The sequence 1, 4, 9, 16, . . . ,whose nth term is n^2 is *not* arithmetic. The difference between the first two terms is

$$a_2 - a_1 = 4 - 1 = 3$$

but the difference between the second and third terms is

$$a_3 - a_2 = 9 - 4 = 5.$$

What You Should Learn:

- How to recognize, write, and find the nth terms of arithmetic sequences
- How to find nth partial sums of arithmetic sequences
- How to use arithmetic sequences to model and solve real-life problems

Why You Should Learn It:

Arithmetic sequences can reduce the amount of time it takes to find the sum of a sequence of numbers with a common difference. In Exercise 85 on page 636, you will use an arithmetic sequence to find the number of bricks needed to lay a brick patio.

Index Stock

In Example 1, notice that each of the arithmetic sequences in parts (a), (b), and (c) has an nth term that is of the form $dn + c$, where the common difference of the sequence is d. This result is summarized as follows.

The nth Term of an Arithmetic Sequence

The nth term of an arithmetic sequence has the form

$$a_n = dn + c$$

where d is the common difference between consecutive terms of the sequence and $c = a_1 - d$.

An arithmetic sequence $a_n = dn + c$ can be thought of as "counting by d's" after a shift of c units from d. For instance, the sequence

$$2, 6, 10, 14, 18, \ldots$$

has a common difference of 4, so you are counting by 4's after a shift of 2 units below 4 (beginning with $a_1 = 2$). So, the nth term is $4n - 2$. Similarly, the nth term of the sequence

$$6, 11, 16, 21, \ldots$$

is $5n + 1$ because you are counting by 5's after a shift of 1 unit above 5 (beginning with $a_1 = 6$).

Exploration

Consider the following sequences.

$1, 4, 7, 10, 13, \ldots, 3n - 2, \ldots$

$-5, 1, 7, 13, 19, \ldots, 6n - 11, \ldots$

$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots, \frac{7}{2} - n, \ldots$

What relationship do you observe between successive terms of these sequences?

The *Interactive* CD-ROM and *Internet* versions of this text offer a built-in graphing calculator, which can be used with the Examples, Explorations, and Exercises.

EXAMPLE 2 Finding the nth Term of an Arithmetic Sequence

Find a formula for the nth term of the arithmetic sequence whose common difference is 3 and whose first term is 2.

Solution

Because the sequence is arithmetic, you know that the formula for the nth term is of the form $a_n = dn + c$. Moreover, because the common difference is $d = 3$, the formula must have the form $a_n = 3n + c$. Because $a_1 = 2$, it follows that

$$c = a_1 - d = 2 - 3 = -1.$$

So, the formula for the nth term is $a_n = 3n - 1$. The sequence therefore has the following form.

$$2, 5, 8, 11, 14, \ldots, 3n - 1, \ldots$$

A graph of the first 15 terms of the sequence is shown in Figure 9.5. Notice that the points lie on a line. This makes sense because a_n is a linear function of n. In other words, the terms "arithmetic" and "linear" are closely connected.

Another way to find a formula for the nth term of the sequence in Example 2 is to begin by writing the terms of the sequence.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	$\ldots$
2	$2 + 3$	$5 + 3$	$8 + 3$	$11 + 3$	$14 + 3$	$17 + 3$	$\ldots$
2	5	8	11	14	17	20	$\ldots$

From these terms, you can reason that the nth term is of the form

$$a_n = dn + c = 3n - 1.$$

STUDY TIP

You can use a graphing utility to generate the arithmetic sequence in Example 2 using the following steps.

2 (enter key)
3 [+] (previous answer key)

Now press the enter key repeatedly to generate the terms of the sequence.

Most graphing utilities have a built-in function that will display the terms of an arithmetic sequence. Consult your user's manual for instructions.

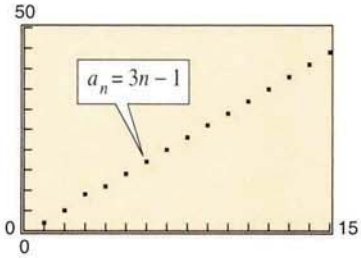


Figure 9.5

EXAMPLE 3 Writing the Terms of an Arithmetic Sequence

The fourth term of an arithmetic sequence is 20, and the 13th term is 65. Write the first several terms of this sequence.

Solution

The fourth and 13th terms of the sequence are related by

$$a_{13} = a_4 + 9d.$$

Using $a_4 = 20$ and $a_{13} = 65$, you can conclude that $d = 5$, which implies that the sequence is as follows.

$$\begin{array}{ccccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & \ldots \\ 5, & 10, & 15, & 20, & 25, & 30, & 35, & 40, & 45, & 50, & 55, & 60, & 65, & \ldots \end{array}$$

If you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the $(n + 1)$th term by using the *recursive formula*

$$a_{n+1} = a_n + d. \qquad \text{Recursive formula}$$

With this formula, you can find any term of an arithmetic sequence, *provided* that you know the previous term. For instance, if you know the first term, you can find the second term. Then, knowing the second term, you can find the third term, and so on.

If you substitute $a_1 - d$ for c in the formula $a_n = dn + c$, the nth term of an arithmetic sequence has the alternative recursive formula

$$a_n = a_1 + (n - 1)d. \qquad \text{Alternative recursive formula}$$

Use this formula to solve Example 4. You should get the same answer.

EXAMPLE 4 Using a Recursive Formula

Find the seventh term of the arithmetic sequence whose first two terms are 2 and 9.

Algebraic Solution

To find the seventh term, first find a formula for the nth term. Because the first term is 2, it follows that

$$c = a_1 - d = 2 - 7 = -5.$$

Therefore, a formula for the nth term is

$$\begin{aligned} a_n &= dn + c \\ &= 7n - 5 \end{aligned}$$

which implies that the seventh term is

$$\begin{aligned} a_7 &= 7(7) - 5 \\ &= 44. \end{aligned}$$

Numerical Solution

For this sequence, the common difference is $d = 9 - 2 = 7$. Use the *table* feature of a graphing utility to create a table that begins at 2 and increases by 7 in each row, as shown in Figure 9.6. The number in the seventh row of the table is 44, so 44 is the seventh term of the arithmetic sequence.

L1	L2	L3
1	2	-----
2	9	
3	16	
4	23	
5	30	
6	37	
7	44	

L2(7)=44

Figure 9.6

The Sum of a Finite Arithmetic Sequence

There is a simple formula for the *sum* of a finite arithmetic sequence. A proof of the formula is given in Appendix A.

The Sum of a Finite Arithmetic Sequence

The sum of a finite arithmetic sequence with n terms is

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Be sure you see that this formula works only for *arithmetic* sequences. Using this formula reduces the amount of time it takes to find the sum of an arithmetic sequence, as you will see in the following example.

EXAMPLE 5 Finding the Sum of a Finite Arithmetic Sequence

Find the sum: $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$.

Solution

To begin, notice that the sequence is arithmetic (with a common difference of 2). Moreover, the sequence has 10 terms. So, the sum of the sequence is

$$\begin{aligned} S_n &= 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 \\ &= \frac{n}{2}(a_1 + a_n) \\ &= \frac{10}{2}(1 + 19) \qquad n = 10,\ a_1 = 1,\ a_{10} = 19 \\ &= 5(20) \\ &= 100. \end{aligned}$$

EXAMPLE 6 Finding the Sum of a Finite Arithmetic Sequence

Find the sum of the integers from 1 to 100.

Solution

The integers from 1 to 100 form an arithmetic sequence that has 100 terms. So, you can use the formula for the sum of an arithmetic sequence, as follows.

$$\begin{aligned} S_n &= 1 + 2 + 3 + 4 + 5 + 6 + \cdots + 99 + 100 \\ &= \frac{n}{2}(a_1 + a_n) \\ &= \frac{100}{2}(1 + 100) \qquad n = 100,\ a_1 = 1,\ a_{100} = 100 \\ &= 50(101) \\ &= 5050 \end{aligned}$$

When **Carl Friedrich Gauss (1777–1855)** was ten years old, his teacher asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments the teacher could only look at him in astounded silence. This is what Gauss did:

$$\begin{array}{rcccccc} 1 + & 2 + & 3 + & \cdots & + & 100 \\ 100 + & 99 + & 98 + & \cdots & + & 1 \\ \hline 101 + & 101 + & 101 + & \cdots & + & 101 \end{array}$$

$$\frac{100 \times 101}{2} = 5050$$

The sum of the first n terms of an infinite sequence is called the ***n*th partial sum.**

EXAMPLE 7 Finding a Partial Sum of an Arithmetic Sequence

Find the 150th partial sum of the arithmetic sequence

$$5, 16, 27, 38, 49, \ldots.$$

Solution

For this arithmetic sequence, you have $a_1 = 5$ and $d = 16 - 5 = 11$. So,

$$c = a_1 - d = 5 - 11 = -6$$

and the nth term is

$$a_n = 11n - 6.$$

Therefore, $a_{150} = 11(150) - 6 = 1644$, and the sum of the first 150 terms is

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ &= \frac{150}{2}(5 + 1644) \qquad n = 150,\ a_1 = 5,\ a_{150} = 1644 \\ &= 75(1649) \\ &= 123{,}675. \end{aligned}$$

Applications

EXAMPLE 8 Seating Capacity

An auditorium has 20 rows of seats. There are 20 seats in the first row, 21 seats in the second row, 22 seats in the third row, and so on. (See Figure 9.7.) How many seats are there in all 20 rows?

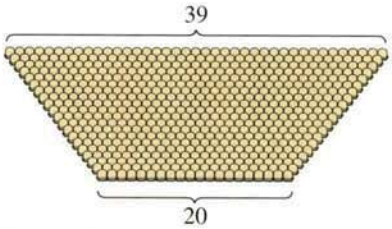


Figure 9.7

Solution

The numbers of seats in the 20 rows form an arithmetic sequence in which the common difference is $d = 1$. Because

$$c = a_1 - d = 20 - 1 = 19$$

you can determine that the formula for the nth term of the sequence is $a_n = n + 19$. So, the 20th term in the sequence is $a_{20} = 20 + 19 = 39$, and the total number of seats is

$$\begin{aligned} S_n &= 20 + 21 + 22 + \cdots + 39 \\ &= \frac{n}{2}(a_1 + a_{20}) \\ &= \frac{20}{2}(20 + 39) \qquad n = 20,\ a_1 = 20,\ a_{20} = 39 \\ &= 10(59) \\ &= 590. \end{aligned}$$

EXAMPLE 9 Total Sales

A small business sells \$10,000 worth of products during its first year. The owner of the business has set a goal of increasing annual sales by \$7500 each year for 19 years. Assuming that this goal is met, find the total sales during the first 20 years this business is in operation.

Algebraic Solution

The annual sales form an arithmetic sequence in which $a_1 = 10{,}000$ and $d = 7500$. So,

$$\begin{aligned} c &= a_1 - d \\ &= 10{,}000 - 7500 \\ &= 2500 \end{aligned}$$

and the nth term of the sequence is

$$a_n = 7500n + 2500.$$

This implies that the 20th term of the sequence is

$$\begin{aligned} a_{20} &= 7500(20) + 2500 \\ &= 152{,}500. \end{aligned}$$

The sum of the first 20 terms of the sequence is

$$S_n = \frac{n}{2}(a_1 + a_{20})$$

$$S_{20} = \frac{20}{2}(10{,}000 + 152{,}500) \qquad n = 20,\ a_1 = 10{,}000,\ a_{20} = 152{,}500$$

$$\begin{aligned} &= 10(162{,}500) \\ &= 1{,}625{,}000. \end{aligned}$$

So, the total sales for the first 20 years is \$1,625,500.

