# PTF #AB 01 – Definition of a Limit

The *intended height* (or $y-value$) of a function, $f(x)$. (Remember that the function doesn't actually have to reach that height.)

Written: $\lim_{x \to c} f(x)$

Read: "the limit of $f(x)$ as $x$ approaches $c$

Methods for finding a limit:

1. Direct substitution
2. Look at the graph

Some reasons why a limit would fail to exist:

1. The function approaches a different number from the left side than from the right side.
2. The function increases or decreases without bound.
3. The function oscillates between 2 fixed values.

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1. Evaluate $\lim_{x \to 3} \left(2x^2 + 7x\right)$

$$\lim_{x \to 3} (2x^2+7x) = 2(3)^2 + 7(3)$$
$$= 18 + 21$$
$$= 39$$

2. Find the limit (if it exists).

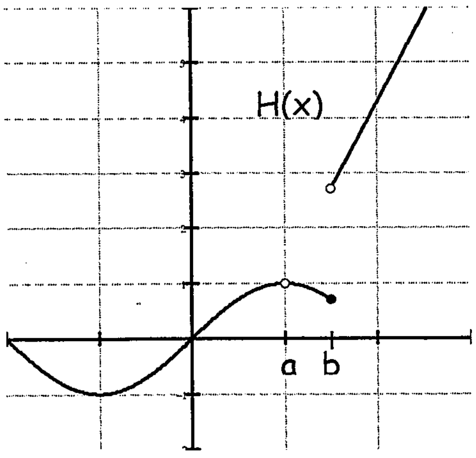
$$\lim_{x \to -2} \frac{|x+2|}{x+2}$$

$$\lim_{x \to -2^-} \frac{|x+2|}{x+2} = -1$$

$$\rightarrow \lim_{x \to -2} \frac{|x+2|}{x+2} \ \underline{\text{DNE}}$$

$$\lim_{x \to -2^+} \frac{|x+2|}{x+2} = 1$$

3. Use the graph below to find the following limits.



a) $\lim_{x \to a} H(x) =$ 1

b) $\lim_{x \to b} H(x) =$ DNE

# PTF #AB 02 – One-Sided Limits

- $\lim_{x \to c^+} f(x)$ means the limit from the right.
- $\lim_{x \to c^-} f(x)$ means the limit from the left.
- A curve has a limit if and only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$. *(Left-hand limit = Right-hand limit)*
- A curve is continuous on a closed interval $[a,b]$ if it is continuous on the open interval $(a,b)$ and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$. *(Limit at the endpoints has to match the function value at the endpoints.)*

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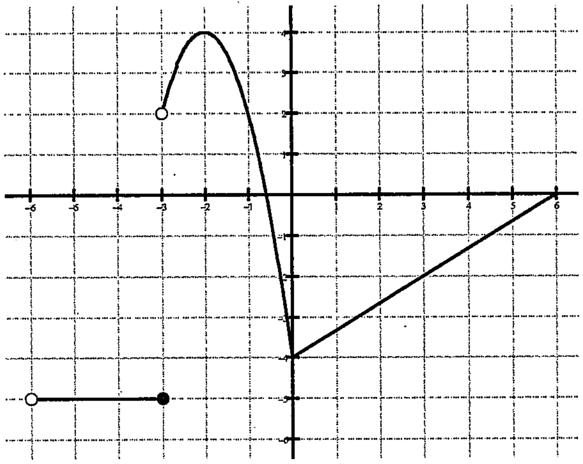
Use the graph to the right to find the following limits, if they exist.



1. $\lim_{x \to -3^-} g(x)$ = -5

2. $\lim_{x \to -3^+} g(x)$ = 2

3. $\lim_{x \to 0^-} g(x)$ = -4

4. $g(-3)$ = -5

5. $\lim_{x \to -6^+} g(x)$ = -5

6. $\lim_{x \to -6} g(x)$ = DNE

7. $\lim_{x \to 0^+} g(x)$ = -4

8. $\lim_{x \to -3} g(x)$ = DNE $\left( \lim_{x \to -3^-} g(x) \neq \lim_{x \to -3^+} g(x) \right)$

9. $\lim_{x \to 0} g(x)$ = -4

# PTF #AB 03 – Horizontal Asymptotes & Limits at Infinity

Horizontal Asymptotes:

1. If $f(x) \to c$ as $x \to \pm\infty$, then $y = c$ is a horizontal asymptote.
2. A horizontal asymptote describes the behavior at the far ends of the graph.
3. It is helpful to think of an End Behavior Function that will mimic the given function (what will dominate as the $x-values$ get large in both directions?)

