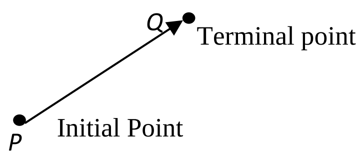
CHA	AP CALCULUS BC	
1	Topic: 9.4	Defining and Differentiating Vector-Valued Functions
Learning Objective CHA-3.H: Calculate derivatives of vector-valued functions.		

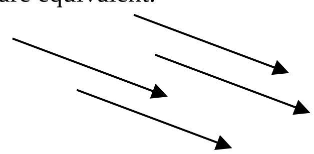
Vectors in a Plane

In geometry and physics, there are many concepts that can be quantified with a single number. These are called **scalar quantities** and the real number associated with it is often referred to as a **scalar**.
However, there are some concepts that require a different representation – mainly because of their need to express both magnitude and direction. These concepts are expressed as a **vector**.

Concepts Expressed by a Single Number	Concepts Expressed by a Vector
temperature, mass, time, length, area, volume	force, velocity, acceleration

To represent a vector, we use a directed line segment as shown below.
The directed line segment PQ has initial point P and terminal point Q and we denote its length by $\|PQ\|$. Two directed line segments that have the same length and direction are called equivalent. For example all the directed line segments below and to the right are equivalent.





We call each a vector in a plane and write $\mathbf{v} = PQ$. Vectors are typically denoted by the lower-case boldfaced letters, **u**, **v** and **w.**

Example 1: Equivalent Vectors
Let **u** be the directed line segment from (0, 0) to (5, 2) and let **v** be represented by the directed line segment from (-1, 3) to (4, 5). Show that **u** = **v**.

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A line segment whose initial point is the origin and whose terminal point is (v_1, v_2) is given by the component form of **v** given by $\mathbf{v} = \langle v_1, v_2 \rangle$. The components v_1 and v_2 are called the components of **v**.

To convert directed line segments to component form or vice versa, use the following:

Converting Directed Line Segments to Component Form

- If $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, then **v** represented by PQ, in component form, is $\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle$ and the length of **v** is $\|v\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$. This is called the magnitude of **v.**

- If $\mathbf{v} = \langle v_1, v_2 \rangle$, then **v** can be represented by the directed line segment in standard position from $P(0,0)$ to Q as $\langle v_1, v_2 \rangle$.

Example 2: **Converting to Component Form**
Find the component form and length of the vector **v** having initial point $(4, -6)$ and terminal point $(-1, 2)$.

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Vector Operations

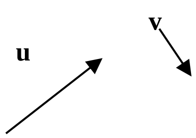
The two basic vector operations are called scalar multiplication and vector addition.
Geometrically, the product of a vector **v** and a scalar *k* is the vector that is *k* times that as long as **v.**
If *k* is positive, then the vector *k***v** has the same direction as **v**. If *k* is negative, then *k***v** has the opposite direction as **v.**

To add vectors, we move one of them so that the initial side of one is the terminal side of the other. The sum **u** + **v**, called the resultant vector, is formed by joining the initial point of the first vector to the terminal side of the second.

Activity

1. Add the vectors **u** + **v**

2. Subtract the vectors **u** - **v**



VECTOR OPERATIONS

For vectors **u** = $\langle u_1, u_2 \rangle$ and **v** = $\langle v_1, v_2 \rangle$ and scalar *k*, the following operations are defined:

1. The scalar multiple of k and vector **u** is the vector *k***u** = $k\langle u_1, u_2 \rangle$.
2. The vector sum of **u** and **v** is the vector **u** + **v** = $\langle u_1 + v_1, u_2 + v_2 \rangle$.
3. The negative of vector **v** is the vector –**v** = (-1)**v** = $\langle -v_1, -v_2 \rangle$.
4. The difference of **u** and **v** is the vector **u** − **v** = $\langle u_1 - v_1, u_2 - v_2 \rangle$.

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Example 3: **Vector Operations**
Given the vectors **u** = $\langle -3, 7 \rangle$ and **v** = $\langle 5, 1 \rangle$, find the following:

a.) $-\frac{1}{2}\mathbf{u}$

b.) $\left\|\frac{-1}{2}u\right\|$

c.) **u** + **v**

d.) $\|u + v\|$

e.) **v** – **u**

f.) $\|u - v\|$

g.) 3**u** - 4**v**

h.) $\|3u - 4v\|$

UNIT VECTORS

A unit vector has the same direction as the original vector but with length 1.

If **v** is a nonzero vector, then the vector $u = \frac{v}{\|v\|}$ is a unit vector.

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Example 4: Finding Unit Vectors
Find a unit vector for the vector $v = \langle 7, -3 \rangle$ and show that it has length 1.

The unit vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are called the standard unit vectors and are denoted by $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

These vectors can be used to represent any vector $\mathbf{v} = \langle v_1, v_2 \rangle = v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$.

We call $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ a linear combination of $\mathbf{i}$ and $\mathbf{j}$.

The scalars v_1 and v_2 are called the components of $\mathbf{v}$.