Numerical Solution

The annual sales form an arithmetic sequence in which $a_1 = 10{,}000$ and $d = 7500$. So, $c = a_1 - d = 10{,}000 - 7500 = 2500$. Use a graphing utility to create a table that shows the sales $u_n = 7500n + 2500$ for each of the 20 years, as shown in Figure 9.8. Then use the graphing utility to find that the sum of the data in the table is 1,625,000. So, the total sales for the first 20 years is \$1,625,000.

n	u(n)
14	107500
15	115000
16	122500
17	130000
18	137500
19	145000
20	152500

n=20

Figure 9.8

Writing About Math *Numerical Relationships*

Decide whether it is possible to fill in the blanks in each of the sequences such that the resulting sequence is arithmetic. If so, find a recursive formula for the sequence. Write a short paragraph explaining how you made your decisions.

a. $-7, \_\_, \_\_, \_\_, \_\_, \_\_, 11$

b. $17, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, 71$

c. $2, 6, \_\_, \_\_, 162$

d. $4, 7.5, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, 39$

e. $8, 12, \_\_, \_\_, \_\_, 60.75$

9.2 Exercises

In Exercises 1–8, determine whether the sequence is arithmetic. If it is, find the common difference.

1. 10, 8, 6, 4, 2, . . .

2. 4, 9, 14, 19, 24, . . .

3. $3, \frac{5}{2}, 2, \frac{3}{2}, 1, \ldots$

4. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \ldots$

5. $-24, -16, -8, 0, 8, \ldots$

6. $\ln 1, \ln 2, \ln 3, \ln 4, \ln 5, \ldots$

7. 3.7, 4.3, 4.9, 5.5, 6.1, . . .

8. $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

In Exercises 9–16, write the first five terms of the sequence. Determine whether the sequence is arithmetic. If it is, find the common difference.

9. $a_n = 8 + 13n$

10. $a_n = (2^n)n$

11. $a_n = \dfrac{1}{n+1}$

12. $a_n = 1 + (n-1)4$

13. $a_n = 150 - 7n$

14. $a_n = 2^{n-1}$

15. $a_n = 3 + \dfrac{(-1)^n 2}{n}$

16. $a_n = (-1)^n$

In Exercises 17–20, write the first five terms of the arithmetic sequence. Find the common difference and write the *n*th term of the sequence as a function of *n*.

17. $a_1 = 15, \quad a_{k+1} = a_k + 9$

18. $a_1 = 200, \quad a_{k+1} = a_k - 20$

19. $a_1 = \frac{7}{2}, \quad a_{k+1} = a_k - \frac{1}{4}$

20. $a_1 = 0.375, \quad a_{k+1} = a_k + 0.25$

In Exercises 21–28, write the first five terms of the arithmetic sequence. Use a graphing utility to verify your results numerically.

21. $a_1 = 5, d = 6$

22. $a_1 = 5, d = -\frac{3}{4}$

23. $a_1 = -2.6, d = -0.4$

24. $a_4 = 16, a_{10} = 46$

25. $a_8 = 26, a_{12} = 42$

26. $a_6 = -38, a_{11} = -73$

27. $a_3 = 19, a_{15} = -1.7$

28. $a_5 = 16, a_{14} = 38.5$

In Exercises 29–34, the first two terms of the arithmetic sequences are given. Find the missing term. Use a graphing utility to verify the result numerically.

29. $a_1 = 5, \quad a_2 = 11, \quad a_{10} =$

30. $a_1 = 3, \quad a_2 = 13, \quad a_9 =$

31. $a_1 = 2, \quad a_2 = -2, \quad a_{14} =$

32. $a_1 = -1, \quad a_2 = -10, \quad a_{25} =$

33. $a_1 = 4.2, \quad a_2 = 6.6, \quad a_7 =$

34. $a_1 = -0.7, \quad a_2 = -13.8, \quad a_8 =$

In Exercises 35–44, find a formula for a_n for the arithmetic sequence.

35. $a_1 = 1, d = 3$

36. $a_1 = 15, d = 4$

37. $a_1 = 100, d = -8$

38. $a_1 = 0, d = -\frac{2}{3}$

39. $4, \frac{3}{2}, -1, -\frac{7}{2}, \ldots$

40. $10, 5, 0, -5, -10, \ldots$

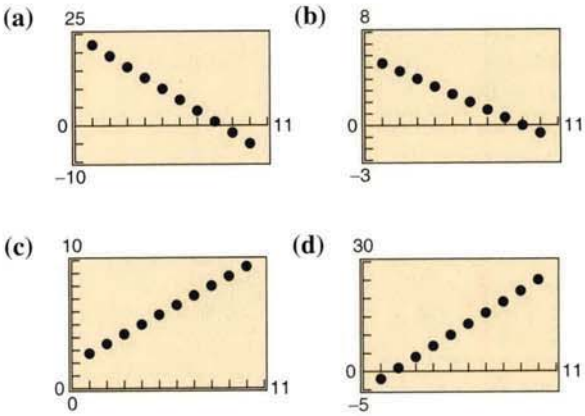
41. $a_1 = 5, a_4 = 15$

42. $a_1 = -4, a_5 = 16$

43. $a_3 = 94, a_6 = 85$

44. $a_5 = 190, a_{10} = 115$

In Exercises 45–48, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]



45. $a_n = -\frac{2}{3}n + 6$

46. $a_n = 3n - 5$

47. $a_n = 2 + \frac{3}{4}n$

48. $a_n = 25 - 3n$

In Exercises 49–52, use a graphing utility in *dot* mode to graph the first ten terms of the sequence.

49. $a_n = 15 - \frac{3}{2}n$

50. $a_n = -5 + 2n$

51. $a_n = 0.2n + 3$

52. $a_n = -0.3n + 8$

In Exercises 53–58, use the *table* feature of a graphing utility to find the first ten terms of the sequence.

53. $a_n = 4n - 5$

54. $a_n = 17 + 3n$

55. $a_n = 20 - \frac{3}{4}n$

56. $a_n = \frac{4}{5}n + 12$

57. $a_n = 1.5 + 0.005n$

58. $a_n = -12.4n + 9$

In Exercises 59–64, find the indicated *n*th partial sum of the arithmetic sequence.

59. 8, 26, 44, 62, . . . , $n = 10$

60. $-6, -2, 2, 6, \ldots, \quad n = 50$

61. 0.5, 1.3, 2.1, 2.9, . . . , $n = 10$

62. 40, 29, 18, 7, . . . , $n = 10$

63. $a_1 = 100, \ a_{25} = 220, \quad n = 25$

64. $a_1 = 15, \ a_{100} = 307, \quad n = 100$

In Exercises 65–72, find the partial sum without using a graphing utility.

65. $\sum_{n=1}^{50} n$

66. $\sum_{n=1}^{100} 2n$

67. $\sum_{n=1}^{100} 5n$

68. $\sum_{n=51}^{100} 7n$

69. $\sum_{n=11}^{30} n - \sum_{n=1}^{10} n$

70. $\sum_{n=51}^{100} n - \sum_{n=1}^{50} n$

71. $\sum_{n=1}^{500} (n + 3)$

72. $\sum_{n=1}^{250} (1000 - n)$

In Exercises 73–78, use a graphing utility to find the partial sum.

73. $\sum_{n=1}^{20} (2n + 5)$

74. $\sum_{n=1}^{100} \frac{n + 4}{2}$

75. $\sum_{n=0}^{50} (1000 - 5n)$

76. $\sum_{n=0}^{100} \frac{8 - 3n}{16}$

77. $\sum_{i=1}^{60} \left(250 - \frac{8}{3}i\right)$

78. $\sum_{j=1}^{200} (4.5 + 0.025j)$

79. Find the sum of the first 100 positive odd integers.

80. Find the sum of the integers from -10 to 50.

Job Offer **In Exercises 81 and 82, consider a job offer with the given starting salary and guaranteed salary increase for the first 5 years of employment.**

(a) Determine the person's salary during the sixth year of employment.

(b) Determine the person's total compensation from the company through 6 full years of employment.

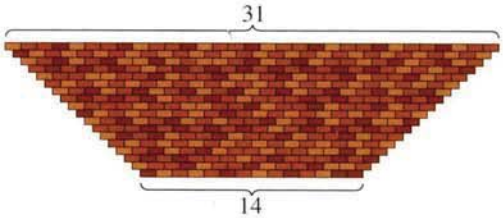
(c) Verify your results in parts (a) and (b) numerically.

	Starting Salary	*Annual Raise*
81.	\$32,500	\$1500
82.	\$36,800	\$1750

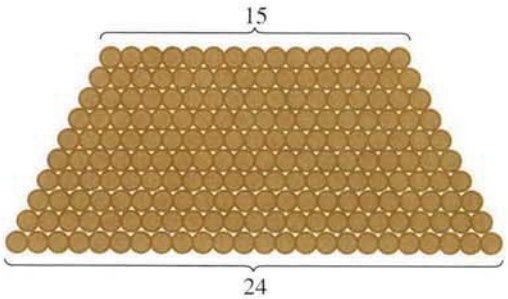
83. ***Seating Capacity*** Determine the seating capacity of an auditorium with 30 rows of seats if there are 20 seats in the first row, 24 seats in the second row, 28 seats in the third row, and so on.

84. ***Seating Capacity*** Determine the seating capacity of an auditorium with 36 rows of seats if there are 15 seats in the first row, 18 seats in the second row, 21 seats in the third row, and so on.

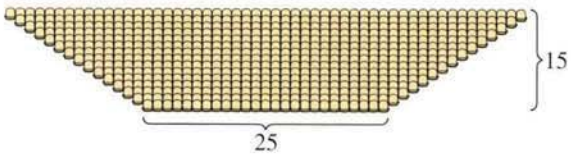
85. ***Brick Pattern*** A brick patio has the approximate shape of a trapezoid, as shown in the figure. The patio has 18 rows of bricks. The first row has 14 bricks and the 18th row has 31 bricks. How many bricks are in the patio?



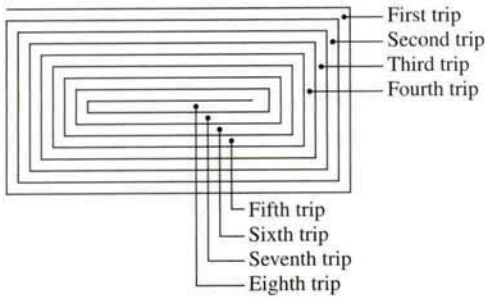
86. ***Number of Logs*** Logs are stacked in a pile, as shown in the figure. The top row has 15 logs and the bottom row has 24 logs. How many logs are in the stack?



87. ***Auditorium Seating*** Each row in a small auditorium has two more seats than the preceding row, as shown in the figure. Find the seating capacity of the auditorium if the front row seats 25 people and there are 15 rows of seats.



88. ***Baling Hay*** In the first two trips around a field baling hay, a farmer makes 93 bales and 89 bales, respectively, as shown in the figure. Because each trip is shorter than the preceding trip, the farmer estimates that the same pattern will continue. Estimate the total number of bales made if there are another six trips around the field.



89. ***Grandfather Clock*** Each hour, a grandfather clock strikes the number of times corresponding to the hour of the day. How many times does the clock strike in a day?

90. ***Falling Object*** An object (with negligible air resistance) is dropped from an airplane. During the first second of fall, the object falls 4.9 meters; during the second second of fall, it falls 14.7 meters; during the third second, it falls 24.5 meters; and during the fourth second, it falls 34.3 meters. If this arithmetic pattern continues, how many meters will the object have fallen after 10 seconds?

Synthesis

True or False? **In Exercises 91 and 92, determine whether the statement is true or false. Justify your answer.**

91. Given an arithmetic sequence for which only the first and second terms are known, it is possible to find the nth term.

92. If the only known information about a finite arithmetic sequence is its first term and its last term, then it is possible to find the sum of the sequence.

In Exercises 93 and 94, find the first ten terms of the sequence.

93. $a_1 = x, d = 2x$ **94.** $a_1 = -y, d = 5y$

95. ***Pattern Recognition***

(a) Compute the following five sums of positive odd integers.