Limits at Infinity:

1. *Graphically*, a limit at infinity will level off at a certain value on one or both ends.
2. *Analytically*, find an End Behavior Function to model the given function. Then use direct substitution to "evaluate" the limit.
3. *Short Cut:*

| | |
|---|---|
| Top Heavy: | limit DNE |
| Bottom Heavy: | limit = 0 |
| Equal: | limit = ratio of leading coefficients |

*Please be careful with the shortcut. Some functions act strange and require some extra thought. Also, watch our for limits at $-\infty$, they can require extra thought.

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Find the horizontal asymptotes and evaluate the limits.

1. $\lim_{n\to\infty} \frac{4n^2}{n^2+10{,}000n}$ $=4$ H.A. at $y=4$

2. $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$ $=0$ H.A. at $y=0$

3. $\lim_{x\to\infty} \frac{2x^5-7}{-5x^2+9}$ $=-\infty$ no H.A.

4. $\lim_{x\to\infty} \frac{x+6}{\sqrt{x^2+1}}$ $=1$ H.A. at $y=1$ and $y=-1$

5. $\lim_{x\to-\infty} \frac{x+6}{\sqrt{x^2+1}}$ $=-1$

6. If the graph of $y = \frac{ax+b}{x+c}$ has a horizontal asymptote $y = 2$ and a vertical asymptote $x = -3$, then $a + c = ?$

$\lim_{x\to\infty} \frac{ax+b}{x+C} = 2$

$a=2$

$x+C=0$

$-3+C=0$

$C=3$

$a+C = \boxed{5}$

7. For $x \geq 0$, the horizontal line $y = 2$ is an asymptote for the graph of the function $f$. Which of the following statements must be true?

(A) $f(0) = 2$

(B) $f(x) \neq 2$ for all $x \geq 0$

(C) $f(2)$ is undefined

(D) $\lim_{x\to 2} f(x) = \infty$

[✓ (E) $\lim_{x\to\infty} f(x) = 2$]

# PTF #AB 04 – Vertical Asymptotes & Infinite Limits

Vertical Asymptotes:

1. If $f(x) \to \pm\infty$ as $x \to c^{\pm}$, then $x = c$ is a vertical asymptote.
2. If a function has a vertical asymptote, then it is not continuous.
3. Vertical asymptotes occur where the denominator = 0, there is no common factor, and the numerator ≠ 0.

Infinite Limits:

1. *Graphically*, an infinite limit increases/decreases without bound at a vertical asymptote.
2. *Analytically*, direct substitution yields a 0 in the denominator only, with no common factor or indeterminate form.
3. *Numerically*, substitute a decimal number approaching the limit to see if the y-values are approaching + or - infinity.

Find the vertical asymptotes and intervals where the function is continuous.

1. $g(x) = \dfrac{2x^2 - x}{x+5}$

V.A. at $x=-5$

$g(x)$ is continuous for $(-\infty,-5)\cup(-5,\infty)$

2. $h(t) = \dfrac{t^2 - 4}{t^2 + 5t + 6}$

$$\frac{t^2-4}{t^2+5t+6} = \frac{(t-2)(t+2)}{(t+2)(t+3)}$$

Hole at $t+2=0 \Rightarrow t=-2$

V.A. at $t+3=0$
$t=-3$

$h(t)$ is continuous for $(-\infty,-3)\cup(-3,-2)\cup$
$(-2,\infty)$

Find $\lim\limits_{x\to 1^+} f(x)$ and $\lim\limits_{x\to 1^-} f(x)$.

4. $y = \dfrac{x^2+5x}{x-1}$

$$\lim_{x\to 1^+} \frac{x^2+5x}{x-1} = \infty$$

$$\lim_{x\to 1^-} \frac{x^2+5x}{x-1} = -\infty$$

5. $f(x) = \dfrac{1}{(x-1)^4}$

$$\lim_{x\to 1^+} f(x) = \infty$$

$$\lim_{x\to 1^-} f(x) = \infty$$

# PTF #AB 05 – The "Weird" Limits

To work these problems you need to be able to visualize the graphs and end behavior for most functions.

$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = \infty$$

$$\lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to\pi/2^+} \tan x = -\infty$$

$$\lim_{x\to\pi/2^-} \tan x = \infty$$

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \qquad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad \lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to 0^+} \frac{1}{x} = \infty$$

Evaluate the limit of the "inside" functions first, and then evaluate the "outside" function at that number.

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Evaluate the following limits.