Example 5: Finding Linear Combinations
Let $\mathbf{u}$ be the vector with initial point $(-3, 7)$ and terminal point $(-5, 2)$ and let $v = 3i - 2j$. Write the following as a linear combination of $\mathbf{i}$ and $\mathbf{j}$.

a.) $\mathbf{u}$

b.) $\mathbf{w} = 4\mathbf{u} - 5\mathbf{v}$

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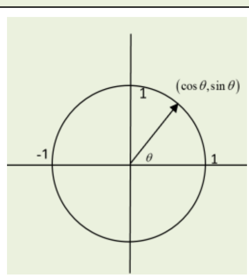

If **u** is a unit vector such that θ is the angle from the *x*-axis to **u**, then the terminal point of **u** lies on the unit circle and is equal to $\langle \cos\theta, \sin\theta \rangle = i\cos\theta + j\sin\theta$.

If **v** is any other vector such that θ is the angle from the *x*-axis to **v**, then $v = \|v\|\langle \cos\theta, \sin\theta \rangle = \|v\| i\cos\theta + \|v\| j\sin\theta$

Example 6: Writing Vectors in Component Form

Write the vector **v** of length 6 making an angle of 60° with the positive *x*-axis in component form.

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Example 7: Application of Vectors

Two tugboats are pulling a disabled cruise ship due east. One pulls with a force of 40 tons at an angle of 12° to northeast and the other at 50 tons at an angle of 18° to the southeast. What is the resultant force on the ship and what angle does it travel?

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Example 8: Application of Vectors

A plane travelling 500 mph in the direction 120° encounters a wind of 45 mph in the direction of 45°. What is the resultant speed and direction of the plane?

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Dot Products

Multiplying two vectors is different from adding or subtracting vectors. When we add or subtract vectors, we get another vector. But when we multiply two vectors, we get a scalar.

The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is given by $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$. Note that the result is a scalar. Simply add the product of the two similar components.

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Example 9: Dot Product

Given $\mathbf{u} = \langle 2, -2 \rangle$ and $\mathbf{v} = \langle 5, 8 \rangle$, find

a.) $\mathbf{u} \cdot \mathbf{v}$ **b.)** $\mathbf{u} \cdot (2\mathbf{v})$ **c.)** $\mathbf{u}^2$ **d.)** $\mathbf{u}(\mathbf{v}^2)$

DEFINITON: Orthogonal Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are called **orthogonal** (perpendicular) if $u \cdot v = 0$.

Perpendicular, normal, and orthogonal all mean the same thing. However, we usually say that vectors are orthogonal, lines are perpendicular, and lines and curves are normal.

Angle Between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{u}$, then $cos\,\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|}$.

Note that by cross multiplying, you get that $u \cdot v = \|u\| \cdot \|v\|\, cos\,\theta$ which is another way of finding the dot product of two vectors, further emphasizing that the dot product of two vectors is scalar.

Example 10: Dot Product

Given $u = \langle 3, -1 \rangle, v = \langle 4{,}3 \rangle$ and $\mathrm{w} = \left\langle \frac{2}{3}, -\frac{8}{9} \right\rangle$, determine the angle between the two vectors and if any of the vectors are orthogonal.

a.) u and v **b.)** u and w **c.)** v and w

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In **Topic 9.1**, we defined a *plane-curve* as the set of points $(f(t), g(t))$satisfying the parametric equations $x = f(t)$and $y = g(t)$ where f and g are continuous functions of t on some interval $\boldsymbol{I}$.

We now define new type of function called a vector-valued function of the form $r(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j}$

A vector-valued function (sometimes called vector functions) takes the variable *t* and returns a vector. The domain of a vector-valued function is the set of all *t* for which both component functions are defined.

Example 11: **Domain of Vector-Valued Functions**
Find the domain of the following vector-valued functions.
a.) $r(t) = \sqrt{t+1}i + ln(4 - t)j$

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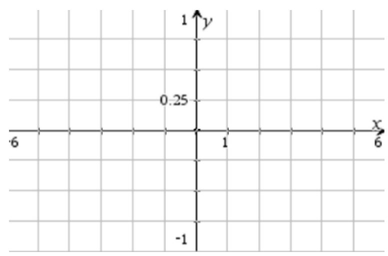
b.) $r(t) = e^{-t}i + (sin^{-1} t)j$

Example 12: **Graphing Vector-Valued Functions**
Sketch the vector-valued function $r(t) = (t - 5)i - j$.



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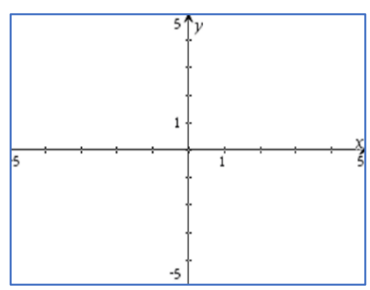
Example 13: **Graphing Vector-Valued Functions.**
Sketch the vector-valued function $r(t) = (4\,cos\,t)i - (2\,sin\,t)j, 0 \leq t \leq 2\pi$



Example 14: Converting a Rectangular Equation to a Vector-Valued Function

Represent the curve $y = x^3 - 2x^2 + 3x - 1$ by a vector-valued function.

