Pre-Calculus Notes   Name: Key

Section 2.1 - Quadratic Functions and Models

DAY ONE:

Let a, b, and c be real numbers with $a \neq 0$. The function $f(x) = ax^2 + bx + c$ is called a quadratic function.

The graph of a quadratic function is a parabola.

Every parabola is symmetrical about a line called the axis of symmetry.

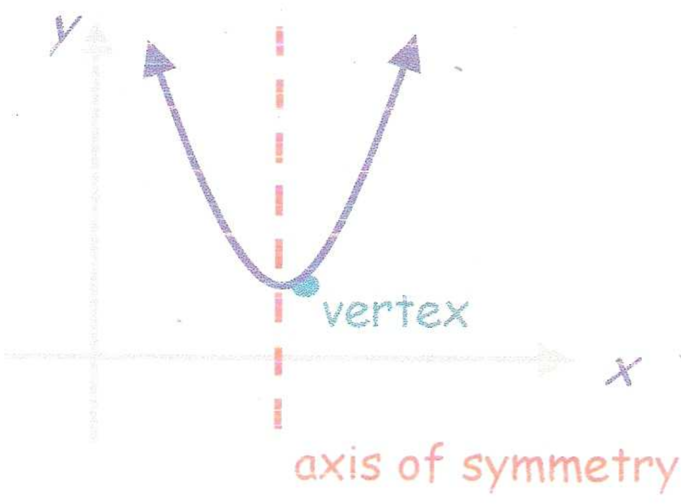
The intersection point of the parabola and the axis is called the vertex of the parabola.

When the leading coefficient is positive, the parabola opens up and the vertex is a minimum.

When the leading coefficient is negative, the parabola opens down and the vertex is a maximum.

A quadratic function can be graphed by comparing it to the parent graph $y = x^2$ and identifying any transformations.

In order to be able to "read" off the transformations, the parabola should be in vertex form, or $y = a(x-h)^2 + k$.



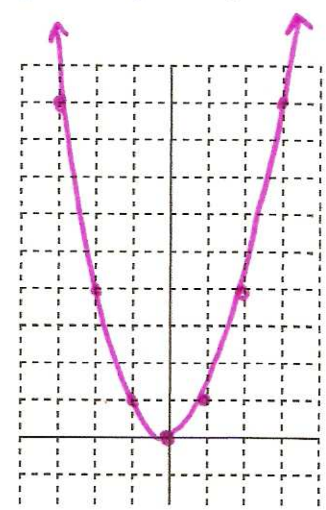
$f(x) = ax^2 + bx + c$

standard form   or   vertex form

$f(x) = a(x-h)^2 + k$

Now, remember the shape of the parent graph $y = x^2$?
Think about the pattern you use from the vertex to find other points on the parabola.

from vertex:
over 1 up 1
over 2 up 4
over 3 up 9



Example 1: Use transformations to graph each of the following parabolas. Identify the vertex and axis of symmetry.