1. $\lim_{x\to\infty} e^{1/x}$

$= e^{\lim_{x\to\infty} \frac{1}{x}}$

$= e^{0}$

$= 1$

2. $\lim_{x\to\infty} \tan^{-1}\left(\frac{x^3+1}{x^2+1}\right)$

$= \tan^{-1}\left(\lim_{x\to\infty} \frac{x^3+1}{x^2+1}\right)$

$= \tan^{-1}(\infty) = \lim_{x\to\infty} \tan^{-1}(x)$

$= \frac{\pi}{2}$

3. $\lim_{x\to\infty} \ln\left(\frac{x^2+2}{x^2-5}\right)$

$= \ln\left(\lim_{x\to\infty} \frac{x^2+2}{x^2-5}\right)$

$= \ln(1)$

$= 0$

4. $\lim_{x\to 0^-} e^{1/x}$

$= e^{\lim_{x\to 0^-} \frac{1}{x}}$

$= e^{-\infty}$ $\left(\lim_{x\to-\infty} e^x = 0\right)$

$= 0$

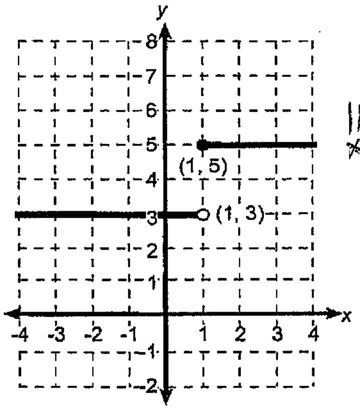
# PTF #AB 06 – Continuity at a Point

To prove a function is continuous at a point, $c$, you must show the following three items are true:

1. $f(c)$ exists *(the function has a y-value for the x-value in question)*
2. $\lim_{x\to c} f(x)$ exists *(the function has a left and right hand limit and they are equal)*
3. $\lim_{x\to c} f(x) = f(c)$ *(the function's value is equal to the limit at that x-value)*

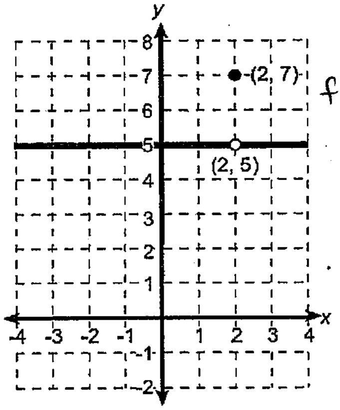
State how continuity is destroyed at $x = c$ for each graph below.

1.



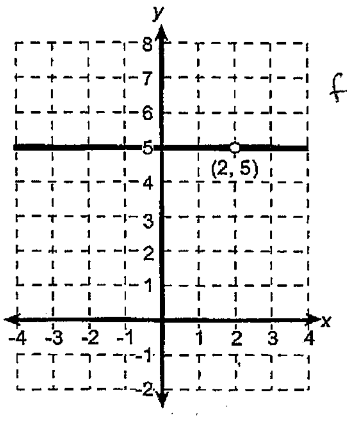
$\lim_{x\to c} f(x)$ DNE

2.



$f(c) \neq \lim_{x\to c} f(x)$

3.



f(c) DNE

4. If the function $f$ is continuous and if $f(x) = \frac{x^2-4}{x+2}$ when $x \neq -2$, then $f(-2) = ?$

then $f(-2) = \lim_{x\to -2} \frac{x^2-4}{x+2}$

$= \lim_{x\to -2} \frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}} = -4$

5. Let $h$ be defined by the following,

$$h(x) = \begin{cases} 3x-7 & x \le -2 \\ -x^4+3 & -2 < x < 2 \\ x^2+9 & x > 2 \end{cases}$$

For what values of $x$ is $h$ not continuous? Justify.

$\lim_{x\to -2^-} 3x-7 = -13$

$\lim_{x\to -2^+} -x^4+3 = -13$

$h(-2) = 3(-2)-7 = -13$

$\lim_{x\to 2^-} -x^4+3 = -13$

$\lim_{x\to 2^+} x^2+9 = 13$

$h(2)$ DNE

discontinuous at $\underline{x=2}$

6. For what value of the constant $c$ is the function $f$ continuous over all reals?

$$f(x) = \begin{cases} cx+1 & x \le 3 \\ cx^2-1 & x > 3 \end{cases}$$

$f(3) = 3c+1$

$\lim_{x\to 3^-} cx+1 = 3c+1$

$\lim_{x\to 3^+} cx^2-1 = 9c-1$

$3c+1 = 9c-1$

$2 = 6c$

$\underline{1/3 = c}$

# PTF #AB 07 – Average Rate of Change

The average rate of change of $f(x)$ over the interval $[a,b]$ can be written as any of the following:

1. $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$
2. $\frac{f(b)-f(a)}{b-a}$
3. Slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.

*Average rate of change is your good old slope formula from Algebra I.