$1 + 3 =$

$1 + 3 + 5 =$

$1 + 3 + 5 + 7 =$

$1 + 3 + 5 + 7 + 9 =$

$1 + 3 + 5 + 7 + 9 + 11 =$

(b) Use the sums in part (a) to make a conjecture about the sums of positive odd integers. Check your conjecture for the sum

$1 + 3 + 5 + 7 + 9 + 11 + 13 =$.

(c) Verify your conjecture algebraically.

96. ***Think About It*** The sum of the first 20 terms of an arithmetic sequence with a common difference of 3 is 650. Find the first term.

97. ***Think About It*** The sum of the first n terms of an arithmetic sequence with first term a_1 and common difference d is S_n. Determine the sum if the first term is increased by 5. Explain.

Review

In Exercises 98 and 99, use Gauss-Jordan elimination to solve the system of equations.

98. $\begin{cases} 2x - y + 7z = -10 \\ 3x + 2y - 4z = 17 \\ 6x - 5y + z = -20 \end{cases}$

99. $\begin{cases} -x + 4y + 10z = 4 \\ 5x - 3y + z = 31 \\ 8x + 2y - 3z = -5 \end{cases}$

In Exercises 100 and 101, use a determinant to find the area of the triangle with the given vertices.

100. $(0, 0), (4, -3), (2, 6)$ **101.** $(-1, 2), (5, 1), (3, 8)$

In Exercises 102 and 103, simplify the ratio of factorials.

102. $\dfrac{6!}{5! \cdot 2!}$ **103.** $\dfrac{6! \cdot 8!}{14!}$

9.3 Geometric Sequences and Series

Geometric Sequences

In Section 9.2, you learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence. In this section, you will study another important type of sequence called a **geometric sequence.** Consecutive terms of a geometric sequence have a common *ratio*.

Definition of Geometric Sequence

A sequence is a **geometric sequence** if the ratios of consecutive terms are the same.

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdot \cdot \cdot = r, \qquad r \neq 0$$

The number r is the **common ratio** of the sequence.

EXAMPLE 1 Examples of Geometric Sequences

a. The sequence whose nth term is 2^n is geometric. For this sequence, the common ratio between consecutive terms is 2.

$$\underbrace{2, 4,}_{\frac{4}{2} = 2} 8, 16, \ldots, 2^n, \ldots$$

b. The sequence whose nth term is $4(3^n)$ is geometric. For this sequence, the common ratio between consecutive terms is 3.

$$\underbrace{12, 36,}_{\frac{36}{12} = 3} 108, 324, \ldots, 4(3^n), \ldots$$

c. The sequence whose nth term is $\left(-\frac{1}{3}\right)^n$ is geometric. For this sequence, the common ratio between consecutive terms is $-\frac{1}{3}$.

$$\underbrace{-\frac{1}{3}, \frac{1}{9},}_{\frac{\frac{1}{9}}{-\frac{1}{3}} = -\frac{1}{3}} -\frac{1}{27}, \frac{1}{81}, \ldots, \left(-\frac{1}{3}\right)^n, \ldots$$

d. The sequence 1, 4, 9, 16, . . . , whose nth term is n^2 is *not* geometric. The ratio of the second term to first term is

$$\frac{a_2}{a_1} = \frac{4}{1} = 4$$

but the ratio of the third term to the second term is

$$\frac{a_3}{a_2} = \frac{9}{4}.$$

What You Should Learn:

- How to recognize, write, and find the nth terms of geometric sequences
- How to find nth partial sums of geometric sequences
- How to find sums of infinite geometric series
- How to use geometric sequences to model and solve real-life problems

Why You Should Learn It:

Geometric sequences can reduce the amount of time it takes to find the sum of a sequence of numbers with a common ratio. For instance, Exercise 92 on page 645 shows how to use a geometric sequence to estimate the population growth of a city.

Jeff Greenberg/PhotoEdit

In Example 1, each of the geometric sequences in parts (a), (b), and (c) has an nth term in the form ar^n, where the common ratio of the sequence is r.

The nth Term of a Geometric Sequence

The nth term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$\begin{array}{ccccccccc} a_1, & a_2, & a_3, & a_4, & a_5, & \ldots\ldots, & a_n, & \ldots\ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots\ldots & \downarrow & \ldots\ldots \\ a_1, & a_1 r, & a_1 r^2, & a_1 r^3, & a_1 r^4, & \ldots, & a_1 r^{n-1}, & \ldots \end{array}$$

If you know the nth term of a geometric sequence, you can find the $(n + 1)$th term by multiplying by r. That is, $a_{n+1} = ra_n$.

EXAMPLE 2 Finding the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence whose first term is $a_1 = 3$ and whose common ratio is $r = 2$.

Solution

Starting with 3, repeatedly multiply by 2 to obtain the following.

$a_1 = 3$	1st term
$a_2 = 3(2^1) = 6$	2nd term
$a_3 = 3(2^2) = 12$	3rd term
$a_4 = 3(2^3) = 24$	4th term
$a_5 = 3(2^4) = 48$	5th term

STUDY TIP

You can use a graphing utility to generate the geometric sequence in Example 2 using the following steps.

3 (enter key)
2 [×] (previous answer key)

Now press the enter key repeatedly to generate the terms of the sequence.

Most graphing utilities have a built-in function that will display the terms of a geometric sequence. Consult your user's manual for instructions.

EXAMPLE 3 Finding a Term of a Geometric Sequence

Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

Algebraic Solution

$a_n = a_1 r^{n-1}$	Formula for a geometric sequence
$a_{15} = 20(1.05)^{15-1}$	Substitute for a_1, r, and n.
≈ 39.599	Use a calculator.

Numerical Solution

For this sequence, $r = 1.05$ and $a_1 = 20$. Use the *table* feature of a graphing utility to create a table that shows the value of $u_n = 20(1.05)^{n-1}$ for $n = 1$ through $n = 15$. From Figure 9.9, the number in the fifteenth row is approximately 39.599, so the 15th term of the geometric sequence is about 39.599.

n	u(n)
9	29.549
10	31.027
11	32.578
12	34.207
13	35.917
14	37.713
15	39.599

u(n)=39.59863199

Figure 9.9

EXAMPLE 4 Finding a Term of a Geometric Sequence

Find a formula for the nth term of the following geometric sequence. What is the 9th term of the sequence?

$$5, 15, 45, \ldots$$

Solution

The common ratio of this sequence is

$$r = \frac{15}{5} = 3.$$

Because the first term is $a_1 = 5$, the formula must have the form

$$a_n = a_1 r^{n-1} = 5(3)^{n-1}.$$

You can determine the 9th term ($n = 9$) to be

$$\begin{aligned} a_9 &= 5(3)^{9-1} && \text{Substitute 9 for } n. \\ &= 5(6561) && \text{Use a calculator.} \\ &= 32{,}805. && \text{Simplify.} \end{aligned}$$

A graph of the first 9 terms of the sequence is shown in Figure 9.10. Notice that the points lie on an exponential curve. This makes sense because a_n is an exponential function of n.

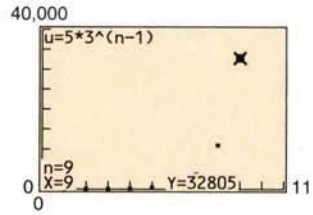


Figure 9.10

If you know *any* two terms of a geometric sequence, you can use that information to find a formula for the nth term of the sequence.

EXAMPLE 5 Finding a Term of a Geometric Sequence

The fourth term of a geometric sequence is 125, and the 10th term is 125/64. Find the 14th term. (Assume that the terms of the sequence are positive.)

Solution

The 10th term is related to the fourth term by the equation

$$a_{10} = a_4 r^6. \quad \text{Multiply 4th term by } r^{10-4}.$$

Because $a_{10} = 125/64$ and $a_4 = 125$, you can solve for r as follows.

$$\frac{125}{64} = 125r^6$$

$$\frac{1}{64} = r^6 \quad \Rightarrow \quad \frac{1}{2} = r$$

You can obtain the 14th term by multiplying the 10th term by r^4.

$$\begin{aligned} a_{14} &= a_{10} r^4 \\ &= \frac{125}{64}\left(\frac{1}{2}\right)^4 = \frac{125}{1024} \end{aligned}$$

STUDY TIP

Remember that r is the common ratio of consecutive terms of a geometric sequence. So, in Example 5,

$$\begin{aligned} a_{10} &= a_1 r^9 \\ &= a_1 \cdot r \cdot r \cdot r \cdot r^6 \\ &= a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot r^6 \\ &= a_4 r^6. \end{aligned}$$

The Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows. A proof of the formula is given in Appendix A.

The Sum of a Finite Geometric Sequence

The sum of the geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4, \ldots, a_1r^{n-1}$$

with common ratio $r \neq 1$ is

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

EXAMPLE 6 Finding the Sum of a Finite Geometric Sequence

Find the sum $\sum_{n=1}^{12} 4(0.3)^n$.

Solution

By writing out a few terms, you have

$$\sum_{n=1}^{12} 4(0.3)^n = 4(0.3) + 4(0.3)^2 + 4(0.3)^3 + \cdots + 4(0.3)^{12}.$$

Now, because $a_1 = 4(0.3)$, $r = 0.3$, and $n = 12$, you can apply the formula for the sum of a finite geometric sequence to obtain

$$\sum_{n=1}^{12} 4(0.3)^n = a_1\left(\frac{1 - r^n}{1 - r}\right) \quad \text{Formula for sum of a finite geometric sequence}$$

$$= 4(0.3)\left[\frac{1 - (0.3)^{12}}{1 - 0.3}\right] \quad \text{Substitute for } a_1, r, \text{ and } n.$$

$$\approx 1.714.$$

When using the formula for the sum of a geometric sequence, be careful to check that the index begins at $i = 1$. If the index begins at $i = 0$, you must adjust the formula for the nth partial sum. For instance, if the index in Example 6 had begun with $n = 0$, the sum would have been

$$\sum_{n=0}^{12} 4(0.3)^n = 4 + \sum_{n=1}^{12} 4(0.3)^n$$

$$\approx 4 + 1.714$$

$$= 5.714.$$

STUDY T!P

Using a graphing calculator, you can calculate the sum of the sequence in Example 6 to be 1.7142848.

Calculate the sum beginning at $n = 0$. You should obtain a sum of 5.7142848.

Geometric Series

The summation of the terms of an infinite geometric sequence is called an **infinite geometric series** or simply a **geometric series.**

The formula for the sum of a *finite* geometric sequence can, depending on the value of r, be extended to produce a formula for the sum of an *infinite* geometric series. Specifically, if the common ratio r has the property that $|r| < 1$, it can be shown that r^n becomes arbitrarily close to zero as n increases without bound. Consequently,

$$a_1\left(\frac{1 - r^n}{1 - r}\right) \longrightarrow a_1\left(\frac{1 - 0}{1 - r}\right) \quad \text{as} \quad n \longrightarrow \infty.$$

This result is summarized as follows.

The Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite geometric series

$$a_1, a_1r, a_1r^2, a_1r^3, \ldots, a_1r^{n-1}, \ldots$$

has the sum

$$S = \frac{a_1}{1 - r}.$$

Exploration

Notice that the formula for the sum of an infinite geometric series requires that $|r| < 1$. What happens if $r = 1$ or $r = -1$? Give examples of infinite geometric series for which $|r| > 1$ and convince yourself that they do not have finite sums.

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 7 Finding the Sum of an Infinite Geometric Series

Find each sum.

a. $\sum_{n=1}^{\infty} 4(0.6)^{n-1}$ **b.** $3 + 0.3 + 0.03 + 0.003 + \cdots$

Solution

a. $\sum_{n=1}^{\infty} 4(0.6)^{n-1} = 4(1) + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdots + 4(0.6)^{n-1} + \cdots$

$$= \frac{4}{1 - (0.6)} \qquad \frac{a_1}{1 - r}$$

$$= 10$$

b. $3 + 0.3 + 0.03 + 0.003 + \cdots = 3 + 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots$

$$= \frac{3}{1 - (0.1)} \qquad \frac{a_1}{1 - r}$$

$$= \frac{10}{3}$$

$$\approx 3.33$$

STUDY TIP

Use a graphing utility to create a table showing the partial sums of the series in Example 7(a). Consult your user's manual for instructions. How can you determine the upper limit of the series from the table?

