PTF #AB 27 – "U-Substitution" Rule Unit 6b

1. Let $u =$ inner function.
2. Find du, then solve for dx.
3. Substitute u & du into the integrand (it should know fit one of the integration rules).
4. Integrate.
5. Substitute the inner function back for u.

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1. Integrate $\int\left[9\left(x^2+3x+5\right)^8(2x+3)\right]dx$

$u = x^2+3x+5$

$du = (2x+3)dx$

$9\int u^8 du$

$\frac{9u^9}{9} + C = (x^2+3x+5)^9 + C$

2. Integrate $\int\left(\sin^2 3x \cos 3x\right)dx$

$u = \sin(3x)$

$du = 3\cos(3x)dx$

$\frac{1}{3}\int u^2 du$

$\frac{1}{3}\cdot\frac{1}{3}u^3 + C = \frac{1}{9}\sin^3(3x) + C$

3. Integrate $\int e^{3x+1}dx$

$\frac{e^u}{3} + C$

$\frac{1}{3}e^{3x+1} + C$

4. Integrate $\int \frac{e^{\tan x}}{\cos^2 x}dx$

$u = \tan x$

$du = \sec^2 x\, dx$

$\int e^u du$

$e^{\tan x} + C$

5. Integrate $\int \frac{e^x}{1+e^x}dx$

$u = 1+e^x$

$du = e^x dx$

$\int \frac{1}{u} du$

$\ln|1+e^x| + C$

6. Using the substitution $u = 2x+1$, $\int_0^2\left(\sqrt{2x+1}\right)dx$ is equal to $\frac{1}{2}\int_1^5 \sqrt{u}\, du$

(A) $\frac{1}{2}\int_{-1/2}^{1/2}\sqrt{u}\,du$

~~(B)~~ $\frac{1}{2}\int_0^2\sqrt{u}\,du$

(C) $\frac{1}{2}\int_1^5\sqrt{u}\,du$

~~(D)~~ $\int_0^2\sqrt{u}\,du$

(E) $\int_1^5\sqrt{u}\,du$

PTF #AB 28 – Approximating Area 6a

Finding a Left or Right Riemann Sum or Trapezoidal Sum:

1. Divide the interval into the appropriate subintervals.
2. Find the y-value of the function at each subinterval.
3. Use the formula for a rectangle (bh) or trapezoid ($\frac{1}{2}b(h_1+h_2)$) to find the area of each individual piece.
4. You must show work to earn credit on these!
5. Always justify a left or right Riemann sum as an over or under approximation using the fact that the function is increasing or decreasing.

	Left Sum	Right Sum
Increasing curve	Under approx.	Over approx.
Decreasing curve	Over approx.	Under approx.

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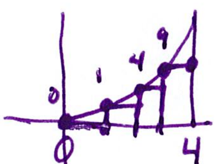
1. Use a left Riemann Sum with 4 equal subdivisions to approximate $\int_0^4 x^2 dx$.



$\Delta x = \frac{4-0}{4} = 1$

$\int_0^4 x^2 dx = (1)[0+1+4+9]$
$= 14$

a. Is this approximation an over or underestimate? Justify.

Underestimate, Increasing function using a left Riemann sum

2. Values of a continuous function $f(x)$ are given below. Use a trapezoidal sum with four subintervals of equal length to approximate $\int_1^{2.2} f(x)$

.3 .3 .3 .3 $\Delta x = .3$

x	1	1.3	1.6	1.9	2.2
$f(x)$	6.0	5.1	4.3	2.0	0.3

$= \frac{1}{2}(0.3)\left[6 + 2(5.1) + 2(4.3) + 2(2) + 0.3\right]$

$= 4.365$

PTF #AB 29 – Fundamental Theorem of Calculus 6a/6b/8a

If f is a continuous function on $[a,b]$ and F is an antiderivative of f on $[a,b]$, then

$$\int_a^b (f(x))\,dx = F(b) - F(a)$$

Graphically this means the **signed area** bounded by $x=a$, $x=b$, $y=f(x)$, and the x-axis.