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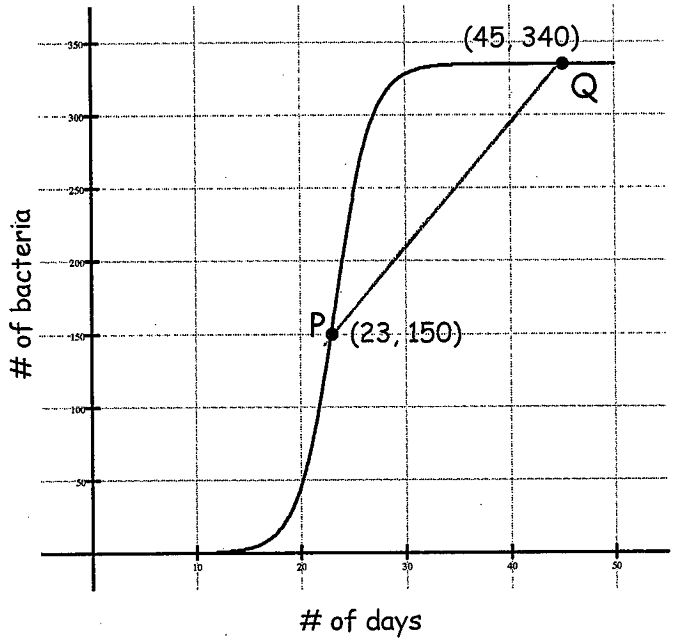
1. In an experiment of population of bacteria, find the average rate of change from P to Q and draw in the secant line.



$\frac{\Delta y}{\Delta x} : \frac{340-150}{45-23} = \frac{190}{22} \frac{\text{bacteria}}{\text{day}}$

2. An equation to model the free fall of a ball dropped from 30 feet high is $f(x) = 30 - 16x^2$. What is the average rate of change for the first 3 minutes? State units.

$\frac{f(3)-f(0)}{3-0} = \frac{-114-30}{3-0} = -48 \frac{\text{feet}}{\text{min}}$

3. Use the table below to

   a) estimate $f'(1870)$

   b) interpret the meaning of the value you found in part (a)

| $t$ (yr) | 1850 | 1860 | 1870 | 1880 |
|---|---|---|---|---|
| $f(t)$ (millions) | 23.1 | 31.4 | 38.6 | 50.2 |

$f'(1870) \approx \frac{38.6-31.4}{1870-1860} = 0.72$

or : $\frac{50.2-38.6}{1880-1870} = 1.16$

or : $\frac{50.2-31.4}{1880-1860} = 0.94$

f(t) is growing at a rate of 0.72 millions/yr in 1870.

# PTF #AB 08 – Instantaneous Rate of Change

The instantaneous rate of change, or the derivative, of $f(x)$ at a point can be written as any of the following:

1. $f'(a)=\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$. This finds the value of the slope of the tangent line at the specific point $x=a$.
2. *Analytically*, find the difference quotient
$$f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$
This finds the generic equation for the slope of the tangent line at any given point on the curve.
3. *Graphically*, it is the slope of the tangent line to the curve through the point $(a, f(a))$.

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1. Set up the limit definition of the derivative at $x=2$ for the function $f(x)=-x^2+2x$?

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{-(2+h)^2+2(2+h)-0}{h}$$

or

$$\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{-x^2+2x-0}{x-2}$$

2. Fill in the blanks:

The $\lim\limits_{h\to 0}\frac{6(x+h)^2-2(x+h)+7-(6x^2-2x+7)}{h}$

finds the derivative at x of

the function f(x)= $6x^2-2x+7$.

3. If $f$ is a differentiable function, then $f'(a)$ is given by which of the following?

I. $\lim\limits_{x\to h}\frac{f(a+h)-f(a)}{h}$

II. $\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}$

III. $\lim\limits_{x\to a}\frac{f(x+h)-f(x)}{h}$

(a) I only
[✓ (b)] II only
(c) I and II only
(d) I and III only
(e) I, II and III

# PTF #AB 09 – Tangent Line

To find the equation of a tangent line to a function through a point, you need both a point and a slope:

1. You may have to find the $y-value$ of the <u>point</u> on the graph by plugging in the given $x-value$ into the *original equation.*
2. Find the derivative of $f$ and evaluate it at the given point to get the <u>slope</u> of the tangent line. (Most times you will plug in just the $x-value$, but sometimes you need to plug in both the $x-value$ and the $y-value$. The slope must be a number and must not contain any variable.)
3. Use the point and the slope to write the equation in point-slope form:

$$y - y_{value} = m\left(x - x_{value}\right)$$

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1. Let $f$ be the function defined by $f(x) = 4x^3 - 5x + 3$. Find the equation of the tangent line to the graph of $f$ at the point where $x = -1$.