Application

EXAMPLE 8 Compound Interest

A deposit of \$50 is made on the first day of each month in a savings account that pays 6% compounded monthly. What is the balance of this annuity at the end of 2 years?

Solution

The first deposit will gain interest for 24 months, and its balance will be

$$A_{24} = 50\left(1 + \frac{0.06}{12}\right)^{24} = 50(1.005)^{24}.$$

The second deposit will gain interest for 23 months, and its balance will be

$$A_{23} = 50\left(1 + \frac{0.06}{12}\right)^{23} = 50(1.005)^{23}.$$

The last deposit will gain interest for only 1 month, and its balance will be

$$A_1 = 50\left(1 + \frac{0.06}{12}\right)^{1} = 50(1.005).$$

The total balance in the account will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $A_1 = 50(1.005)$ and $r = 1.005$, you have

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$$ Formula for sum of a finite geometric sequence

$$S_{24} = 50(1.005)\left[\frac{1 - (1.005)^{24}}{1 - 1.005}\right]$$ Substitute values for a_1, r, and n.

$$= \$1277.96.$$ Simplify.

Writing About Math *An Experiment*

You will need a piece of string or yarn, a pair of scissors, and a tape measure. Measure out any length of string at least 5 feet long. Double over the string and cut it in half. Take one of the resulting halves, double it over, and cut it in half. Continue this process until you are no longer able to cut a length of string in half. How many cuts were you able to make? Construct a sequence of the resulting string lengths after each cut, starting with the original length of the string. Find a formula for the nth term of this sequence. How many cuts could you theoretically make? Write a short paragraph discussing why you were not able to make that many cuts.

9.3 Exercises

In Exercises 1–10, determine whether the sequence is geometric. If it is, find the common ratio.

1. 5, 15, 45, 135, . . .
2. 3, 15, 75, 375, . . .
3. 6, 18, 30, 42, . . .
4. 1, −2, 4, −8, . . .
5. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$
6. 3, 0.6, 0.12, 0.024, . . .
7. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
8. $9, -6, 4, -\frac{8}{3}, \ldots$
9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
10. $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \frac{4}{11}, \ldots$

In Exercises 11–20, write the first five terms of the geometric sequence.

11. $a_1 = 8, \quad r = 3$
12. $a_1 = 10, \quad r = 2$
13. $a_1 = 1, \quad r = \frac{1}{2}$
14. $a_1 = 2, \quad r = \frac{1}{3}$
15. $a_1 = 5, \quad r = -\frac{1}{10}$
16. $a_1 = 6, \quad r = -\frac{1}{4}$
17. $a_1 = 3.5, \quad r = 5$
18. $a_1 = 0.4, \quad r = \frac{5}{2}$
19. $a_1 = 1, \quad r = e$
20. $a_1 = 4, \quad r = \sqrt{3}$

In Exercises 21–26, write the first five terms of the geometric sequence. Determine the common ratio and write the *n*th term of the sequence as a function of *n*.

21. $a_1 = 64, \quad a_{k+1} = \frac{1}{2}a_k$
22. $a_1 = 81, \quad a_{k+1} = \frac{1}{3}a_k$
23. $a_1 = 4, \quad a_{k+1} = 3a_k$
24. $a_1 = 5, \quad a_{k+1} = -2a_k$
25. $a_1 = 6, \quad a_{k+1} = -\frac{3}{2}a_k$
26. $a_1 = 36, \quad a_{k+1} = -\frac{2}{3}a_k$

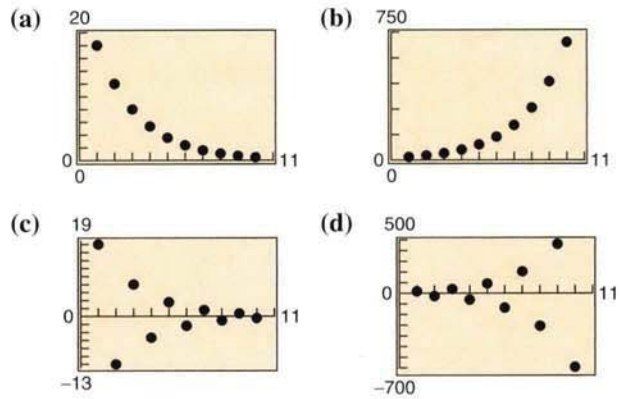
In Exercises 27–36, find the *n*th term of the geometric sequence. Use a graphing utility to verify your answer numerically.

27. $a_1 = 4, \quad r = \frac{1}{2}, \quad n = 10$
28. $a_1 = 5, \quad r = \frac{3}{2}, \quad n = 8$
29. $a_1 = 6, \quad r = -\frac{1}{3}, \quad n = 12$
30. $a_1 = 8, \quad r = \sqrt{5}, \quad n = 9$
31. $a_1 = 500, \quad r = 1.02, \quad n = 14$
32. $a_1 = 1000, \quad r = 1.005, \quad n = 11$
33. $a_1 = 16, \quad a_4 = \frac{27}{4}, \quad n = 3$
34. $a_2 = 3, \quad a_5 = \frac{3}{64}, \quad n = 1$
35. $a_2 = -18, \quad a_5 = \frac{2}{3}, \quad n = 6$
36. $a_3 = \frac{16}{3}, \quad a_5 = \frac{64}{27}, \quad n = 7$

In Exercises 37–42, find the indicated *n*th term of the geometric sequence.

37. 9th term: 7, 21, 63, . . .
38. 7th term: 3, 36, 432, . . .
39. 10th term: 5, 30, 180, . . .
40. 22nd term: 4, 8, 16, . . .
41. 12th term: $\frac{3}{16}, \frac{3}{4}, 3, \ldots$
42. 8th term: $\frac{1}{2}, 8, 128, \ldots$

In Exercises 43–46, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]



43. $a_n = 18\left(\frac{2}{3}\right)^{n-1}$
44. $a_n = 18\left(-\frac{2}{3}\right)^{n-1}$
45. $a_n = 18\left(\frac{3}{2}\right)^{n-1}$
46. $a_n = 18\left(-\frac{3}{2}\right)^{n-1}$

In Exercises 47–50, use a graphing utility to graph the first ten terms of the sequence.

47. $a_n = 12(-0.75)^{n-1}$
48. $a_n = 12(-0.4)^{n-1}$
49. $a_n = 2(1.3)^{n-1}$
50. $a_n = 2(-1.4)^{n-1}$

In Exercises 51 and 52, find the first four terms of the sequence of partial sums of the geometric series. In a sequence of partial sums, the term S_n is the sum of the first n terms of the sequence. For instance, S_2 is the sum of the first two terms.

51. $8, -4, 2, -1, \frac{1}{2}, \ldots$
52. $8, 12, 18, 27, \frac{81}{2}, \ldots$

In Exercises 53 and 54, use a graphing utility to create a table showing the sequence of partial sums of the first ten terms of the series.

53. $\sum_{n=1}^{\infty} 16\left(\frac{1}{2}\right)^{n-1}$

54. $\sum_{n=1}^{\infty} 4(0.2)^{n-1}$

In Exercises 55–64, find the sum. Use a graphing utility to verify your result.

55. $\sum_{n=1}^{9} 2^{n-1}$

56. $\sum_{n=1}^{9} (-2)^{n-1}$

57. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1}$

58. $\sum_{i=1}^{6} 32\left(\frac{1}{4}\right)^{i-1}$

59. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^{n}$

60. $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^{n}$

61. $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1}$

62. $\sum_{i=1}^{10} 5\left(-\frac{1}{3}\right)^{i-1}$

63. $\sum_{n=0}^{5} 300(1.06)^{n}$

64. $\sum_{n=0}^{6} 500(1.04)^{n}$

In Exercises 65–68, use summation notation to express the sum.

65. $5 + 15 + 45 + \cdots + 3645$

66. $7 + 14 + 28 + \cdots + 896$

67. $2 - \frac{1}{2} + \frac{1}{8} - \cdots + \frac{1}{2048}$

68. $15 - 3 + \frac{3}{5} - \cdots - \frac{3}{625}$

In Exercises 69–84, find the sum of the infinite geometric series.

69. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n}$

70. $\sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^{n}$

71. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n}$

72. $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^{n}$

73. $\sum_{n=0}^{\infty} 4\left(\frac{1}{4}\right)^{n}$

74. $\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^{n}$

75. $\sum_{n=1}^{\infty} 2\left(\frac{7}{3}\right)^{n-1}$

76. $\sum_{n=1}^{\infty} \frac{1}{2}(2)^{n}$

77. $\sum_{n=0}^{\infty} (0.4)^{n}$

78. $\sum_{n=0}^{\infty} 4(0.2)^{n}$

79. $\sum_{n=0}^{\infty} -3(0.9)^{n}$

80. $\sum_{n=0}^{\infty} -10(0.2)^{n}$

81. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$

82. $9 + 6 + 4 + \frac{8}{3} + \cdots$

83. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$

84. $-6 + 5 - \frac{25}{6} + \frac{125}{36} - \cdots$

In Exercises 85–88, find the rational number representation of the repeating decimal.

85. $0.\overline{36}$

86. $0.\overline{297}$

87. $0.3\overline{18}$

88. $1.3\overline{8}$

89. ***Compound Interest*** A principal of \$1000 is invested at 8% interest. Find the amount after 10 years if the interest is compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

90. ***Compound Interest*** A principal of \$2500 is invested at 7% interest. Find the amount after 20 years if the interest is compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

91. ***Depreciation*** A company buys a machine for \$155,000 and it depreciates at a rate of 30% per year. (In other words, at the end of each year the depreciated value is 70% of what it was at the beginning of the year.) Find the depreciated value of the machine after 5 full years.

92. ***Population Growth*** A city of 350,000 people is growing at a rate of 1.3% per year. Estimate the population of the city 30 years from now.

93. ***Annuities*** A deposit of \$100 is made at the beginning of each month in an account that pays 6% interest, compounded monthly. The balance A in the account at the end of 5 years is

$$A = 100\left(1 + \frac{0.06}{12}\right)^{1} + \cdots + 100\left(1 + \frac{0.06}{12}\right)^{60}.$$

Find A.

94. ***Annuities*** A deposit of \$50 is made at the beginning of each month in an account that pays 8% interest, compounded monthly. The balance A in the account at the end of 5 years is

$$A = 50\left(1 + \frac{0.08}{12}\right)^{1} + \cdots + 50\left(1 + \frac{0.08}{12}\right)^{60}.$$

Find A.

95. ***Annuities*** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r, compounded monthly. The balance A after t years is

$$A = P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^{2} + \cdots + P\left(1 + \frac{r}{12}\right)^{12t}.$$

Show that the balance is

$$A = P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right).$$

96. ***Annuities*** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r, compounded continuously. The balance A after t years is

$$A = Pe^{r/12} + Pe^{2r/12} + \cdots + Pe^{12tr/12}.$$

Show that the balance is $A = \dfrac{Pe^{r/12}(e^{rt} - 1)}{e^{r/12} - 1}$.

