How to Tell if a Graph Represents a Function: Vertical Line Test

What does it mean graphically for y to be a function of x? Look at the graph of y against x. For a function, each x-value corresponds to exactly one y-value. This means that the graph intersects any vertical line at most once. If a vertical line cuts the graph twice, the graph would contain two points with different y-values but the same x-value; this would violate the definition of a function. Thus, we have the following criterion:

Vertical Line Test: If there is a vertical line that intersects a graph in more than one point, then the graph does not represent a function.

Example 6 In which of the graphs in Figures 1.2 and 1.3 could y be a function of x?

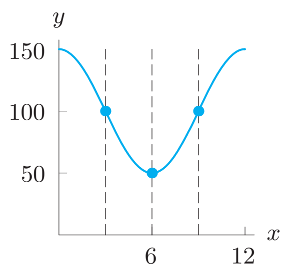


Figure 1.2: Since no vertical line intersects this curve at more than one point, y could be a function of x

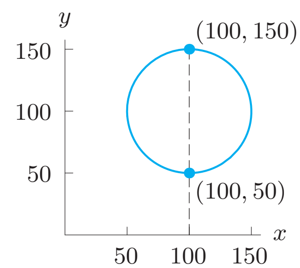


Figure 1.3: Since one vertical line intersects this curve at more than one point, y is not a function of x

Solution The graph in Figure 1.2 could represent y as a function of x because no vertical line intersects this curve in more than one point. The graph in Figure 1.3 does not represent a function because the vertical line shown intersects the curve at two points.

A graph fails the vertical line test if at least one vertical line cuts the graph more than once, as in Figure 1.3. However, if a graph represents a function, then *every* vertical line must intersect the graph at no more than one point.

Exercises and Problems for Section 1.1

Skill Refresher

In Exercises S1–S4, simplify each expression.

S1. $c + \frac{1}{2}c$

S2. $P + 0.07P + 0.02P$

S3. $2\pi r^2 + 2\pi r \cdot 2r$

S4. $\dfrac{12\pi - 2\pi}{6\pi}$

In Exercises S5–S8, find the value of the expressions for the given value of x and y.

S5. $x - 5y$ for $x = \frac{1}{2}$, $y = -5$.

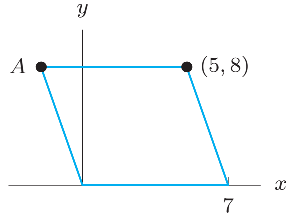
S6. $1 - 12x + x^2$ for $x = 3$.

S7. $\dfrac{3}{2 - x^3}$ for $x = -1$.

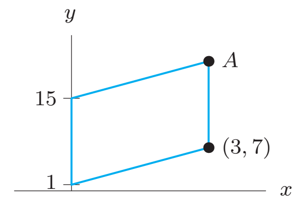
S8. $\dfrac{4}{1 + 1/x}$ for $x = -\frac{3}{4}$.

The figures in Exercises S9–S10 are parallelograms. Find the coordinates of the labeled point(s).

S9.



S10.



Exercises

1. Figure 1.4 gives the depth of the water at Montauk Point, New York, for a day in November.

(a) How many high tides took place on this day?
(b) How many low tides took place on this day?
(c) How much time elapsed in between high tides?

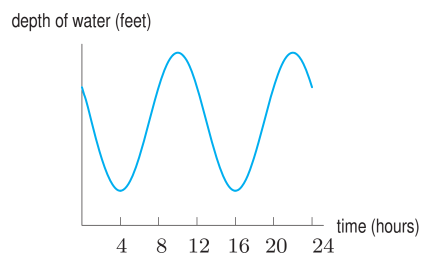


Figure 1.4

In Exercises 2–3, write the relationship using function notation (that is, y is a function of x is written $y = f(x)$).

2. Number of molecules, m, in a gas, is a function of the volume of the gas, v.

3. Weight, w, is a function of caloric intake, c.

In Exercises 4–7, label the axes for a sketch to illustrate the given statement.

4. "Over the past century we have seen changes in the population, P (in millions), of the city. . ."

5. "Sketch a graph of the cost of manufacturing q items. . ."

6. "Graph the pressure, p, of a gas as a function of its volume, v, where p is in pounds per square inch and v is in cubic inches."

7. "Graph D in terms of y. . ."

8. Using Table 1.4, graph $n = f(A)$, the number of gallons of paint needed to cover a house of area A. Identify the independent and dependent variables.