$f'(x) = 12x^2 - 5$

$f'(-1) = 12 - 5 = 7$

$f(-1) = -4 + 5 + 3$

$= 4$

$y - 4 = 7(x + 1)$

2. If the line tangent to the graph of the function $f$ at the point (1,7) passes through (-2, -2), then $f'(1) = ?$

$$f'(1) = \frac{7-(-2)}{1-(-2)} = \frac{9}{3} = 3$$

3. Find the equation of the line tangent to the graph of $f(x) = x^4 + 2x^2$ at the point where $f'(x) = 1$. You will need to use your calculator for this problem.

$f'(x) = 4x^3 + 4x$

$4x^3 + 4x = 1$

$X = 0.237$

$y - 0.115 = 1(x - 0.237)$

# PTF #AB 10 – Horizontal Tangent Lines

To find the point(s) where a function has a *horizontal tangent line*:

1. Find $f'(x)$ and set it equal to zero. (Remember that a fraction is zero only if the numerator equals zero.)
2. Solve for $x$.
3. Substitute the value(s) for $x$ into the original function to find the $y-value$ of the point of tangency.
4. Not all $x-values$ will yield a $y-value$. If you cannot find a $y-value$, then that point gets thrown out.
5. Write the equation of your tangent line. Remember that since it is horizontal, it will have the equation $y = y_{value}$.

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1. Find the point(s), if any, where the function has horizontal tangent lines.

a) $f(x) = x^3 + 2x^2 - 15x + 14$

$f'(x) = 3x^2 + 4x - 15$

$3x^2 + 4x - 15 = 0$

$(3x - 5)(x + 3) = 0$

at $x = \frac{5}{3}$ $\quad x = -3$

b) $g(t) = \frac{2}{t^3}$

$g'(t) = -\frac{6}{t^4} = 0$

no solution

no horizontal tangent lines

2. Let $h$ be a function defined for all $x \neq 0$ and the derivative of $h$ is given by $h'(x) = \frac{x^2 - 2}{x}$ for all $x \neq 0$. Find all values of $x$ for which the graph of $h$ has a horizontal tangent.

$\frac{x^2 - 2}{x} = 0$

$x^2 - 2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

3. If a function $f$ has a derivative $f'(x) = \sqrt{3} - 2\sin x$ for $0 \le x \le 2\pi$, find the x-coordinates of the points where the function has horizontal tangent lines.

$\sqrt{3} - 2\sin x = 0$

$\sin x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{3}$ and $\frac{2\pi}{3}$

# PTF #AB 11 – Linear Approximation

**<u>Standard Linear Approximation</u>**: an approximate value of a function at a specified $x$-coordinate.

To find a linear approximation:

1. Write the equation of the tangent line at a "*nice*" $x$-value close to the one you want.
2. Plug in your $x$-value into the tangent line and solve for $y$.

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1. Find a linear approximation for $f(2.1)$ if $f(x) = \frac{6}{x^2}$?

$f'(x) = -\frac{12}{x^3}$

$f(2) = \frac{3}{2}$ $\quad f'(2) = -\frac{3}{2}$

$y = \frac{3}{2} - \frac{3}{2}(x-2)$

$y = \frac{3}{2} - \frac{3}{2}(0.1)$

$y = 1.5 - 0.15$

$y = 1.35$

2. Evaluate $\sqrt{39}$ without a calculator (use linear approximation).

$y = \sqrt{x+36}$ $\quad \sqrt{36} = 6$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x+36}}$ $\quad \frac{1}{2\sqrt{36}} = \frac{1}{12}$

$y = 6 + \frac{1}{12}(x-0)$

$y = 6 + \frac{1}{12}(3-0)$

$y = 6 + \frac{1}{4} = 6.25$

3. Find a linear approximation for $f(1.67)$ if $f(x) = \sin x$?

$f'(x) = \cos x$ $\quad f(0) = 0$

$f'(0) = 1$

$y - 0 = 1(x-0)$

$y = 1(1.67)$

$y = 1.67$

# PTF #AB 12 – Derivatives of Inverse Functions

1. Find $f'(x)$.
2. Make sure that you have figured out which value is the $x$ and $y$ values for each function ($f(x)$ and $f^{-1}(x)$)
3. Substitute the $x$-value for $f$ into $f'(x)$.
4. The solution is $\dfrac{1}{\text{the value you found in step \#3}}$

1. If $g$ is the inverse function of $F$ and $F(2)=3$, find the value of $g'(3)$ for $F(x)=\dfrac{x^3}{4}+x-1$.

$F'(x) = \frac{3x^2}{4}+1$

$F'(2) = 3+1=4$

$g'(3) = \frac{1}{F'(2)} = \boxed{\frac{1}{4}}$

2. Let $f$ be the function defined by $f(x)=x^5+2x-1$. If $g(x)=f^{-1}(x)$ and $(1,2)$ is on $f$, what is the value of $g'(2)$?