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1. Evaluate: $\int_1^2 (4x^3+6x)\,dx$

$x^4+3x^2\Big|_1^2$

$[2^4+3(2)^2]-[1^4-3(1)^2]$

$(16+12)-(1-3)$

$28-4=24$

2. Evaluate: $\int_0^{\pi/4} (\sin x)\,dx$

$-\cos x\Big|_0^{\pi/4}$

$-[\cos \pi/4 - \cos 0]$

$-[\sqrt{2}/2 - 1]$

$= 1-\sqrt{2}/2$

3. Evaluate: $\int_0^1 e^{-4x}\,dx$

$\dfrac{e^{-4x}}{-4}\Big|_0^1$

$\dfrac{e^{-4}}{-4} - \dfrac{e^0}{-4}$

$\dfrac{-1}{4e^4} + \dfrac{1}{4}$

4. Evaluate: $\int_{\ln 2}^3 (5e^x)\,dx$

$5e^x\Big|_{\ln 2}^3$

$5(e^3 - e^{\ln 2})$

$5e^3 - 10$

5. Evaluate: $\int_1^2 \left(\frac{x-4}{x^2}\right)dx$ $\quad \int_1^2 \left(\frac{1}{x}-\frac{4}{x^2}\right)dx$

$\ln|x| + \frac{4}{x}\Big|_1^2$

$(\ln|2|+2)-(\ln|1|+4)$

$\ln(2)-2$

6. What are all the values of k for which $\int_{-3}^k x^2\,dx = 0$?

$\dfrac{x^3}{3}\Big|_{-3}^K$

$\dfrac{K^3}{3} - \dfrac{(-3)^3}{3} = 0$

$K^3+27=0$

$K^3=-27$

$K=-3$

PTF #AB 30 – Properties of Definite Integrals 6a/6b

1. If f is defined at $x=a$, then $\int_a^a (f(x))\,dx=0$
2. If f is integrable on $[a,b]$, then $\int_b^a (f(x))\,dx=-\int_a^b (f(x))\,dx$
3. If f is integrable, then $\int_a^b (f(x))\,dx=\int_a^c (f(x))\,dx+\int_c^b (f(x))\,dx$

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1. If $\int_1^{10} (f(x))\,dx=4$ and $\int_{10}^3 (f(x))\,dx=7$, then $\int_1^3 (f(x))\,dx=?$

$\int_3^{10} f(x)dx=-7$

$\int_1^3 f(x)dx + -7 = 4$

$\int_1^3 f(x)dx = 11$

2. Which, if any, of the following are *false*?

I. $\int_a^b (f(x)+g(x))\,dx=\int_a^b (f(x))\,dx+\int_a^b (g(x))\,dx$ T

II. $\int_a^b (f(x)g(x))\,dx=\left(\int_a^b (f(x))\,dx\right)\left(\int_a^b (g(x))\,dx\right)$ F

III. $\int_a^b (cf(x))\,dx=c\int_a^b (f(x))\,dx$ T

PTF #AB 31 – Average Value of a Function

6b/8a

If f is integrable on $[a,b]$, then the average value from the interval is

$$\frac{1}{b-a}\int_a^b (f(x))\,dx$$

To find where this height occurs in the interval:

1. Set $f(x) =$ answer (average value).
2. Solve for x.
3. Check to see if the $x-$value in the given interval.

1. Find the average value of $f(x)=\sin x$ over $[0,\pi]$.

$$\frac{1}{\pi-0}\int_0^{\pi} \sin x\,dx$$

$$= \frac{-1}{\pi}\cos x\Big|_0^{\pi}$$

$$=\frac{-1}{\pi}\left[\cos\pi - \cos 0\right] = \frac{-1}{\pi}\left[-1-1\right]$$

$$= 2/\pi$$

2. Find the average value of $y = x^2\sqrt{x^3+1}$ on the interval $[0,2]$, then find where this value occurs in the interval.

$$\frac{1}{2-0}\int_0^2 x^2\sqrt{x^3+1}\,dx$$

$u = x^3+1$

$du = 3x^2dx$

$$\frac{1}{3}\cdot\frac{1}{2}\int_1^9 \sqrt{u}\,du$$

$$\frac{2}{3}\cdot\frac{1}{6}u^{1/2+2/2}\Big|_1^9 = \frac{1}{9}\left[9^{3/2}-1^{3/2}\right]$$

$$= \frac{1}{9}\left[27-1\right]$$

$$= 26/9$$

$$\frac{26}{9} = x^2\sqrt{x^3+1} \Rightarrow x = 1.281$$

3. The function

$$f(t) = 6+\cos\left(\frac{t}{10}\right)+3\sin\left(\frac{7t}{40}\right)$$

is used to model the velocity of a plane in miles per minute. According to this model, what is the average velocity of the plane for $0 \le t \le 40$? (*calculator*)

$$\frac{1}{40}\int_0^{40} f(t)\,dt \approx 5.916\text{ mi/min}$$

PTF #AB 32 – 2nd Fundamental Theorem of Calculus

6a

To find the derivative of an integral:

$$\frac{d}{dx}\left[\int_a^x (f(t))\,dt\right] = f(x)\bullet dx$$

*Remember that a must be a constant. If it is not, then you must use your properties of integrals to make it a constant.