Table 1.4

A	0	250	500	750	1000	1250	1500
n	0	1	2	3	4	5	6

9. Use Table 1.5 to fill in the missing values. (There may be more than one answer.)

(a) $f(0) = ?$ **(b)** $f(?) = 0$
(c) $f(1) = ?$ **(d)** $f(?) = 1$

Table 1.5

x	0	1	2	3	4
$f(x)$	4	2	1	0	1

10. Use Figure 1.5 to fill in the missing values:

(a) $f(0) = ?$ **(b)** $f(?) = 0$

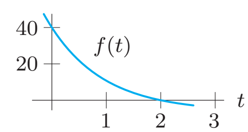


Figure 1.5

Exercises 11–14 use Figure 1.6.

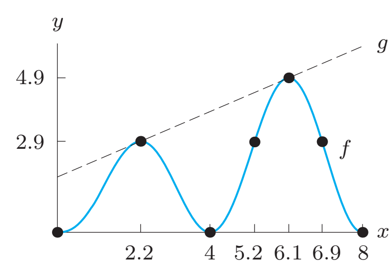


Figure 1.6

11. Find $f(6.9)$.

12. Give the coordinates of two points on the graph of g.

13. Solve $f(x) = 0$ for x.

14. Solve $f(x) = g(x)$ for x.

15. **(a)** You are going to graph $p = f(w)$. Which variable goes on the horizontal axis?
(b) If $10 = f(-4)$, give the coordinates of a point on the graph of f.
(c) If 6 is a solution of the equation $f(w) = 1$, give a point on the graph of f.

16. **(a)** Suppose x and y are the coordinates of a point on the circle $x^2 + y^2 = 1$. Is y a function of x? Why or why not?
(b) Suppose x and y are the coordinates of a point on the part of the circle $x^2 + y^2 = 1$ that is above the x-axis. Is y a function of x? Why or why not?

17. **(a)** Is the area, A, of a square a function of the length of one of its sides, s?
(b) Is the area, A, of a rectangle a function of the length of one of its sides, s?

18. Which of the following graphs represent functions?

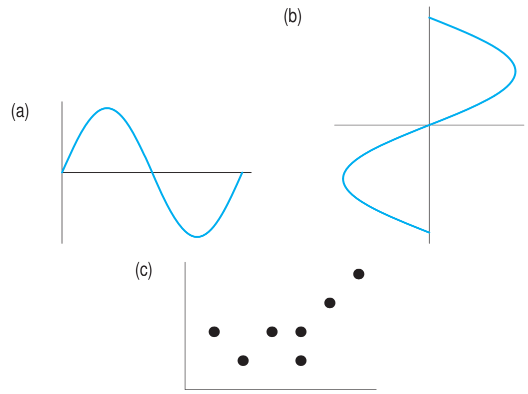


Figure 1.7

Problems

19. A buzzard is circling high overhead when it spies some road kill. It swoops down, lands, and eats. Later it takes off sluggishly, and resumes circling overhead, but at a lower altitude. Sketch a possible graph of the height of the buzzard as a function of time.

20. A person's blood sugar level at a particular time of the day is partially determined by the time of the most recent meal. After a meal, blood sugar level increases rapidly, then slowly comes back down to a normal level. Sketch a person's blood sugar level as a function of time over the course of a day. Label the axes to indicate normal blood sugar level and the time of each meal.

21. Let $f(t)$ be the number of people, in millions, who own cell phones t years after 1990. Explain the meaning of the following statements.

(a) $f(10) = 100.3$ **(b)** $f(a) = 20$
(c) $f(20) = b$ **(d)** $n = f(t)$

22. At the end of a semester, students' math grades are listed in a table which gives each student's ID number in the left column and the student's grade in the right column. Let N represent the ID number and the G represent the grade. Which quantity, N or G, must necessarily be a function of the other?

23. Table 1.6 gives the ranking r for three different names—Hannah, Alexis, and Madison. Of the three names, which was most popular and which was least popular in

(a) 1995? **(b)** 2004?

Table 1.6 *Ranking of names—Hannah (r_h), Alexis (r_a), and Madison (r_m)—for girls born between 1995 ($t = 0$) and 2004 ($t = 9$)*[1]

t	0	1	2	3	4	5	6	7	8	9
r_h	7	7	5	2	2	2	3	3	4	5
r_a	14	8	8	6	3	6	5	5	7	11
r_m	29	15	10	9	7	3	2	2	3	3

24. Table 1.6 gives information about the popularity of the names Hannah, Madison, and Alexis. Describe in words what your answers to parts (a)–(c) tell you about these names.