Annuities **In Exercises 97–100, consider making monthly deposits of P dollars in a savings account earning an annual interest rate r. Use the results of Exercises 95 and 96 to find the balance A after t years if the interest is compounded (a) monthly and (b) continuously.**

97. $P = \$50,\ r = 7\%,\ t = 20$ years

98. $P = \$75,\ r = 9\%,\ t = 25$ years

99. $P = \$100,\ r = 10\%,\ t = 40$ years

100. $P = \$20,\ r = 6\%,\ t = 50$ years

101. ***Annuities*** Consider an initial deposit of P dollars in an account earning an annual interest rate r, compounded monthly. At the end of each month, a withdrawal of W dollars will occur and the account will be depleted in t years. The amount of the initial deposit required is

$$P = W\left(1 + \frac{r}{12}\right)^{-1} + W\left(1 + \frac{r}{12}\right)^{-2} + \cdots + W\left(1 + \frac{r}{12}\right)^{-12t}.$$

Show that the initial deposit is

$$P = W\left(\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right].$$

102. ***Annuities*** Determine the amount required in an individual retirement account for an individual who retires at age 65 and wants an income of \$2000 from the account each month for 20 years. Use the result of Exercise 101, and assume that the account earns 9% compounded monthly.

103. ***Geometry*** The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles are shaded. If this process is repeated five more times, determine the total area of the shaded region.

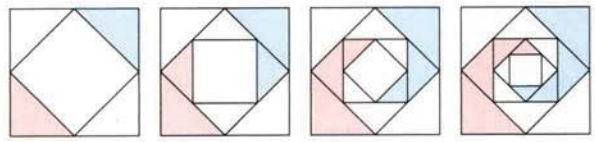
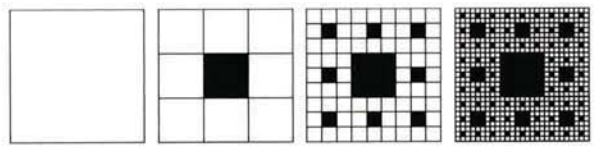


FIGURE FOR 103

104. ***Geometry*** The sides of a square are 27 inches in length. New squares are formed by dividing the original square into nine squares. The center square is then shaded. If this process is repeated three more times, determine the total area of the shaded region.



105. ***Corporate Revenue*** The annual revenues a_n (in billions of dollars) for the Coca-Cola Company for 1990 through 1996 can be approximated by the model

$$a_n = 3.978e^{0.11n}, \quad n = 0, 1, \ldots, 6$$

where $n = 0$ represents 1990. Use this model and the formula for the sum of a finite geometric sequence to approximate the total revenue earned during this 7-year period. (Source: Coca-Cola Enterprises, Inc.)

106. ***Distance*** A ball is dropped from a height of 16 feet. Each time it drops h feet, it rebounds $0.81h$ feet.

(a) Find the total distance traveled by the ball.

(b) The ball takes the following time for each fall.

$$\begin{aligned}
s_1 &= -16t^2 + 16, & s_1 &= 0 \text{ if } t = 1\\
s_2 &= -16t^2 + 16(0.81), & s_2 &= 0 \text{ if } t = 0.9\\
s_3 &= -16t^2 + 16(0.81)^2, & s_3 &= 0 \text{ if } t = (0.9)^2\\
s_4 &= -16t^2 + 16(0.81)^3, & s_4 &= 0 \text{ if } t = (0.9)^3\\
&\vdots & &\vdots\\
s_n &= -16t^2 + 16(0.81)^{n-1}, & s_n &= 0 \text{ if } t = (0.9)^{n-1}
\end{aligned}$$

Beginning with s_2, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is

$$t = 1 + 2\sum_{n=1}^{\infty}(0.9)^n.$$

Find this total.

107. ***Salary*** Suppose you go to work for a company that pays \$0.01 the first day, \$0.02 the second day, \$0.04 the third day, and so on. If the daily wage keeps doubling, what will your total income be for working (a) 29 days? (b) 30 days? (c) 31 days?

108. ***Salary*** A company is offering a job with a salary of \$30,000 for the first year. Suppose that during the next 39 years, there is a 5% raise each year. Determine the total compensation over the 40-year period.

Synthesis

True or False? **In Exercises 109–111, determine whether the statement is true or false. Justify your answer.**

109. A sequence is geometric if the ratios of consecutive differences of consecutive terms are the same.

110. You can find the nth term of a geometric sequence by multiplying its common ratio by the first term of the sequence raised to the $(n - 1)$ power.

111. A geometric sequence with a common ratio of 1 is also an arithmetic sequence.

In Exercises 112–115, write the first five terms of the geometric sequence.

112. $a_1 = 3, r = \frac{x}{2}$

113. $a_1 = 8, r = \frac{2x}{3}$

114. $a_1 = 5, r = 2x$

115. $a_1 = \frac{1}{2}, r = 7x$

In Exercises 116–119, find the nth term of the geometric sequence.

116. $a_1 = 100, r = e^x, n = 9$

117. $a_1 = 6, r = 3e^x, n = 8$

118. $a_1 = 1, r = -\frac{x}{3}, n = 7$

119. $a_1 = 4, r = \frac{4x}{3}, n = 6$

Graphical Reasoning **In Exercises 120 and 121, use a graphing utility to graph the function. Identify the horizontal asymptote of the graph and determine its relationship to the sum.**

120. $f(x) = 6\left[\frac{1 - (0.5)^x}{1 - (0.5)}\right], \quad \sum_{n=0}^{\infty} 6\left(\frac{1}{2}\right)^n$

121. $f(x) = 2\left[\frac{1 - (0.8)^x}{1 - (0.8)}\right], \quad \sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$

122. ***Writing*** Write a brief paragraph explaining why the terms of a geometric sequence decrease in magnitude when $-1 < r < 1$.

123. ***Writing*** Write a brief paragraph explaining how to use the first two terms of a geometric sequence to find the nth term.

Review

124. The ratio of cement to sand in a 90-pound bag of dry mix is 1 to 4. Find the number of pounds of sand in the bag.

125. A truck traveled at an average speed of 50 miles per hour on a 200-mile trip. On the return trip, the average speed was 42 miles per hour. Find the average speed for the round trip.

126. Find two consecutive positive even integers whose product is 624.

127. Suppose your friend can mow a lawn in 4 hours and you can mow it in 6 hours. How long will it take both of you to mow the lawn?

In Exercises 128–131, perform the matrix operation.

128. $\begin{bmatrix} 4 & -1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

129. $-4\begin{bmatrix} 7 & 2 \\ -6 & 0 \end{bmatrix} + \begin{bmatrix} -5 & 5 \\ 3 & 1 \end{bmatrix}$

130. $\begin{bmatrix} -1 & 3 & 4 \\ -2 & 8 & 0 \\ 2 & 5 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 4 \\ -4 & 3 & 5 \\ 0 & 2 & -3 \end{bmatrix}$

131. $2\begin{bmatrix} -1 & 3 & 4 \\ -2 & 8 & 0 \\ 2 & 5 & -1 \end{bmatrix} - 4\begin{bmatrix} -1 & 0 & 4 \\ -4 & 3 & 5 \\ 0 & 2 & -3 \end{bmatrix}$

In Exercises 132–135, find the sum.

132. $\sum_{i=1}^{4} (3i + 4)$

133. $\sum_{i=0}^{6} 4i^2$

134. $\sum_{k=1}^{5} 12$

135. $\sum_{k=0}^{4} \frac{2}{k^2 + 2}$

9.4 Mathematical Induction

Introduction

In this section you will study a form of mathematical proof called **mathematical induction.** It is important that you clearly see the logical need for it, so let's take a closer look at a problem discussed in Example 5 on page 632.

$$S_1 = 1 = 1^2$$

$$S_2 = 1 + 3 = 2^2$$

$$S_3 = 1 + 3 + 5 = 3^2$$

$$S_4 = 1 + 3 + 5 + 7 = 4^2$$

$$S_5 = 1 + 3 + 5 + 7 + 9 = 5^2$$

Judging from the pattern formed by these first five sums, it appears that the sum of the first n odd integers is

$$S_n = 1 + 3 + 5 + 7 + 9 + \cdot \cdot \cdot + (2n - 1) = n^2.$$

Although this particular formula is valid, it is important for you to see that recognizing a pattern and then simply *jumping to the conclusion* that the pattern must be true for all values of n is *not* a logically valid method of proof. There are many examples in which a pattern appears to be developing for small values of n but then fails at some point. One of the most famous cases of this was the conjecture by the French mathematician Pierre de Fermat (1601–1665), who speculated that all numbers of the form

$$F_n = 2^{2^n} + 1, \quad n = 0, 1, 2, \ldots$$

are prime. For $n = 0, 1, 2, 3,$ and 4, the conjecture is true.

$$F_0 = 3$$

$$F_1 = 5$$

$$F_2 = 17$$

$$F_3 = 257$$

$$F_4 = 65{,}537$$

The size of the next Fermat number ($F_5 = 4{,}294{,}967{,}297$) is so great that it was difficult for Fermat to determine whether or not it was prime. However, another well-known mathematician, Leonhard Euler (1707–1783), later found a factorization

$$\begin{aligned} F_5 &= 4{,}294{,}967{,}297 \\ &= 641(6{,}700{,}417) \end{aligned}$$

which proved that F_5 is not prime. Therefore Fermat's conjecture was false.

Just because a rule, pattern, or formula seems to work for several values of n, you cannot simply decide that it is valid for *all* values of n without going through a *legitimate proof.*

What You Should Learn:

- How to use mathematical induction to prove statements involving a positive integer n
- How to find sums of powers of integers
- How to find finite differences of sequences

Why You Should Learn It:

Mathematical induction can be used to prove statements involving positive integers. For instance, in Exercises 27–35 on page 654, you are asked to use mathematical induction to prove properties of positive integers.

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. If

1. P_1 is true, and
2. the truth of P_k implies the truth of P_{k+1}, for every positive k,

then P_n must be true for all positive integers n.

STUDY TIP

It is important to recognize that *both* parts of the Principle of Mathematical Induction are necessary.

To apply the Principle of Mathematical Induction, you need to be able to determine the statement P_{k+1} for a given statement P_k.

EXAMPLE 1 A Preliminary Example

Find P_{k+1} for the following.

a. $P_k : S_k = \dfrac{k^2(k+1)^2}{4}$

b. $P_k : S_k = 1 + 5 + 9 + \cdots + [4(k-1) - 3] + (4k - 3)$

c. $P_k : 3^k \geq 2k + 1$

Solution

a. $P_{k+1} : S_{k+1} = \dfrac{(k+1)^2(k+1+1)^2}{4}$ Substitute $k + 1$ for k.

$= \dfrac{(k+1)^2(k+2)^2}{4}$ Simplify.

b. $P_{k+1} : S_{k+1} = 1 + 5 + 9 + \cdots + \{4[(k+1) - 1] - 3\} + [4(k+1) - 3]$

$= 1 + 5 + 9 + \cdots + (4k - 3) + (4k + 1)$

c. $P_{k+1} : 3^{k+1} \geq 2(k+1) + 1$

$3^{k+1} \geq 2k + 3$

A well-known illustration used to explain why the Principle of Mathematical Induction works is the unending line of dominoes represented by Figure 9.11. If the line actually contains infinitely many dominoes, it is clear that you could not knock down the entire line by knocking down only *one domino* at a time. However, suppose it were true that each domino would knock down the next one as it fell. Then you could knock them all down simply by pushing the first one and starting a chain reaction. Mathematical induction works in the same way. If the truth of P_k implies the truth of P_{k+1} and if P_1 is true, the chain reaction proceeds as follows: P_1 implies P_2, P_2 implies P_3, P_3 implies P_4, and so on.

Figure 9.11

When using mathematical induction to prove a *summation* formula (such as the one in Example 2), it is helpful to think of S_{k+1} as

$$S_{k+1} = S_k + a_{k+1}$$

where a_{k+1} is the $(k + 1)$ term of the original sum.

EXAMPLE 2 Using Mathematical Induction

Use mathematical induction to prove the following formula.

$$\begin{aligned} S_n &= 1 + 3 + 5 + 7 + \cdots + (2n - 1) \\ &= n^2 \end{aligned}$$

Solution

Mathematical induction consists of two distinct parts. First, you must show that the formula is true when $n = 1$.

1. When $n = 1$, the formula is valid, because

$$S_1 = 1 = 1^2.$$

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for *some* integer k. The second step is to use this assumption to prove that the formula is valid for the next integer, $k + 1$.