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Rules for Vector-Valued Functions

Adding:

$$\mathbf{r}_1(t) + \mathbf{r}_2(t) = [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] + [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}]$$
$$= [f_1(t) + f_2(t)]\mathbf{i} + [g_1(t) + g_2(t)]\mathbf{j}$$

Scalar Multiplication:

$$k\mathbf{r}(t) = k[f(t)\mathbf{i} + g(t)\mathbf{j}]$$
$$= kf(t)\mathbf{i} + kg(t)\mathbf{j}$$

Limits:

$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j}$$

Note: If the limit of either (or both) component(s) of a vector-valued function fails to exist, then limit of the overall vector-valued function fails to exit.

Continuity:

A vector-valued function **r** is continuous at $t = a$ if

$$\lim_{t \to a} r(t) = r(a)$$

Derivatives:

$$r'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j}$$

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Example 15: Operations with Vector-Valued Functions

If $r_1(t) = t^2 i - (4t - 1)j, r_2(t) = (2t^2 + 3)i + 2tj$, and $h(t) = \frac{1}{t}$, find

a.) $\boldsymbol{r_1(t) - r_2(t)}$

b.) $\|\boldsymbol{r_1(t) - r_2(t)}\|$

c.) $\dfrac{\boldsymbol{r_1(t)}}{\|\boldsymbol{r_1(t)}\|}$

d.) $\boldsymbol{r_1(t) \cdot r_2(t)}$

e.) $\boldsymbol{h(t) \cdot r_1(t)}$

f.) $\boldsymbol{h(t) \cdot (r_2(t))^2}$

Example 16: Limits of Vector-Valued Functions
Find the following limits.

a.) $\lim_{t\to 3}\left(t\mathbf{i}+\frac{t^2-9}{t-3}\mathbf{j}\right)$

b.) $\lim_{t\to 0}\left(e^{-t}\mathbf{i}+\frac{\sin t}{2t}\mathbf{j}\right)$

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c.) $\lim_{t\to\infty}\left(e^{-t}\mathbf{i}+\frac{5t+1}{t}\mathbf{j}\right)$

Example 17: Vector-Valued Functions and Continuity
Determine the intervals on which the vector-valued function is continuous.

a.) $\mathbf{r}(t)=\frac{1}{t}\mathbf{i}-t\mathbf{j}$

b.) $\mathbf{r}(t)=\ln(t-1)\mathbf{i}+\tan t\,\mathbf{j}$

Scan the QR Code above to watch a video covering Example 17

c.) $\mathbf{r}(t)=\cos t\,\mathbf{i}+\cos^{-1}(t)\mathbf{j}$

<u>Derivative Rules for Vector Valued Functions</u>

If u and **v** are differentiable vector-valued functions of *t*, *f* is a differentiable, real-valued function of *t* and *k* is a scalar, then the following properties are true:

Scalar Multiplication: $\frac{d}{dt}[ku(t)] = ku'(t)$

Addition/Subtraction: $\frac{d[\boldsymbol{u}(\boldsymbol{t}) \pm \boldsymbol{v}(\boldsymbol{t})]}{dt} = \boldsymbol{u}'(\boldsymbol{t}) \pm \boldsymbol{v}'(\boldsymbol{t})$

Multiplication by a Function: $\frac{d}{dt}[\boldsymbol{f}(\boldsymbol{t}) \cdot \boldsymbol{u}(\boldsymbol{t})] = \boldsymbol{f}(\boldsymbol{t}) \cdot \boldsymbol{u}'(\boldsymbol{t}) + \boldsymbol{u}(\boldsymbol{t}) \cdot \boldsymbol{f}'(\boldsymbol{t})$

Product Rule Involving Two V-V F's: $\frac{d}{dt}[\boldsymbol{u}(\boldsymbol{t}) \cdot \boldsymbol{v}(\boldsymbol{t})] = \boldsymbol{u}(\boldsymbol{t}) \cdot \boldsymbol{v}'(\boldsymbol{t}) + \boldsymbol{v}(\boldsymbol{t}) \cdot \boldsymbol{u}'(\boldsymbol{t})$

Example 18: Vector-Valued Functions and Differentiation

If $f(t) = t^2 + t$, $\mathbf{u}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j}$, $\mathbf{v}(t) = t^2\mathbf{i} - \ln t\,\mathbf{j}$, find each of the following in two different ways.

a) $\frac{d}{dt}[\boldsymbol{f}(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})]$

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Example 18: Vector-Valued Functions and Differentiation

If $f(t) = t^2 + t$, $\mathbf{u}(t) = \dfrac{1}{t}\mathbf{i} - \mathbf{j}$, $\mathbf{v}(t) = t^2\mathbf{i} - \ln t\,\mathbf{j}$, find each of the following in two different ways.

b) $\frac{d}{dt}[\boldsymbol{u}(\boldsymbol{t})\boldsymbol{v}(\boldsymbol{t})]$

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