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1. For $F(x)=\int_2^x \sqrt{1+t^2}\,dt$, find

(a) $F(2)$ $= 0$

(b) $F'(3)$

$F'(x) = \sqrt{1+x^2}$

$F'(3) = \sqrt{1+9}$

$= \sqrt{10}$

2. Evaluate: $\frac{d}{dx}\int_{2x}^{3}(e^t+3)\,dt$

$= -\frac{d}{dx}\int_3^{2x}(e^t+3)\,dt$

$= -\left[(e^{2x}+3)\cdot 2 - \cancel{(e^3+3)\cdot 0}\right]$

$= -2(e^{2x}+3)$

3. Find $F'(x)$ if $F(x)=\int_x^{x^3}(\sec^2 t)\,dt$

$\frac{d}{dx}\int_x^{x^3}(\sec^2 t)\,dt$

$3x^2\sec^2(x^3) - \sec^2(x)$

4. Given $f(x)=\int_0^{3x}(4-2t)\,dt$ and $g(x)=f(e^x)$, find $\int_0^{3e^x}(4-2t)\,dt = g(x)$

(a) $f'(-1)$

$f'(x) = (4-2(3t))\cdot 3$

$f'(1) = 3(4-6(-1)) = 30$

(b) $g(x)$ in terms of an integral

$g(x) = \int_0^{3e^x}(4-2t)\,dt$

(c) $g'(x)$

$[4-2(3e^x)]\cdot 3e^x$

$12e^x - 18e^{2x}$

(d) $g'(0)$

$g'(0) = 12e^0 - 18e^0$

$= 12-18$

$= -6$

(e) Write the equation for the tangent line to $g(x)$ at $x=0$ $(0, g(0))$

$g(0) = \int_0^{3e^0}(4-2t)\,dt = \int_0^3 (4-2t)\,dt$

$= 4t - t^2\Big|_0^3$

$= (12-9)-(0)$

$= 3$

$g'(0) = -6$ from (d)

$y-3 = -6(x-0)$

or $y = -6x+3$

PTF #AB 33 – Extensions of FTC 66/8a

1. FTC as Accumulation ("Integrate removes the rate!"):
   a. Change in Population: $\int_a^b (P'(t))\,dt = P(b) - P(a)$ (gives total population *added* between time a and b)
   b. Change in Amount: $\int_a^b (R'(t))\,dt = R(b) - R(a)$ (gives total amount *added* of water, sand, traffic, etc. between time a and b)
2. FTC as Final Position ("Integrate to find the end!"):
   - Particle Position: $S(b) = S(a) + \int_a^b (v(t))\,dt$ (gives particle position at a certain time, b)
   - Total Amount: $R(b) = R(a) + \int_a^b (R'(t))\,dt$ (gives total amount of water, sand, traffic, etc. at a given time, b)

---

1. A particle moves along the y-axis so that $v(t) = t\sin(t^2)$ for $t \geq 0$. Given that $s(t)$ is the position of the particle and that $s(0) = 3$, find $s(2)$.

$$s(2) = s(0) + \int_0^2 t\sin(t^2)\,dt$$

$$= 3 + \int_0^2 t\cdot\sin(t^2)\,dt$$

$u = t^2$
$du = 2t\,dt$

$$= 3 + \frac{1}{2}\int_0^4 \sin u\,du$$

$$= 3 - \frac{1}{2}\cos u \Big|_0^4$$

$$= 3 - \frac{1}{2}\left[\cos 4 - \cos 0\right]$$

$$= 3 - \frac{1}{2}\cos 4 + \frac{1}{2}$$

$$= \frac{7}{2} - \frac{1}{2}\cos(4)$$

$$\approx 3.827$$

2. ~~A metal~~ A metal of length 8 cm is heated at one end. The function $T'(x) = 2x + 3$ gives the temperature, in $°C$, of the wire x cm from the heated end. Find $\int_0^8 (T'(x))\,dx$ and indicate units of measure. Explain the meaning of the temperature of the wire.