2. Assuming that the formula

$$\begin{aligned} S_k &= 1 + 3 + 5 + 7 + \cdots + (2k - 1) \\ &= k^2 \end{aligned}$$

is true, you must show that the formula $S_{k+1} = (k + 1)^2$ is true.

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + 7 + \cdots + (2k - 1) + [2(k + 1) - 1] \\ &= [1 + 3 + 5 + 7 + \cdots + (2k - 1)] + (2k + 2 - 1) \\ &= S_k + (2k + 1) && \text{Group terms to form } S_k. \\ &= k^2 + 2k + 1 && \text{Substitute } k^2 \text{ for } S_k. \\ &= (k + 1)^2 \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* positive integer values of n.

It occasionally happens that a statement involving natural numbers is not true for the first $k - 1$ positive integers but is true for all values of $n \geq k$. In these instances, you use a slight variation of the Principle of Mathematical Induction in which you verify P_k rather than P_1. This variation is called the *extended principle of mathematical induction*. To see the validity of this, note from Figure 9.11 that all but the first $k - 1$ dominoes can be knocked down by knocking over the kth domino. This suggests that you can prove a statement P_n to be true for $n \geq k$ by showing that P_k is true and that P_k implies P_{k+1}. In Exercises 21–26 in this section, you are asked to apply this extension of mathematical induction.

EXAMPLE 3 Using Mathematical Induction

Use mathematical induction to prove the formula

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$$
$$= \frac{n(n+1)(2n+1)}{6}$$

for all $n \geq 1$.

Solution

1. When $n = 1$, the formula is valid, because

$$S_1 = 1^2 = \frac{1(2)(3)}{6}.$$

2. Assuming that

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2$$
$$= \frac{k(k+1)(2k+1)}{6}$$

you must show that

$$S_{k+1} = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}.$$

To do this, write the following.

$$\begin{aligned} S_{k+1} &= S_k + a_{k+1} \\ &= (1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* $n \geq 1$.

When proving a formula with mathematical induction, the only statement that you *need* to verify is P_1. As a check, however, it is a good idea to try verifying some of the other statements. For instance, in Example 3, try verifying P_2 and P_3.

Sums of Powers of Integers

The formula in Example 3 is one of a collection of useful summation formulas. This and other formulas dealing with the sums of various powers of the first n positive integers are summarized as follows.

Sums of Powers of Integers

1. $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \cdots + n = \dfrac{n(n+1)}{2}$
2. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
3. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$
4. $\sum_{i=1}^{n} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
5. $\sum_{i=1}^{n} i^5 = 1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5 = \dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$

Each of these formulas for sums can be proven by mathematical induction. (See Exercises 13, 14, 17, and 18 in this section.)

EXAMPLE 4 Proving an Inequality by Mathematical Induction

Prove that $n < 2^n$ for all positive integers n.

Solution

1. For $n = 1$ and $n = 2$, the formula is true, because

$$1 < 2^1 \text{ and } 2 < 2^2.$$

2. Assuming that

$$k < 2^k$$

you need to show that $k + 1 < 2^{k+1}$. Note first that

$$2^{k+1} = 2(2^k) > 2(k) = 2k. \qquad \text{By assumption } 2^k > k.$$

Because $2k = k + k > k + 1$ for all $k > 1$, it follows that

$$2^{k+1} > 2k > k + 1$$

or

$$k + 1 < 2^{k+1}.$$

Therefore, $n < 2^n$ for all integers $n \geq 1$.

Finite Differences

The **first differences** of a sequence are found by subtracting consecutive terms. The **second differences** are found by subtracting consecutive first differences. The first and second differences of the sequence 3, 5, 8, 12, 17, 23, . . . are as follows.

n:	1		2		3		4		5		6
a_n:	3		5		8		12		17		23
First differences:		2		3		4		5		6	
Second differences:			1		1		1		1		

For this sequence, the second differences are all the same. When this happens, and the second differences are nonzero, the sequence has a perfect quadratic model. If the first differences are all the same nonzero number, the sequence has a linear model—that is, it is arithmetic.

EXAMPLE 5 Finding a Quadratic Model

Find the quadratic model for the sequence 3, 5, 8, 12, 17, 23, ...

Solution

You know from the second differences shown above the model is quadratic and has the form

$$a_n = an^2 + bn + c.$$

By substituting 1, 2, and 3 for n, you can obtain a system of three linear equations in three variables.

$a_1 = a(1)^2 + b(1) + c = 3$ Substitute 1 for n.

$a_2 = a(2)^2 + b(2) + c = 5$ Substitute 2 for n.

$a_3 = a(3)^2 + b(3) + c = 8$ Substitute 3 for n.

You now have a system of three equations in a, b, and c.

$$\begin{cases} a + b + c = 3 & \text{Equation 1} \\ 4a + 2b + c = 5 & \text{Equation 2} \\ 9a + 3b + c = 8 & \text{Equation 3} \end{cases}$$

Solving this system of equations using techniques discussed in Chapter 7, you can find the solution to be $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = 2$. So, the quadratic model is

$$a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 2.$$

Check the values of a_1, a_2, and a_3 as follows.

Check

$a_1 = \frac{1}{2}(1)^2 + \frac{1}{2}(1) + 2 = 3$ Solution checks. ✓

$a_2 = \frac{1}{2}(2)^2 + \frac{1}{2}(2) + 2 = 5$ Solution checks. ✓

$a_3 = \frac{1}{2}(3)^2 + \frac{1}{2}(3) + 2 = 8$ Solution checks. ✓

9.4 Exercises

In Exercises 1–6, find P_{k+1} for the given P_k.

1. $P_k = \dfrac{5}{k(k+1)}$

2. $P_k = \dfrac{4}{(k+2)(k+3)}$

3. $P_k = \dfrac{k^2(k+3)^2}{6}$

4. $P_k = \dfrac{k}{2}(5k-3)$

5. $P_k = 1 + 6 + 11 + \cdots + [5(k-1) - 4] + (5k-4)$

6. $P_k = 7 + 13 + 19 + \cdots + [6(k-1) + 1] + (6k+1)$

In Exercises 7–20, use mathematical induction to prove the formula for every positive integer n.

7. $2 + 4 + 6 + 8 + \cdots + 2n = n(n+1)$

8. $1 + 5 + 9 + 13 + \cdots + (4n-3) = n(2n-1)$

9. $3 + 8 + 13 + 18 + \cdots + (5n-2) = \dfrac{n}{2}(5n+1)$

10. $1 + 4 + 7 + 10 + \cdots + (3n-2) = \dfrac{n}{2}(3n-1)$

11. $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$

12. $2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-1}) = 3^n - 1$

13. $1 + 2 + 3 + 4 + \cdots + n = \dfrac{n(n+1)}{2}$

14. $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

15. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

16. $\left(1 + \dfrac{1}{1}\right)\left(1 + \dfrac{1}{2}\right)\left(1 + \dfrac{1}{3}\right)\cdots\left(1 + \dfrac{1}{n}\right) = n + 1$

17. $\displaystyle\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

18. $\displaystyle\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

19. $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

20. $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

In Exercises 21–26, prove the inequality for the indicated integer values of n.

21. $n! > 2^n, \quad n \geq 4$

22. $\left(\frac{4}{3}\right)^n > n, \quad n \geq 7$

23. $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}, \quad n \geq 2$

24. $\left(\dfrac{x}{y}\right)^{n+1} < \left(\dfrac{x}{y}\right)^n, \quad n \geq 1$ and $0 < x < y$

25. $(1 + a)^n \geq na, \quad n \geq 1, \quad a > 1$

26. $3^n > n\,2^n, \quad n \geq 1$

In Exercises 27–35, use mathematical induction to prove the given property for all positive integers n.

27. $(ab)^n = a^n b^n$

28. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

29. If $x_1 \neq 0,\ x_2 \neq 0, \ldots, x_n \neq 0$, then
$(x_1 x_2 x_3 \cdots x_n)^{-1} = x_1^{-1} x_2^{-1} x_3^{-1} \cdots x_n^{-1}$.

30. If $x_1 > 0,\ x_2 > 0, \ldots, x_n > 0$, then
$\ln(x_1 x_2 x_3 \cdots x_n) = \ln x_1 + \ln x_2 + \ln x_3 + \cdots + \ln x_n$.

31. Generalized Distributive Law:
$x(y_1 + y_2 + \cdots + y_n) = xy_1 + xy_2 + \cdots + xy_n$

32. $(a + bi)^n$ and $(a - bi)^n$ are complex conjugates for all $n \geq 1$.

33. A factor of $(n^3 + 3n^2 + 2n)$ is 3.

34. A factor of $(2^{2n-1} + 3^{2n-1})$ is 5.

35. A factor of $(9^n - 8n - 1)$ is 64 for all $n \geq 2$.

In Exercises 36–39, write the first five terms of the sequence.

36. $a_0 = 1$
$a_n = a_{n-1} + 2$

37. $a_0 = 10$
$a_n = 4a_{n-1}$

38. $a_0 = 4$
$a_1 = 2$
$a_n = a_{n-1} - a_{n-2}$

39. $a_0 = 0$
$a_1 = 2$
$a_n = a_{n-1} + 2a_{n-2}$

In Exercises 40–49, write the first five terms of the sequence beginning with the given term. Then calculate the first and second differences of the sequence. Does the sequence have a linear model, a quadratic model, or neither?

40. $a_1 = 0$
$a_n = a_{n-1} + 3$

41. $a_1 = 2$
$a_n = n - a_{n-1}$

42. $a_1 = 3$
$a_n = a_{n-1} - n$

43. $a_2 = -3$
$a_n = -2a_{n-1}$

44. $a_0 = 0$
$a_n = a_{n-1} + n$

45. $a_0 = 2$
$a_n = (a_{n-1})^2$

46. $a_1 = 2$
$a_n = a_{n-1} + 2$

47. $a_1 = 0$
$a_n = a_{n-1} + 2n$

48. $a_0 = 1$
$a_n = a_{n-1} + n^2$

49. $a_0 = 0$
$a_n = a_{n-1} - 1$

In Exercises 50–53, find a quadratic model for the sequence with the indicated terms.

50. $a_0 = 3,\ a_1 = 3, a_4 = 15$
51. $a_0 = 7,\ a_1 = 6,\ a_3 = 10$
52. $a_0 = -3,\ a_2 = 1,\ a_4 = 9$
53. $a_0 = 3,\ a_2 = 0,\ a_6 = 36$

Synthesis

True or False? **In Exercises 54–56, determine whether the statement is true or false. Justify your answer.**

54. If the statement P_k is true and P_k implies P_{k+1}, then P_1 is also true.
55. If a sequence is arithmetic, then the first differences of the sequence are all zero.
56. A sequence with n terms has $n - 1$ second differences.
57. ***Writing*** In your own words, explain what is meant by a proof by mathematical induction.
58. ***Think About It*** What conclusion can be drawn from the given information about the sequence of statements P_n?
    (a) P_3 is true and P_k implies P_{k+1}.
    (b) $P_1, P_2, P_3, \ldots, P_{50}$ are all true.
    (c) P_1, P_2, and P_3 are all true, but the truth of P_k does not imply that P_{k+1} is true.
    (d) P_2 is true and P_{2k} implies P_{2k+2}.

Review

In Exercises 59–62, solve the system of equations.

59. $\begin{cases} x - y = 2 \\ -4x + 5y = -3 \end{cases}$

60. $\begin{cases} x - 3y = 1 \\ 7x - 6y = -38 \end{cases}$

61. $\begin{cases} y = x^2 \\ -3x + 2y = 2 \end{cases}$

62. $\begin{cases} x - y^3 = 0 \\ x - 2y^2 = 0 \end{cases}$

In Exercises 63–66, use Gauss-Jordan elimination to solve the system.