(a) Evaluate $r_m(0) - r_h(0)$.
(b) Evaluate $r_m(9) - r_h(9)$.
(c) Solve $r_m(t) < r_a(t)$.

25. Figure 1.8 shows the fuel consumption (in miles per gallon, mpg) of a car traveling at various speeds.

(a) How much gas is used on a 300-mile trip at 40 mph?
(b) How much gas is saved by traveling 60 mph instead of 70 mph on a 200-mile trip?
(c) According to this graph, what is the most fuel-efficient speed to travel? Explain.

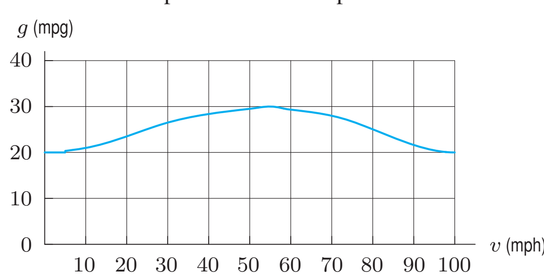


Figure 1.8

[1]Data from the SSA website at www.ssa.gov, accessed January 12, 2006.

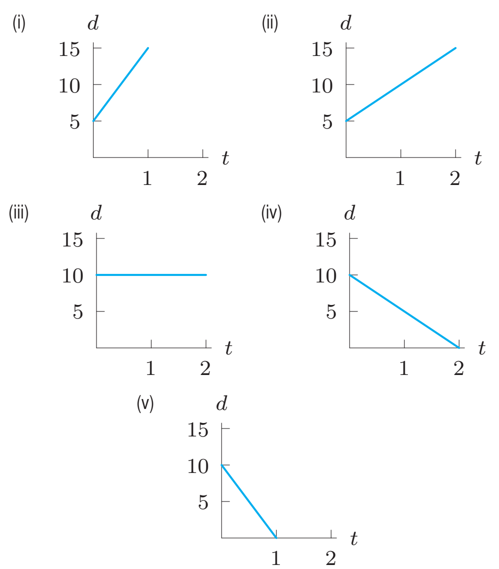
26. **(a)** Ten inches of snow is equivalent to about one inch of rain.[2] Write an equation for the amount of precipitation, measured in inches of rain, $r = f(s)$, as a function of the number of inches of snow, s.
(b) Evaluate and interpret $f(5)$.
(c) Find s such that $f(s) = 5$ and interpret your result.

27. An 8-foot-tall cylindrical water tank has a base of diameter 6 feet.

(a) How much water can the tank hold?
(b) How much water is in the tank if the water is 5 feet deep?
(c) Write a formula for the volume of water as a function of its depth in the tank.

28. Match each story about a bike ride to one of the graphs (i)–(v), where d represents distance from home and t is time in hours since the start of the ride. (A graph may be used more than once.)

(a) Starts 5 miles from home and rides 5 miles per hour away from home.
(b) Starts 5 miles from home and rides 10 miles per hour away from home.
(c) Starts 10 miles from home and arrives home one hour later.
(d) Starts 10 miles from home and is halfway home after one hour.
(e) Starts 5 miles from home and is 10 miles from home after one hour.



29. Table 1.7 shows the daily low temperature for a one-week period in New York City during July.

(a) What was the low temperature on July 19?
(b) When was the low temperature 73°F?
(c) Is the daily low temperature a function of the date?
(d) Is the date a function of the daily low temperature?

Table 1.7

Date	17	18	19	20	21	22	23
Low temp (°F)	73	77	69	73	75	75	70

30. Use the data from Table 1.3 on page 5.

(a) Plot R on the vertical axis and t on the horizontal axis. Use this graph to explain why you believe that R is a function of t.
(b) Plot F on the vertical axis and t on the horizontal axis. Use this graph to explain why you believe that F is a function of t.
(c) Plot F on the vertical axis and R on the horizontal axis. From this graph show that F is not a function of R.
(d) Plot R on the vertical axis and F on the horizontal axis. From this graph show that R is not a function of F.