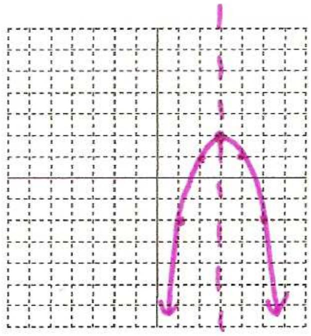
a. $f(x)=-(x-3)^2+2 \leftarrow$ up 2

Opens down

3 right

vertex $(3,2)$

axis of symm.: $x=3$



b. $f(x)=\frac{+1}{2}(x+2)^2-4 \leftarrow$ down 4

opens up

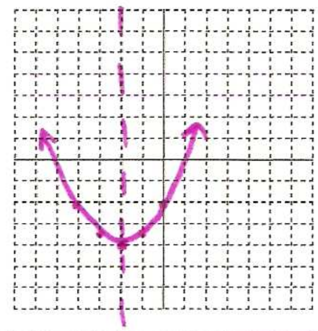
left 2

over 1 up $\frac{1}{2}$

over 2 up 2

vertex $(-2,-4)$

axis: $x=-2$



We learned to graph conics by using "standard form", and for the parabola, the equations looked like $4c(x-h)=(y-k)^2$ or $4c(y-k)=(x-h)^2$. In reality, when talking about a parabola outside of the context of conic sections, most books designate standard form of a quadratic as $f(x)=ax^2+bx+c$. Confused, yet? In this unit, we will be focusing on quadratic functions, which means the the parabola only opens up or down (not left or right). We will be using vertex form, or $f(x)=a(x-h)^2+k$, in this unit of study. To reiterate what we saw in example 1, the graph of $f(x)=a(x-h)^2+k$ is a parabola opening upward if $a>0$ and downward if $a<0$. It has an axis of symmetry $x=h$ and vertex (h,k).

Example 2: Find an equation for the parabola with vertex $\overset{h\ k}{(2,-1)}$ and passing through the point $\overset{x\ y}{(0,1)}$.

$y=a(x-h)^2+k$

$1=a(0-2)^2+-1$

$1=a(2)^2-1$

$2=4a$

$a=\frac{1}{2}$

$\boxed{y=\frac{1}{2}(x-2)^2-1}$

Before we delve any further, we need to review a skill necessary for later on... solving quadratic equations by using the quadratic formula. Ready? Remember that $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Example 3: Use the quadratic formula to find the zeros of the function (when $f(x)=0$).

a. $f(x)=-x^2-8x-14$

$0=-x^2-8x-14$ $a=-1$ $b=-8$ $c=-14$

$x=\dfrac{8\pm\sqrt{64-4(14)}}{-2}$

$x=\dfrac{8\pm\sqrt{8}}{-2}$

$x=\dfrac{\cancel{8}\pm\cancel{2}\sqrt{2}}{\cancel{-2}}$

$\boxed{x=-4\pm\sqrt{2}}$

b. $f(x)=9x^2-6x-4$ $a=9$ $b=-6$ $c=-4$

$0=9x^2-6x-4$

$x=\dfrac{6\pm\sqrt{36-4(-36)}}{18}$

$x=\dfrac{6\pm\sqrt{180}}{18}$

$x=\dfrac{\cancel{6}\pm\cancel{6}\sqrt{5}}{\cancel{18}}$

$\boxed{x=\dfrac{1\pm\sqrt{5}}{3}}$

Now... ready to finish up this lesson? We can rewrite a parabola from the form $f(x) = ax^2 + bx + c$ to vertex form $f(x) = a(x-h)^2 + k$ by a couple of methods. The first is by completing the square.

Example 4: Rewrite the parabola in vertex form by completing the square.

a. $f(x) = 2x^2 + 4x - 1$

$f(x) = 2(x^2 + 2x + \underline{1}) - 1 - 2(\underline{1})$

$f(x) = 2(x+1)^2 - 1 - 2$

$\boxed{f(x) = 2(x+1)^2 - 3}$

b. $y = -x^2 + 6x - 5$

$y = -(x^2 - 6x + \underline{9}) - 5 + (\underline{9})$

$y = -(x-3)^2 - 5 + 9$

$\boxed{y = -(x-3)^2 + 4}$

The vertex of $f(x) = ax^2 + bx + c$, where $a \neq 0$, will ALWAYS be $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. In other words, the x-coordinate of the vertex can be found using $x = -\frac{b}{2a}$. Then, to find the y-coordinate of the vertex you plug x into $f(x) = ax^2 + bx + c$. Ready?