$f(1)=2 \rightarrow g(2)=1$

$f'(x)=5x^4+2$

$f'(1)=7$

$g'(2) = \frac{1}{f'(1)} = \boxed{\frac{1}{7}}$

# PTF #AB 13 – Differentiability Implies Continuity

Differentiability means that you can find the slope of the tangent line at that point or that the derivative exists at that point.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. (Remember what is means to be continuous at a point.)
2. It is possible for a function to be continuous at $x = c$ and <u>not differentiable</u> at $x = c$.

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1. Let $f$ be a function such that

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 5.$$

Which of the following must be true?

I. $f$ is continuous at $x = 2$?

II. $f$ is differentiable at $x = 2$?

III. The derivative of $f$ is continuous at $x = 2$?

(a) I only
(b) II only
[✓ (c) I and II only]
(d) I and III only
(e) II and III only

$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 5$ means: $f'(2) = 5$

Differentiable $\Rightarrow$ Continuous

2. Let $f$ be a function defined by

$$f(x) = \begin{cases} 2x - x^2 & x \le 1 \\ x^2 + kx + p & x > 1 \end{cases}$$

For what values of $k$ and $p$ will $f$ be continuous and differentiable at $x = 1$?

$\lim_{x \to 1^-} 2x - x^2 = 1$ $\qquad$ $\lim_{x \to 1^+} x^2 + kx + p = 1 + k + p$

$\underline{1 = 1 + k + p}$

$f'(x) = \begin{cases} 2 - x \\ 2x + k \end{cases}$

$2 - (1) = 2(1) + k$ $\qquad$ (for $f'(x)$ to exist at $x = 1$)

$1 = 2 + k$

$-1 = k$

$1 = 1 + (-1) + p$

$1 = p$

$\boxed{k = -1 \quad p = 1}$

# PTF #AB 14 – Conditions that Destroy Differentiability

Remember for a function to be differentiable, the slopes on the right hand side must be equal to the slopes on the left hand side. There are four conditions that destroy differentiability:

1. Discontinuities in the graph. (Function is not continuous.)
2. Corners in the graph. (Left and right-hand derivatives are not equal.)
3. Cusps in the graph. (The slopes approach ± ∞ on either side of the point.)
4. Vertical tangents in the graph. (The slopes approach ± ∞ on either side of the point.)

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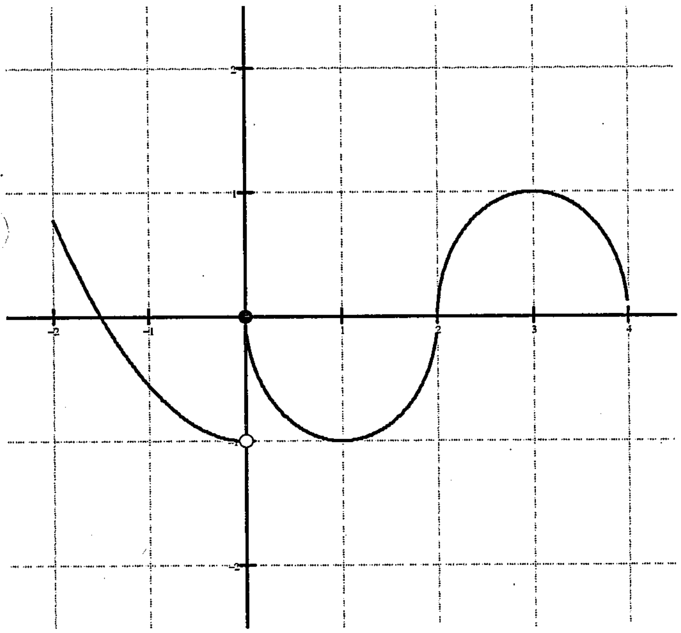
1. The graph shown below has a vertical tangent at (2,0) and horizontal tangents at (1,-1) and (3,1). For what values of $x$ in the interval $(-2,4)$ is $f$ not differentiable?



$x = 0$ not continuous

$x = 2$ vertical tangent

2. Let $f$ be a function defined by

$$f(x) = \begin{cases} x^2 + 1 & x < 0 \\ -x^2 + 4 & x \geq 0 \end{cases}$$

a) Show that $f$ is/is not continuous at $x = 0$.

