Day 5: Motion Along a Curve — Vectors (continued)

Example (calculator):

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dx}{dt} = \sin(t^3)$, $\frac{dy}{dt} = \cos(t^2)$. At time $t = 2$, the object is at the position (1, 4).

(a) Find the acceleration vector for the particle at $t = 2$.

(b) Write the equation of the tangent line to the curve at the point where $t = 2$.

(c) Find the speed of the vector at $t = 2$.

(d) Find the position of the particle at time $t = 1$.

Solution:

(a) Students should use their calculators to numerically differentiate both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ when $t = 2$ to get $a(2) = \langle -1.476, 3.027 \rangle$.

(b) When $t = 2$, $\frac{dy}{dx} = \frac{\cos 4}{\sin 8}$ or -0.661, so the tangent line equation is

$y - 4 = \frac{\cos 4}{\sin 8}(x - 1)$ or $y - 4 = -0.661(x - 1)$.

Notice that it is fine to leave the slope as the exact value or to write it as a decimal correct to three decimal places.

(c) Speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\sin 8)^2 + (\cos 4)^2}$ or 1.186

Notice that it is fine to leave the speed as the exact value or to write it as a decimal correct to three decimal places.

(d) Students should apply the Fundamental Theorem of Calculus to find the x and y components of the position.

$$x(1) = x(2) - \int_1^2 x'(t)\,dt \qquad y(1) = y(2) - \int_1^2 y'(t)\,dt$$

$$= 1 - \int_1^2 \sin(t^3)\,dt \qquad = 4 - \int_1^2 \cos(t^2)\,dt$$

$$= 0.782 \qquad = 4.443$$

Therefore the position at time $t = 1$ is (0.782, 4.443).

Day 5 Homework

Use your calculator on problems 7–11 only.

1. If $x = e^{2t}$ and $y = \sin(3t)$, find $\frac{dy}{dx}$ in terms of t.

2. Write an integral expression to represent the length of the path described by the parametric equations $x = \cos^3 t$ and $y = \sin^2 t$ for $0 \le t \le \frac{\pi}{2}$.

3. For what value(s) of t does the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ have a vertical tangent?

4. For any time $t \ge 0$, if the position of a particle in the xy-plane is given by $x = t^2 + 1$ and $y = \ln(2t + 3)$, find the acceleration vector.

5. Find the equation of the tangent line to the curve given by the parametric equations $x(t) = 3t^2 - 4t + 2$ and $y(t) = t^3 - 4t$ at the point on the curve where $t = 1$.

6. If $x(t) = e^t + 1$ and $y = 2e^{2t}$ are the equations of the path of a particle moving in the xy-plane, write an equation for the path of the particle in terms of x and y.

7. A particle moves in the xy-plane so that its position at any time t is given by $x = \cos(5t)$ and $y = t^3$. What is the speed of the particle when $t = 2$?

8. The position of a particle at time $t \ge 0$ is given by the parametric equations $x(t) = \frac{(t-2)^3}{3} + 4$ and $y(t) = t^2 - 4t + 4$.

(a) Find the magnitude of the velocity vector at $t = 1$.

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 1$.

(c) When is the particle at rest? What is its position at that time?

9. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \ge 0$ with $\frac{dx}{dt} = 1 + \tan(t^2)$ and $\frac{dy}{dt} = 3e^{\sqrt{t}}$. Find the acceleration vector and the speed of the object when $t = 5$.

10. A particle moves in the xy-plane so that the position of the particle is given by $x(t) = t + \cos t$ and $y(t) = 3t + 2\sin t$, $0 \le t \le \pi$. Find the velocity vector when the particle's vertical position is $y = 5$.

11. An object moving along a curve in the xy-plane has position $\left(x(t), y(t)\right)$ at time t with $\dfrac{dx}{dt} = 2\sin\left(t^3\right)$ and $\dfrac{dy}{dt} = \cos\left(t^2\right)$ for $0 \le t \le 4$. At time $t = 1$, the object is at the position (3, 4).

(a) Write an equation for the line tangent to the curve at (3, 4).

(b) Find the speed of the object at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.

(d) Find the position of the object at time $t = 2$.

Answers to Day 5 Homework

1. $\dfrac{dy}{dx} = \dfrac{3\cos(3t)}{2e^{2t}}$

2. Length = $\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4 t \sin^2 t + 4\sin^2 t \cos^2 t}\, dt$

3. $\dfrac{dy}{dx} = \dfrac{4t^3 + 4t - 8}{3t^2 - 2t}$ is undefined when $3t^2 - 2t = 0$.

So the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ has a vertical tangent when $t = 0$ and $t = \dfrac{2}{3}$.

4. $v(t) = \left\langle 2t, \dfrac{2}{2t+3} \right\rangle$, $a(t) = \left(2, -\dfrac{4}{(2t+3)^2} \right)$

5. $\left.\dfrac{dy}{dx}\right|_{t=1} = \left.\dfrac{3t^2 - 4}{6t - 1}\right|_{t=1} = -\dfrac{1}{2}$. When $t = 1, x = 1, y = -3$.

Tangent line equation: $y + 3 = -\dfrac{1}{2}(x - 1)$

6. $e^t = x - 1$ so $e^{2t} = x^2 - 2x + 1$. Then $y = 2e^{2t}$ so $y = 2x^2 - 4x + 2$.

7. Speed = $\left.\sqrt{\left(-5\sin(5t)\right)^2 + \left(3t^2\right)^2}\right|_{t=2} = 12.304$