31. Since Roger Bannister broke the 4-minute mile on May 6, 1954, the record has been lowered by over sixteen seconds. Table 1.8 shows the year and times (as min:sec) of new world records for the one-mile run.[3] The last time the record was broken was in 1999.

(a) Is the time a function of the year? Explain.
(b) Is the year a function of the time? Explain.
(c) Let $y(r)$ be the year in which the world record, r, was set. Explain what is meant by the statement $y(3{:}47.33) = 1981$.
(d) Evaluate and interpret $y(3{:}51.1)$.

Table 1.8

Year	Time	Year	Time	Year	Time
1954	3:59.4	1966	3:51.3	1981	3:48.53
1954	3:58.0	1967	3:51.1	1981	3:48.40
1957	3:57.2	1975	3:51.0	1981	3:47.33
1958	3:54.5	1975	3:49.4	1985	3:46.32
1962	3:54.4	1979	3:49.0	1993	3:44.39
1964	3:54.1	1980	3:48.8	1999	3:43.13
1965	3:53.6				

[2] http://mo.water.usgs.gov/outreach/rain, accessed May 7, 2006.
[3] www.infoplease.com/ipsa/A0112924.html, accessed January 15, 2006.

32. Table 1.9 gives $A = f(d)$, the amount of money in bills of denomination d circulating in US currency in 2008.[4] For example, there were \$64.7 billion worth of \$50 bills in circulation.

(a) Find $f(100)$. What does this tell you about money?
(b) Are there more \$1 bills or \$5 bills in circulation?

Table 1.9

Denomination (\$)	1	2	5	10	20	50	100
Circulation (\$bn)	9.5	1.7	11	16.3	125.1	64.7	625

33. There are x male job-applicants at a certain company and y female applicants. Suppose that 15% of the men are accepted and 18% of the women are accepted. Write an expression in terms of x and y representing each of the following quantities:

(a) The total number of applicants to the company.
(b) The total number of applicants accepted.
(c) The percentage of all applicants accepted.

34. The sales tax on an item is 6%. Express the total cost, C, in terms of the price of the item, P.

35. Write a formula for the area of a circle as a function of its radius and determine the percent increase in the area if the radius is increased by 10%.

36. A price increases 5% due to inflation and is then reduced 10% for a sale. Express the final price as a function of the original price, P.

37. A chemical company spends \$2 million to buy machinery before it starts producing chemicals. Then it spends $\$0.5$ million on raw materials for each million liters of chemical produced.

(a) The number of liters produced ranges from 0 to 5 million. Make a table showing the relationship between the number of million liters produced, l, and the total cost, C, in millions of dollars, to produce that number of million liters.
(b) Find a formula that expresses C as a function of l.

38. A person leaves home and walks due west for a time and then walks due north.

(a) The person walks 10 miles in total. If w represents the (variable) distance west she walks, and D represents her (variable) distance from home at the end of her walk, is D a function of w? Why or why not?
(b) Suppose now that x is the distance that she walks in total. Is D a function of x? Why or why not?

1.2 RATE OF CHANGE

Sales of digital video disc (DVD) players have been increasing since they were introduced in early 1998. To measure how fast sales were increasing, we calculate a *rate of change* of the form

$$\frac{\text{Change in sales}}{\text{Change in time}}.$$

At the same time, sales of video cassette recorders (VCRs) have been decreasing. See Table 1.10.

Let us calculate the rate of change of DVD player and VCR sales between 1998 and 2003. Table 1.10 gives

$$\begin{array}{c}\text{Average rate of change of DVD}\\ \text{player sales from 1998 to 2003}\end{array} = \frac{\text{Change in DVD player sales}}{\text{Change in time}} = \frac{3050-421}{2003-1998} \approx 525.8 \begin{array}{l}\text{mn \$/}\\ \text{year.}\end{array}$$

Thus, DVD player sales increased on average by $\$525.8$ million per year between 1998 and 2003. See Figure 1.9. Similarly, Table 1.10 gives

$$\begin{array}{c}\text{Average rate of change of VCR sales}\\ \text{from 1998 to 2003}\end{array} = \frac{\text{Change in VCR sales}}{\text{Change in time}} = \frac{407-2409}{2003-1998} \approx -400.4 \begin{array}{l}\text{mn \$/}\\ \text{year.}\end{array}$$

[4] www.visualeconomics.com/the-value-of-united-states-currency-in-circulation, The Value of United States Currency in Circulation, 2008, accessed November 16, 2009.