$\lim_{x \to 0^-} x^2 + 1 = 1 \quad \lim_{x \to 0^+} -x^2 + 4 = 4$

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Since $\lim_{x \to 0} f(x)$ DNE, $f(x)$ is <u>not</u> continuous at $x=0$.

b) Prove that $f$ is/is not differentiable at $x = 0$.

Since $f(x)$ is not continuous at $x=0$, $f(x)$ is not differentiable at $x=0$.

# PTF #AB 15 – Implicit Differentiation

1. Differentiate both sides with respect to $x$.
2. Collect all $\frac{dy}{dx}$ terms on one side and the others on the other side.
3. Factor out the $\frac{dy}{dx}$.
4. Solve for $\frac{dy}{dx}$ by dividing by what's left in the parenthesis.

Errors to watch out for:

- Remember to use the product rule
- Remember to use parenthesis so that you distribute any negative signs
- Remember that the derivative of a constant is zero

---

1. Find $\frac{dy}{dx}$ for $y^4+x^3y^5-2x^7=13$.

$$4y^3\frac{dy}{dx}+3x^2y^5+x^3 5y^4\frac{dy}{dx}-14x^6=0$$

$$4y^3\frac{dy}{dx}+5x^3y^4\frac{dy}{dx}=14x^6-3x^2y^5$$

$$\frac{dy}{dx}\left(4y^3+5x^3y^4\right)=14x^6-3x^2y^5$$

$$\frac{dy}{dx}=\frac{14x^6-3x^2y^5}{4y^3+5x^3y^4}$$

2. Find the instantaneous rate of change at $(1,1)$ for $x+2xy-y^2=2$.

$$1+2y+2x\frac{dy}{dx}-2y\frac{dy}{dx}=0$$

$$2x\frac{dy}{dx}-2y\frac{dy}{dx}=-1-2y$$

$$\frac{dy}{dx}(2x-2y)=-1-2y$$

$$\frac{dy}{dx}=\frac{-1-2y}{2x-2y}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)}=\frac{-1-2(1)}{2(1)-2(1)}=\frac{-3}{0}=\text{undefined slope}$$

3. If $x^2+y^2=25$, what is the value of $\frac{d^2y}{dx^2}$ at the point $(4,3)$?

$$2x+2y\frac{dy}{dx}=0$$

$$x+y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-x}{y}$$

$$\frac{d^2y}{dx^2}=\frac{(-1)(y)-(-x)\left(\frac{dy}{dx}\right)}{y^2}$$

$$=\frac{-y+x\left(\frac{-x}{y}\right)}{y^2}$$

$$=\frac{-y-\frac{x^2}{y}}{y^2}\cdot\frac{y}{y}$$

$$=\frac{-y^2-x^2}{y^3}$$

$$=\frac{-(y^2+x^2)}{y^3}$$

$$=\frac{-25}{y^3}\rightarrow\frac{-25}{(3)^3}=\boxed{-\frac{25}{27}}$$

# PTF #AB 16 – Vertical Tangent Lines

To find the point(s) where a function has a *vertical tangent line*:

1. Find $f'(x)$ and set the denominator equal to zero. (Remember that the slope of a vertical line is undefined therefore must have a zero on the bottom.)
2. Solve for $x$.
3. Substitute the value(s) for $x$ into the original function to find the $y-value$ of the point of tangency.
4. Not all $x-values$ will yield a $y-value$. If you cannot find a $y-value$, then that point gets thrown out.
5. Write the equation of your tangent line. Remember that since it is vertical, it will have the equation $x = x_{value}$.

---

1. Find the point(s), if any, where the function has vertical tangent lines. Then write the equation for those tangent lines.

a) $g(x) = 3 - \sqrt[3]{x}$

$$g'(x) = -\frac{1}{3}x^{-2/3} = -\frac{1}{3\sqrt[3]{x^2}}$$

$\underline{g'(x) = \text{und.}}$ :

$$3\sqrt[3]{x^3} = 0$$

$$\boxed{x = 0}$$

b) $f(x) = \sqrt{4 - x^2}$

$$f'(x) = \frac{-2x}{2\sqrt{4-x^2}}$$

$$2\sqrt{4-x^2} = 0$$

$$\sqrt{4-x^2} = 0$$

$$4 - x^2 = 0$$

$$x = \pm 2$$

$$\boxed{x = 2 \quad x = -2}$$

2. Consider the function defined by $xy^2 - x^3y = 6$. Find the $x$-coordinate of each point on the curve where the tangent line is vertical.