63. $\begin{cases} x - y = -1 \\ x + 2y - 2z = 3 \\ 3x - y + 2z = 3 \end{cases}$

64. $\begin{cases} 2x + y - 2z = 1 \\ x - z = 1 \\ 3x + 3y + z = 12 \end{cases}$

65. $\begin{cases} -3x + y + 5z = 25 \\ x - 2y + 3z = 7 \\ 2x + 3y - z = 0 \end{cases}$

66. $\begin{cases} 2x - y + 4z = 21 \\ -4x + 3y + z = -14 \\ -x - 4y + 7z = 12 \end{cases}$

In Exercises 67 and 68, find the determinant of the matrix.

67. $\begin{bmatrix} 7 & 6 \\ -4 & 2 \end{bmatrix}$

68. $\begin{bmatrix} 2 & 4 & 8 \\ 0 & 6 & -9 \\ 4 & -3 & 8 \end{bmatrix}$

In Exercises 69–72, expand the expression.

69. $(2x^2 - 1)^2$
70. $(2x - y)^2$
71. $(5 - 4x)^3$
72. $(2x - 4y)^3$

9.5 The Binomial Theorem

Binomial Coefficients

Recall that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that provides a quick method of raising a binomial to a power. To begin, look at the expansion of

$$(x + y)^n$$

for several values of n.

$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

There are several observations you can make about these expansions.

1. In each expansion, there are $n + 1$ terms.
2. In each expansion, x and y have symmetric roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.
3. The sum of the powers of each term is n. For instance, in the expansion of $(x + y)^5$, the sum of the powers of each term is 5.

$$(x + y)^5 = x^5 + 5\overbrace{x^4y^1}^{4+1=5} + 10\overbrace{x^3y^2}^{3+2=5} + 10x^2y^3 + 5x^1y^4 + y^5$$

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.** To find them, you can use the **Binomial Theorem.** A proof of this theorem is given in Appendix A.

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r\, x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote binomial coefficients.

What You Should Learn:

- How to use the Binomial Theorem to calculate binomial coefficients
- How to use Pascal's Triangle to calculate binomial coefficients
- How to use binomial coefficients to write binomial expansions

Why You Should Learn It:

You can use binomial coefficients to predict future behavior. For instance, in Exercise 84 on page 662, you are asked to use binomial coefficients to find the probability that a baseball player gets 3 hits during the next 10 times at bat.

Jonathan Daniel/Allsport

EXAMPLE 1 Finding Binomial Coefficients

Find the binomial coefficients.

a. ${}_8C_2$ **b.** $\binom{10}{3}$ **c.** ${}_7C_0$ **d.** $\binom{8}{8}$

Solution

a. ${}_8C_2 = \dfrac{8!}{6! \cdot 2!} = \dfrac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} = \dfrac{8 \cdot 7}{2 \cdot 1} = 28$

b. $\binom{10}{3} = \dfrac{10!}{7! \cdot 3!} = \dfrac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} = \dfrac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$

c. ${}_7C_0 = \dfrac{\cancel{7!}}{\cancel{7!} \cdot 0!} = 1$

d. $\binom{8}{8} = \dfrac{\cancel{8!}}{0! \cdot \cancel{8!}} = 1$

When $r \neq 0$ and $r \neq n$, as in parts (a) and (b) above, there is a simple pattern for evaluating binomial coefficients.

$${}_8C_2 = \frac{\overbrace{8 \cdot 7}^{\text{2 factors}}}{\underbrace{2 \cdot 1}_{\text{2 factorial}}} \quad \text{and} \quad \binom{10}{3} = \frac{\overbrace{10 \cdot 9 \cdot 8}^{\text{3 factors}}}{\underbrace{3 \cdot 2 \cdot 1}_{\text{3 factorial}}}$$

EXAMPLE 2 Finding Binomial Coefficients

Find the binomial coefficients.

a. ${}_7C_3$ **b.** ${}_7C_4$ **c.** ${}_{12}C_1$ **d.** ${}_{12}C_{11}$

Solution

a. ${}_7C_3 = \dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

b. ${}_7C_4 = \dfrac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35$

c. ${}_{12}C_1 = \dfrac{12}{1} = 12$

d. ${}_{12}C_{11} = \dfrac{12!}{1! \cdot 11!} = \dfrac{(12) \cdot \cancel{11!}}{1! \cdot \cancel{11!}} = \dfrac{12}{1} = 12$

It is not a coincidence that the results in parts (a) and (b) of Example 2 are the same and that the results in parts (c) and (d) are the same. In general, it is true that

$${}_nC_r = {}_nC_{n-r}.$$

This shows the symmetric property of binomial coefficients that was identified earlier.

STUDY TIP

Most graphing calculators are programmed to evaluate ${}_nC_r$. Consult your user's manual to evaluate the binomial coefficients in Example 1.

Exploration

Find the following pairs of binomial coefficients.

a. ${}_7C_0$, ${}_7C_7$
b. ${}_8C_0$, ${}_8C_8$
c. ${}_{10}C_0$, ${}_{10}C_{10}$
d. ${}_7C_1$, ${}_7C_6$
e. ${}_8C_1$, ${}_8C_7$
f. ${}_{10}C_1$, ${}_{10}C_9$

What do you observe about the pairs in (a), (b), and (c)? What do you observe about the pairs in (d), (e), and (f)? Write two conjectures from your observations. Develop a convincing argument for your two conjectures.

Pascal's Triangle

There is a convenient way to remember a pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal's Triangle.** This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).

$$
\begin{array}{ccccccccccccccc}
 & & & & & & 1 & & & & & & & & \\
 & & & & & 1 & & 1 & & & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & & & & \vdots & & & & & & & &
\end{array}
$$

$4 + 6 = 10$

$15 + 6 = 21$

The first and last number in each row of Pascal's Triangle is 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that numbers in this triangle are precisely the same numbers as the coefficients of binomial expansions.

$$
\begin{aligned}
(x + y)^0 &= 1 \\
(x + y)^1 &= 1x + 1y \\
(x + y)^2 &= 1x^2 + 2xy + 1y^2 \\
(x + y)^3 &= 1x^3 + 3x^2y + 3xy^2 + 1y^3 \\
(x + y)^4 &= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \\
(x + y)^5 &= 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 \\
(x + y)^6 &= 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6 \\
(x + y)^7 &= 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7 \\
&\vdots
\end{aligned}
$$

The top row in Pascal's Triangle is called the *zero row* because it corresponds to the binomial expansion $(x + y)^0 = 1$.

Similarly, the next row is called the *first row* because it corresponds to the binomial expansion $(x + y)^1 = 1(x) + 1(y)$.

In general, the *nth row* in Pascal's Triangle gives the coefficients of $(x + y)^n$.

Exploration

Complete the following table and describe the result.

n	r	${}_nC_r$	${}_nC_{n-r}$
9	5		
7	1		
12	4		
6	0		
10	7		

What characteristics of Pascal's Triangle are illustrated by the table?

EXAMPLE 3 Using Pascal's Triangle

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

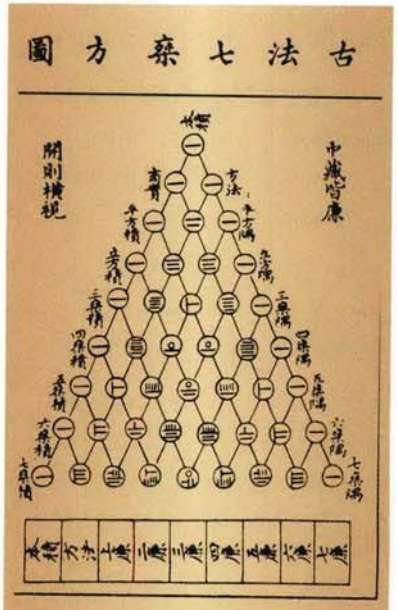
${}_8C_0, {}_8C_1, {}_8C_2, {}_8C_3, {}_8C_4, {}_8C_5, {}_8C_6, {}_8C_7, {}_8C_8$

Solution

$$
\begin{array}{ccccccccccccccccc}
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \longleftarrow \text{Seventh row} \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 \\
{}_8C_0 & & {}_8C_1 & & {}_8C_2 & & {}_8C_3 & & {}_8C_4 & & {}_8C_5 & & {}_8C_6 & & {}_8C_7 & & {}_8C_8
\end{array}
$$

Binomial Expansions

As mentioned at the beginning of this section, when you write out the coefficients for a binomial that is raised to a power, you are **expanding a binomial.** The formulas for binomial coefficients give you an easy way to expand binomials, as demonstrated in the next four examples.



"Pascal's" Triangle and forms of the Binomial Theorem were known in Eastern cultures prior to the Western "discovery" of the theorem. The Chinese text *Precious Mirror* contains a triangle of binomial expansions through the eighth power.

EXAMPLE 4 Expanding a Binomial

Write the expansion for the expression $(x + 1)^3$.

Solution

The binomial coefficients from the third row of Pascal's Triangle are

1, 3, 3, 1.

Therefore, the expansion is as follows.

$$(x + 1)^3 = (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3)$$
$$= x^3 + 3x^2 + 3x + 1$$

To expand binomials representing *differences*, rather than sums, you alternate signs. Here are two examples.

$$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$$
$$(x - 1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$$

A computer animation of this concept appears in the *Interactive* CD-ROM and *Internet* versions of this text.

EXAMPLE 5 Expanding a Binomial

Write the expansion for the expression

$(2x - 3)^4$.

Solution

The binomial coefficients from the fourth row of Pascal's Triangle are

1, 4, 6, 4, 1.

Therefore, the expansion is as follows.

$$(2x - 3)^4 = (1)(2x)^4 - (4)(2x)^3(3) + (6)(2x)^2(3^2) - (4)(2x)(3^3) + (1)(3^4)$$
$$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$$

You can use a graphing utility to check the expansion by graphing the original binomial expression and the expansion in the same viewing window. The graphs should coincide, as shown in Figure 9.12.

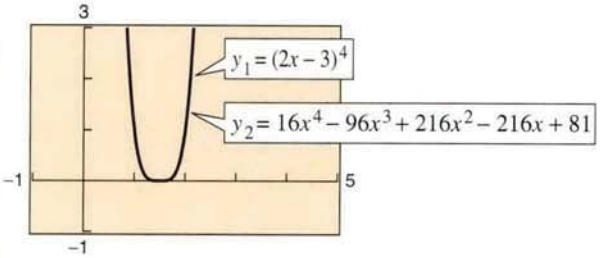


Figure 9.12

EXAMPLE 6 Expanding a Binomial

Write the expansion for $(x - 2y)^4$.

Solution

Use the fourth row of Pascal's Triangle, as follows.

$$\begin{aligned}(x - 2y)^4 &= (1)x^4 - (4)x^3(2y) + (6)x^2(2y)^2 - (4)x(2y)^3 + (1)(2y)^4\\ &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4\end{aligned}$$

EXAMPLE 7 Expanding a Binomial

Write the expansion for $(x^2 + 4)^3$.

Solution

Use the third row of Pascal's Triangle, as follows.

$$\begin{aligned}(x^2 + 4)^3 &= (1)(x^2)^3 + (3)(x^2)^2(4) + (3)x^2(4^2) + (1)(4^3)\\ &= x^6 + 12x^4 + 48x^2 + 64\end{aligned}$$

To find a specific term in a binomial expansion, use the fact that from the Binomial Theorem, the $(r + 1)$st term is

$${}_nC_r x^{n-r} y^r.$$

EXAMPLE 8 Finding a Term in a Binomial Expansion

Find the sixth term of $(a + 2b)^8$.