$T'(x)$ = Temp in °C/cm

$$\int_0^8 (2x+3)\,dx = x^2 + 3x \Big|_0^8$$

$$= (8^2 + 24) - (0)$$

$$= 64 + 24$$

$$= 88°C$$

8 cm from the end of the wire, the Temperature is 88°C

PTF #AB 34 – Accumulating Rates 6b/8a

- Identify the rate going in and the rate going out.
- To find a max or min point, set the two rates equal to each other and solve.
- To find the total amount

$$\text{Total} = \text{Initial Amt} + \int_a^b \textit{Rate Added} - \int_a^b \textit{Rate Removed}$$

- Remember to think of different blocks of time for piece-wise functions. Try to visualize what is happening in the situation before you try to put the math to work.

A factory produces bicycles at a rate of $p(w) = 95 + 0.1w^2 - w$ bikes per week for $0 \le w \le 25$. They can ship bicycles out at a rate of $s(w) = \begin{cases} 90 & 0 \le w < 3 \\ 95 & 3 \le w \le 25 \end{cases}$ bikes/week.

1. How many bicycles (To the nearest whole) are produced in the first 2 weeks?

$\int_0^2 p(w)\,dw \approx 188.27$ bikes

$= 188$ bikes

2. How many bicycles are in the warehouse at the end of week 3?

$\int_0^3 p(w)\,dw - \int_0^3 90\,dw$

≈ 11.4 bikes

$= 11$ bikes

3. Find when the number of bicycles in the warehouse is at a minimum.

$p(w) = s(w)$

$95 + 0.1w^2 - w = 90$ Never

$95 + 0.1w^2 - w = 95 \Rightarrow w = 0,\ w = 10$

w	$b = \int_0^w p(w)\,dw - \int_0^w s(w)\,dw$
0	0
10	$933 - 270 - 665 = -2$
25	

bad question

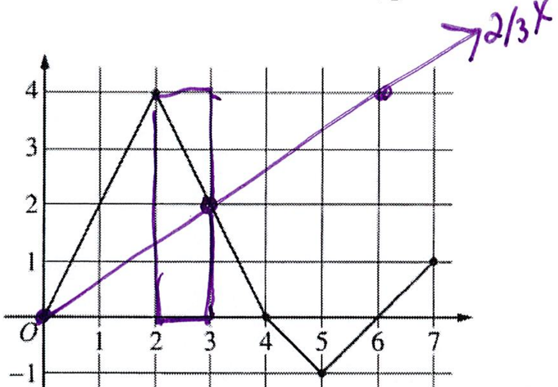
4. The factory needs to stop production if the number of bicycles stored in the warehouse reaches 20 or more. Does the factory need to stop production at any time during the first 25 weeks? If so, when?

PTF #AB 35 – Functions Defined by Integrals 6a

$$F(x)=\int_a^x (f(t))\,dt$$

- $F'(x)=f(t)$ (The function in the integrand is the derivative equation!)
- These problems work just like curve sketching problems – you are looking at a derivative graph so answer accordingly.
- To evaluate $F(b)$, find the area under the curve from where it tells you to start (a) to the number given (b).

Let f be a function defined on the closed interval $[0,7]$. The graph of f, consisting of four line segments, is shown below. Let g be the function given by $g(x)=\int_2^x f(t)\,dt$.



Graph of f $=g'(x)$

1. Find $g(3)$, $g'(3)$, and $g''(3)$.

$g(3)=\int_2^3 f(t)dt=\frac{1}{2}(1)(2+4)=3$

$g'(3)=2$

$g''(3)=-2$ "slope at 3"

2. Find the average rate of change of g on the interval $0\le x\le 3$.

$$\text{AROC}=\frac{\boxed{-\frac{1}{2}(2)(4)}-3}{0-3}=\frac{-7}{-3}=\frac{7}{3}$$

3. Find the x-coordinate of each point of inflection of the graph of g on the interval $0<x<7$. Justify your answer.

$g'(x)=f(x)$

g has inflection points where $g''(x)$ changes signs or $g'(x)$ changes from inc/dec or vice versa

$x=2$ & $x=5$

4. Let $h(x)=\left(\int_2^x f(t)dt\right)-\left(\frac{1}{3}x^2\right)$. Find all critical values for $h(x)$ and classify them as a minimum, maximum or neither.

$h'(x)=0$ are critical values

$h'(x)=f(x)-\frac{2}{3}x$

$0=f(x)-\frac{2}{3}x$

$f(x)=\frac{2}{3}x$

h' + | − at 3

h(x) has a rel. max at x=3