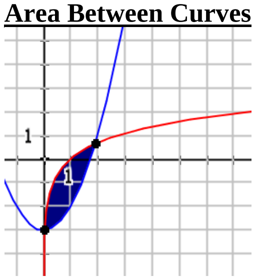
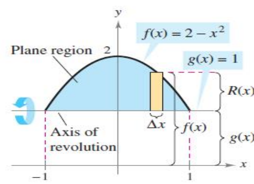
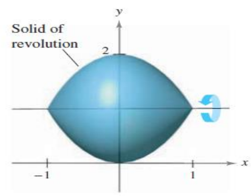
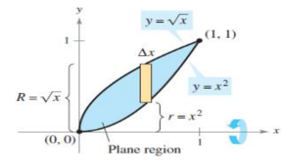
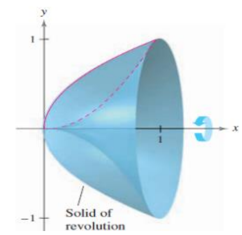
CHA	AP CALCULUS BC		
3	Topic: 8.13	The Arc Length of a Smooth, Planar Curve and Distance Traveled	
Learning Objective CHA-6.A: Determine the length of a curve in the plane defined by a function, using a definite integral.			

Hopefully you recall some of the applications of the definite integral from Calculus AB.

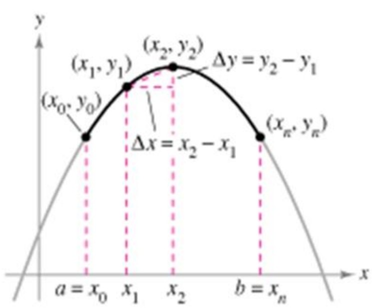
Area Between Curves	Volume of Solid of Revolution (Disc Method)	Volume of Solid of Revolution (Washer Method)
 $\int_{0.499}^{1.909} (\ln x - (x^2 - 3))\,dx = 2.783$	  $V = \pi\int_{-1}^{1}\left(\left(2-x^2\right)-1\right)^2 dx = \frac{16\pi}{15}$	  $V = \pi\int_{0}^{1}\left(\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right)dx = \frac{3\pi}{10}$

Arc Length

In this section we will use integration to find the arc length and the surface area of a solid of revolution. In either case, we will heavily rely upon a familiar friend, the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

A **rectifiable** curve is one that has a finite arc length. In order for this to be true, the function's derivative must be continuous over the interval on which we will find its length. We say that this function is **continuously differentiable** on the interval and that its graph is a **smooth curve**.



Scan the Code above to view a video of this demonstration.

Consider a function $f(x)$ that is continuously differentiable on the interval $[a,b]$. We can approximate the graph of f using n line segments whose endpoints are determined by the partition: $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

So, by letting $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$ as shown in the figure to the right, we can approximate the length of the curve of the graph by

$$s \approx \sum_{i=1}^{n}\sqrt{\left(x_i - x_{i-1}\right)^2 + \left(y_i - y_{i-1}\right)^2}$$

$$= \sum_{i=1}^{n}\sqrt{\left(\Delta x_i\right)^2 + \left(\Delta y_i\right)^2}$$

$$= \sum_{i=1}^{n}\sqrt{\frac{\left(\Delta x_i\right)^2 + \left(\Delta y_i\right)^2}{\left(\Delta x_i\right)^2}\cdot\left(\Delta x_i\right)^2}$$

$$= \sum_{i=1}^{n}\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\cdot\left(\Delta x_i\right)$$

As $n \to \infty$ and we use more and more subdivisions we obtain

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\cdot\left(\Delta x_i\right)$$

Which ultimately results in the following integral expression

$$\int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Definition of Arc Length

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a,b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

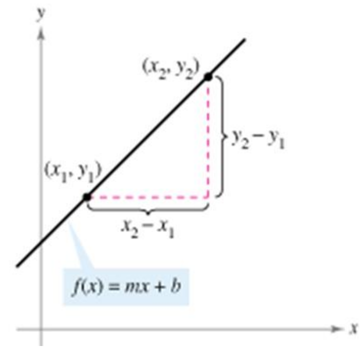
Similarly, for a smooth curve given by $x = g(y)$, The **arc length** of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy$$

Note: Very few of these integrals can be solved without using a calculator.

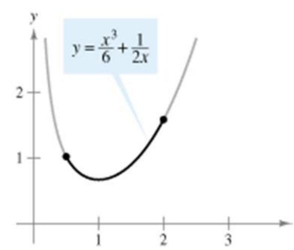
Example 1: The Length of a Line Segment.

Find the arc length from (x_1, y_1) to (x_2, y_2) on the graph of $f(x) = mx + b$ as shown in the figure to the right using calculus.



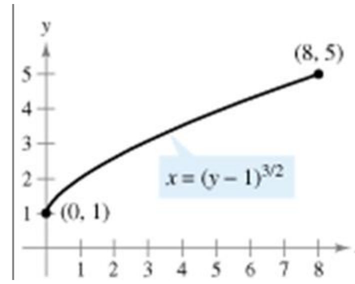
Example 2: Finding Arc Length With Respect to *x*

Find the arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\left[\frac{1}{2}, 2\right]$ as shown to the right.



Example 3: Finding Arc Length Finding Arc Length With Respect to *x* or *y*

Find the arc length of the graph of $(y-1)^3 = x^2$ on the interval $[0,8]$ as shown to the right.



Example 4: Finding Arc Length.

Find the arc length of the graph of $y = ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$ as shown to the right.