Topic: 7.8 Exponential Models with Differential Equations

Learning Objective FUN-7.F: Interpret the meaning of a differential equation and its variables in context
Learning Objective FUN-7.G: Determine general and particular solutions for problems involving differential equations in context.

Often times, physical phenomena can be described by a differential equation. For example, problems involving radioactive decay, population growth, and Newton's Law of Cooling can be formulated in terms of an equation that contains a derivative (or rate of change).

Growth and Decay

In many applications, the rate of change of the variable *y* is directly proportional to the value of *y*. If *y* is a function of time *t*, the proportion can be written as follows:

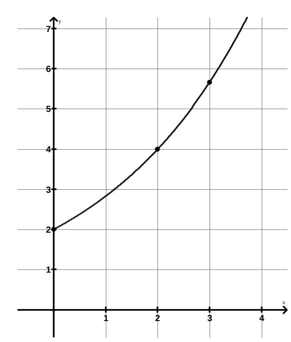
$$\frac{dy}{dt} = ky$$

Rate of change of *y* — is — proportional to *y*

The general solution of this differential equation is given in the following theorem.

> **Exponential Growth and Decay Model Theorem**
> If *y* is a differentiable function of *t* such that $y > 0$ and $y' = ky$, for some constant *k*, then
> $$y = Ce^{kt}$$
> *C* is the **initial value** of *y*, and *k* is the **proportionality constant. Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

Example 1: The rate of change of *y* is proportional to *y*. When $t = 0,\ y = 2.$ When $t = 2,\ y = 4$. What is the value of *y* when $t = 3$?



Graphically, this is what Example 1 looks like

Applications of Growth and Decay

Half-Lives

You may recall this topic in previous math or science courses. We will take a slightly different approach with these problems now that we have a formal Growth/Decay Model with which to work. We work on the assumption that the rate of decay is proportional to the amount of the material.

The half-lives of some common radioactive isotopes are as follows:

Uranium (^{236}U)	4,510,000,000 years	Plutonium (^{239}P)	24,360 years
Carbon (^{14}C)	5730 years	Caesium (^{137}Cs)	30 years
Einsteinium (^{254}Es)	270 days	Nobelium (^{257}No)	23 seconds

Radioactivity in Chernobyl, U.S.S.R.

Example 2: Suppose that 15 grams of the Caesium isotope Cs-137 that was released during the 1986 Chernobyl nuclear accident and absorbed into the groundwater. How long will it take for the 15 grams to decay to 1 gram and become safer to drink?

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

In other words, $\frac{dT_o}{dt} = k \cdot (T_o - T_E)$, where T_o represents the temperature of the object and T_E represents the temperature of the surrounding environment.

Note: Newton's Law of Cooling isn't specifically tested on the AP Calculus Exam, although models that closely resemble its form are sometimes tested.

Example 3: The world famous pastry chef, Bernie D. Rolz removes a completely baked cake from an oven and places it upon a cooling rack in a room whose temperature is kept at a constant 60° F. Using a cooking thermometer, he determines the cake cools from 100° F to 90° F in 10 minutes. The icing that is to be used for the cake cannot be applied until the cake's temperature reaches 80° F. How much longer must Bernie wait to ice the cake?

Another Type of Growth Model

Example 4: Fish are being introduced into a man-made lake. The change in the rate of fish, *F*, with respect to time, *t*, is directly proportional to $900 - F$, where *t* is measured in years. When $t = 0$, there are 400 fish in the lake and 3 years later, there are 600 fish in the lake.

a. Write and solve the differential equation that describes this situation.

b. Find the fish population in another 3 years.

c. Find $\lim_{t \to \infty} F(t)$ and explain what the answer means.