Solution

For the sixth term in this binomial expansion, use $n = 8$ and $r = 5$ to get

$$\begin{aligned}{}_8C_5 a^{8-5}(2b)^5 &= 56 \cdot a^3 \cdot (2b)^5\\ &= 56(2^5)a^3b^5\\ &= 1792a^3b^5.\end{aligned}$$

Writing About Math *Error Analysis*

Suppose you are a math instructor and receive the following solutions from one of your students on a quiz. Find the error(s) in each solution and write a short paragraph discussing ways that your student could avoid the error(s) in the future.

a. Find the second term in the expansion of $(2x - 3y)^5$.

$$5(2x)^4(3y)^2 = 720x^4y^2 \quad \times$$

b. Find the fourth term in the expansion of $\left(\frac{1}{2}x + 7y\right)^6$.

$${}_6C_4\left(\tfrac{1}{2}x\right)^2(7y)^4 = 9003.75x^2y^4 \quad \times$$

9.5 Exercises

In Exercises 1–12, evaluate ${}_nC_r$.

1. ${}_7C_5$
2. ${}_9C_6$
3. $\binom{12}{0}$
4. $\binom{20}{20}$
5. ${}_{20}C_{15}$
6. ${}_{12}C_3$
7. ${}_{14}C_1$
8. ${}_{18}C_{17}$
9. $\binom{100}{98}$
10. $\binom{10}{7}$
11. ${}_{100}C_2$
12. ${}_{13}C_8$

In Exercises 13–18, use a graphing utility to evaluate ${}_nC_r$.

13. ${}_{32}C_{28}$
14. ${}_{17}C_4$
15. ${}_{22}C_9$
16. ${}_{52}C_{47}$
17. ${}_{41}C_{36}$
18. ${}_{34}C_4$

In Exercises 19–22, evaluate using Pascal's Triangle.

19. ${}_7C_2$
20. ${}_6C_4$
21. ${}_8C_5$
22. ${}_8C_6$

In Exercises 23–44, use the Binomial Theorem to expand and simplify the expression.

23. $(x + 1)^4$
24. $(x + 1)^6$
25. $(a + 3)^3$
26. $(a + 2)^4$
27. $(y - 2)^4$
28. $(y - 2)^5$
29. $(x + y)^5$
30. $(x + y)^6$
31. $(r + 2s)^6$
32. $(x + 3y)^4$
33. $(x - y)^5$
34. $(2x - y)^5$
35. $(1 - 4x)^3$
36. $(5 - 2y)^3$
37. $(x^2 + 5)^4$
38. $(x^2 + y^2)^6$
39. $\left(\frac{1}{x} + y\right)^5$
40. $\left(\frac{1}{x} + 2y\right)^6$
41. $2(x - 3)^4 + 5(x - 3)^2$
42. $3(x + 1)^5 - 4(x + 1)^3$
43. $-3(x - 2)^3 - 4(x + 1)^6$
44. $6(x + 2)^5 - 2(x - 1)^2$

In Exercises 45–48, expand the binomial using Pascal's Triangle to determine the coefficients.

45. $(3t - s)^5$
46. $(x + 2y)^5$
47. $(3 - 2z)^4$
48. $(3y + 2)^5$

In Exercises 49–56, find the coefficient a of the given term in the expansion of the binomial.

	Binomial	*Term*
49.	$(x + 3)^{12}$	ax^4
50.	$(x^2 + 3)^{12}$	ax^{10}
51.	$(x - 2y)^{10}$	ax^8y^2
52.	$(4x - y)^{10}$	ax^2y^8
53.	$(3x - 2y)^9$	ax^6y^3
54.	$(2x - 3y)^8$	ax^4y^4
55.	$(x^2 + y)^{10}$	ax^8y^6
56.	$(z^2 - 1)^{12}$	az^6

In Exercises 57–60, use the Binomial Theorem to expand and simplify the expression.

57. $\left(\sqrt{x} + 5\right)^4$
58. $\left(4\sqrt{t} - 1\right)^3$
59. $(x^{2/3} - y^{1/3})^3$
60. $(u^{3/5} + 2)^5$

In Exercises 61–64, expand the binomial in the difference quotient and simplify.

$$\frac{f(x + h) - f(x)}{h}$$

61. $f(x) = x^3$
62. $f(x) = x^4$
63. $f(x) = \sqrt{x}$
64. $f(x) = \frac{1}{x}$

In Exercises 65–70, use the Binomial Theorem to expand the complex number. Simplify your result.

65. $(1 + i)^4$
66. $(4 - i)^5$
67. $(2 - 3i)^6$
68. $\left(5 + \sqrt{-9}\right)^3$
69. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$
70. $\left(5 - \sqrt{3}i\right)^4$

Approximation **In Exercises 71–74, use the Binomial Theorem to approximate the given quantity accurate to three decimal places. For example, in Exercise 71, use the expansion**

$$(1.02)^8 = (1 + 0.02)^8 = 1 + 8(0.02) + 28(0.02)^2 + \cdots.$$

71. $(1.02)^8$ **72.** $(2.005)^{10}$

73. $(2.99)^{12}$ **74.** $(1.98)^9$

Graphical Reasoning **In Exercises 75–78, use a graphing utility to obtain the graph of f and g in the same viewing window. What is the relationship between the two graphs? Use the Binomial Theorem to write the polynomial function g in standard form.**

75. $f(x) = x^3 - 4x, \quad g(x) = f(x + 6)$

76. $f(x) = -x^4 + 4x^2 - 1, \quad g(x) = f(x - 4)$

77. $f(x) = -x^2 + 3x + 2, \quad g(x) = f(x - 2)$

78. $f(x) = 2x^2 - 4x + 1, \quad g(x) = f(x + 3)$

Exploration **In Exercises 79 and 80, use a graphing utility to evaluate and determine which two are equal.**

79. (a) ${}_{12}C_5$ (b) $({}_6C_5)^2$

(c) ${}_{11}C_5 + {}_{11}C_4$ (d) ${}_6C_5 + {}_6C_5$

80. (a) ${}_{25}C_6$ (b) $2({}_{25}C_2 + {}_{25}C_4)$

(c) $\sum_{k=0}^{5}[({}_{10}C_k)({}_8C_{5-k})]$ (d) ${}_{18}C_5$

Graphical Reasoning **In Exercises 81 and 82, use a graphing utility to obtain the graphs of the functions in the given order and in the same viewing window. Compare the graphs. Which two functions have identical graphs and why?**

81. (a) $f(x) = (1 - x)^3$

(b) $g(x) = 1 - 3x$

(c) $h(x) = 1 - 3x + 3x^2$

(d) $p(x) = 1 - 3x + 3x^2 - x^3$

82. (a) $f(x) = \left(1 - \frac{1}{2}x\right)^4$

(b) $g(x) = 1 - 2x + \frac{3}{2}x^2$

(c) $h(x) = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3$

(d) $p(x) = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

Probability **In Exercises 83–86, consider n independent trials of an experiment where each trial has two possible outcomes, called a success and a failure, respectively. The probability of a success on each trial is p and the probability of a failure is $q = 1 - p$. In this context, the term ${}_nC_k\, p^k q^{n-k}$ in the expansion of $(p + q)^n$ gives the probability of k successes in the n trials of the experiment.**

83. A fair coin is tossed seven times. To find the probability of obtaining 4 heads, evaluate the term

$${}_7C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^3$$

in the expansion $\left(\frac{1}{2} + \frac{1}{2}\right)^7$.

84. The probability of a baseball player getting a hit on any given time at bat is $\frac{1}{4}$. To find the probability that the player gets 3 hits during the next 10 times at bat, evaluate the term

$${}_{10}C_3\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^7$$

in the expansion $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$.

85. The probability of a sales representative making a sale with any one customer is $\frac{1}{3}$. The sales representative makes 8 contacts a day. To find the probability of making 4 sales, evaluate the term

$${}_8C_4\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^4$$

in the expansion $\left(\frac{1}{3} + \frac{2}{3}\right)^8$.

86. To find the probability that the sales representative in Exercise 85 makes 4 sales if the probability of a sale with any one customer is $\frac{1}{2}$, evaluate the term

$${}_8C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^4$$

in the expansion $\left(\frac{1}{2} + \frac{1}{2}\right)^8$.

87. ***Life Insurance*** The average amount of life insurance per household f (in thousands of dollars) from 1980 through 1996 can be approximated by

$$f(t) = 0.0348t^2 + 5.1083t + 41.0250, \quad 0 \le t \le 16$$

where $t = 0$ represents 1980. You want to adjust this model so that $t = 0$ corresponds to 1990 rather than 1980. To do this, you shift the graph of f 10 units *to the left* and obtain

$$g(t) = f(t + 10).$$

(Source: American Council of Life Insurance)

(a) Write $g(t)$ in standard form.

(b) Use a graphing utility to graph f and g in the same viewing window.

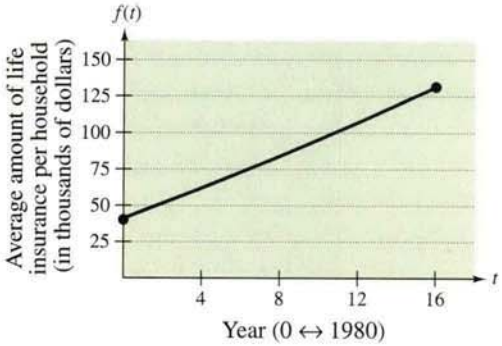


FIGURE FOR 87

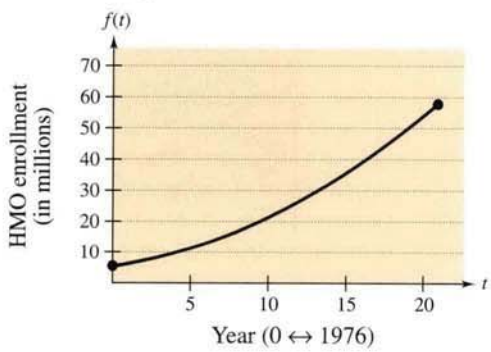
88. ***Health Maintenance Organizations*** The number of people f (in millions) enrolled in health maintenance organizations in the United States from 1976 through 1997 can be approximated by the model

$$f(t) = 0.0834t^2 + 0.7657t + 5.3680, \quad 0 \le t \le 21$$

where $t = 0$ represents 1976. You want to adjust this model so that $t = 0$ corresponds to 1980 rather than 1976. To do this, you shift the graph of f 4 units *to the left* and obtain $g(t) = f(t + 4)$. (Source: Group Health Association of America, Interstudy)

(a) Write $g(t)$ in standard form.

(b) Use a graphing utility to graph f and g in the same viewing window.



Synthesis

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.**

89. One of the terms in the expansion of $(x - 2y)^{12}$ is $7920x^4y^8$.

90. The x^{10}-term and the x^{14}-term of the expansion of $(x^2 + 3)^{12}$ have identical coefficients.

91. ***Writing*** In your own words, explain how to form the rows of Pascal's Triangle.

92. Form the first nine rows of Pascal's Triangle.

93. ***Think About It*** How many terms are in the expansion of $(x + y)^n$?

94. ***Think About It*** How do the expansions of $(x + y)^n$ and $(x - y)^n$ differ?

In Exercises 95–98, prove the given property for all integers r and n where $0 \le r \le n$.

95. ${}_nC_r = {}_nC_{n-r}$

96. ${}_nC_0 - {}_nC_1 + {}_nC_2 - \cdots \pm {}_nC_n = 0$

97. ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

98. The sum of the numbers in the nth row of Pascal's Triangle is 2^n.

Review

In Exercises 99–102, describe the relationship between the graphs of f and g.

99. $g(x) = f(x) + 8$

100. $g(x) = f(x - 3)$

101. $g(x) = f(-x)$

102. $g(x) = -f(x)$

In Exercises 103–108, perform the matrix operation using the given matrices.

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 5 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -4 & -3 \\ 3 & 1 & 2 \\ -5 & -2 & 6 \end{bmatrix}$$

103. $A + B$

104. $4A - B$

105. $-3A - 5B$

106. $6A + 10B$

107. AB

108. BA

In Exercises 109 and 110, find the inverse of the matrix.

109. $\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

110. $\begin{bmatrix} 1.2 & -2.3 \\ -2 & 4 \end{bmatrix}$