Example 5: Rewrite the parabola in vertex form by using $x = -\frac{b}{2a}$.

a. $f(x) = -x^2 + 10x - 22$

$a = -1 \quad b = 10 \quad c = -22$

$x = \frac{-10}{-2}$

$x = 5$

$f(5) = -(5)^2 + 10(5) - 22$

$f(5) = -25 + 50 - 22$

$f(5) = 3$

vertex $(5, 3)$

$\boxed{f(x) = -(x-5)^2 + 3}$

b. $f(x) = \frac{1}{4}x^2 + 4x + 11$

$a = \frac{1}{4} \quad b = 4 \quad c = 11$

$x = \frac{-4}{\frac{1}{2}}$

$x = -4 \cdot 2$

$x = -8$

$f(-8) = \frac{1}{4} \cdot 64 + -32 + 11$

$f(-8) = -5$

vertex $(-8, -5)$

$\boxed{f(x) = \frac{1}{4}(x+8)^2 - 5}$

DAY TWO:

If a parabola is given to us in vertex form, it would be an inefficient use of our time to rewrite it in standard form, set it equal to zero, and then use the quadratic formula to find the zeros (also known as roots or x-intercepts). And we don't want that, do we? So instead, use the fact that you know how to take the square root of both sides of an equation. ☺

Example 6: Find the zeros of each function.

a. $f(x) = -3(x+2)^2 - 9$

$0 = -3(x+2)^2 - 9$

$\frac{9}{-3} = \frac{-3(x+2)^2}{-3}$

$\sqrt{-3} = \sqrt{(x+2)^2}$

$x + 2 = \pm i\sqrt{3}$

$\boxed{x = -2 \pm i\sqrt{3}}$

b. $f(x) = \frac{1}{2}(x-4)^2 - 2$

$0 = \frac{1}{2}(x-4)^2 - 2$

$2(2) = \left(\frac{1}{2}(x-4)^2\right)2$

$\sqrt{4} = \sqrt{(x-4)^2}$

$x - 4 = \pm 2$

$x = 4 \pm 2$ $\quad \boxed{x = 6, 2}$

We already saw yesterday how to find the zeros of a parabola in the form $f(x) = ax^2 + bx + c$. We use the Quadratic Formula. No need to revisit that concept here, right?

Instead, let's look at another concept we need to be comfortable with... maximum and minimum values. If a parabola opens upward, then it has a MINIMUM value. If a parabola opens downward, it has MAXIMUM value. How do you find the maximum or minimum value? Always start with the vertex!

Example 7: For each parabola, find its maximum or minimum value and where it occurs.

a. $f(x) = 5x^2 - 10x + 5$

$a = 5 \quad b = -10 \quad c = 5$ opens up

$x = \frac{10}{10}$

$x = 1$

$f(1) = 5 - 10 + 5$

$f(1) = 0$

vertex $(1, 0)$

$\boxed{\text{min value of } 0 \text{ @ } x = 1}$

b. $f(x) = -\frac{1}{3}x^2 - 2x - 5$

$a = -\frac{1}{3} \quad b = -2 \quad c = -5$ opens down

$x = \frac{2}{-\frac{2}{3}}$

$x = 2 \cdot \frac{3}{-2}$

$x = -3$

$f(-3) = -\frac{1}{3} \cdot 9 + 6 - 5$

$f(-3) = -2$

max value of -2 @ $x = -3$

And now... time for some applications!

Example 8:

a. A basketball is thrown from the free throw line from a height of six feet. What is the maximum height of the ball if the path of the ball is $y = -\frac{1}{9}x^2 + 2x + 6$?

$x = \frac{-2}{-\frac{2}{9}}$

$x = -2 \cdot \frac{9}{-2}$

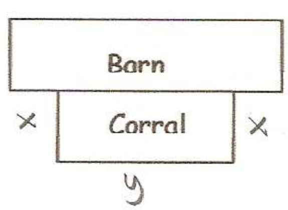
$x = 9$

$y = -\frac{1}{9} \cdot 81 + 18 + 6$

$y = 15$

$\boxed{15 \text{ feet}}$

b. A fence is to be built to form a rectangular corral along the side of a barn 65 feet long. If 120 feet of fencing are available, what are the dimensions of the corral of maximum area?



$2x + y = 120$

$y = 120 - 2x$

$y = 120 - 2(30)$

$\boxed{30 \text{ ft by } 60 \text{ ft}}$

Area $= xy$

$A = x(120 - 2x)$

$A = -2x^2 + 120x$

$x = \frac{-120}{-4}$

$x = 30$