$$xy^2 - x^3y = 6$$

$$y^2 + 2xy\frac{dy}{dx} - 3x^2y - x^3\frac{dy}{dx} = 0$$

$$2xy\frac{dy}{dx} - x^3\frac{dy}{dx} = 3x^2y - y^2$$

$$\frac{dy}{dx}\left(2xy - x^3\right) = 3x^2y - y^2$$

$$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$$

$$2xy - x^3 = 0$$

$$2xy = x^3$$

$$y = \frac{x^2}{2} \rightarrow x\left(\frac{x^2}{2}\right)^2 - x^3\left(\frac{x^2}{2}\right) = 6$$

$$\frac{1}{4}x^5 - \frac{1}{2}x^5 = 6$$

$$-\frac{1}{4}x^5 = 6$$

$$x^5 = -24$$

$$\boxed{x = \sqrt[5]{-24}}$$

# PTF #AB 17 – Strategies for Finding Limits/L'Hospital's Rule

Steps to evaluating limits:

1. Try direct substitution. (this will work unless you get an indeterminate answer: 0/0)
2. Try L'Hospital's Rule (take derivative of top and derivative of bottom and evaluate again.)
3. Try L'Hopital's Rule again (as many times as needed.)
4. Use factoring and canceling or rationalizing the numerator.

---

Find the following limits if they exist.

1. $\lim_{x\to 3}(2x^3 - x^2 + 5)$

$\lim_{x\to 3} (2x^3 - x^2 + 5) = 54 - 9 + 5 = \boxed{50}$

2. $\lim_{x\to \frac{\pi}{6}}(x\cos x)$

$\lim_{x\to \frac{\pi}{6}} x\cos x = \boxed{\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2}}$

3. $\lim_{x\to 1}\frac{x^2+5x-6}{x-1}$

$\lim_{x\to 1}\frac{x^2+5x-6}{x-1} = \frac{0}{0}$

$\hookrightarrow \lim_{x\to 1}\frac{(x+6)\cancel{(x-1)}}{\cancel{x-1}} = 1+6 = \boxed{7}$

(L'Hospital) 4. $\lim_{x\to a}\frac{x-a}{x^4-a^4}$, $(a \neq 0)$

$\lim_{x\to a}\frac{x-a}{x^4-a^4} = \frac{0}{0}$

$\hookrightarrow \lim_{x\to a}\frac{1}{4x^3} = \boxed{\frac{1}{4a^3}}$

(L'Hospital) 5. $\lim_{x\to 0}\frac{g(x)-g(0)}{x^3}$, $g(x) = 3x^3 - 5$

$\lim_{x\to 0}\frac{g(x)-g(0)}{x^3} =$

6. $\lim_{x\to 0}\frac{x+\tan x}{\sin x}$

$\lim_{x\to 0}\frac{x+\tan x}{\sin x} = \frac{0+0}{0} = \frac{0}{0}$

$\lim_{x\to 0}\frac{1+\sec^2 x}{\cos x} = \lim_{x\to 0}\frac{1+\left(\frac{1}{\cos x}\right)^2}{\cos x} = \frac{1+1}{1} = \boxed{2}$

7. $\lim_{x\to 0}\frac{\sin(5x)}{2x}$

$\lim_{x\to 0}\frac{\sin(5x)}{2x} = \frac{0}{0}$

$\lim_{x\to 0}\frac{5\cos(5x)}{2} = \boxed{\frac{5}{2}}$

8. $\lim_{\theta\to 0}\frac{1-\cos\theta}{2\sin^2\theta}$

$\lim_{\theta\to 0}\frac{1-\cos\theta}{2\sin^2\theta} = \frac{0}{0}$

$\lim_{\theta\to 0}\frac{\cancel{\sin\theta}}{4\cancel{\sin\theta}\cos\theta} = \frac{1}{4(1)} = \boxed{\frac{1}{4}}$

OR: $\lim_{\theta\to 0}\frac{1-\cos\theta}{2\sin^2\theta} = \lim_{\theta\to 0}\frac{1-\cos\theta}{2(1-\cos^2\theta)} = \lim_{\theta\to 0}\frac{1}{2(1+\cos\theta)} = \frac{1}{4}$

9. $\lim_{\theta\to 0}\theta\sec\theta$

$\lim_{\theta\to 0}\theta\sec\theta = \lim_{\theta\to 0}\frac{\theta}{\cos\theta} = \frac{0}{1} = \boxed{0}$

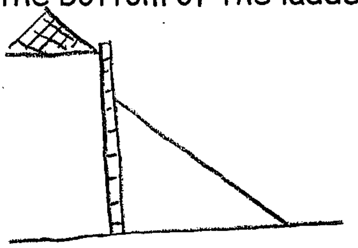
# PTF #AB18 – Related Rates

Set up the related rate problem by:

1. Drawing a diagram and label.
2. Read the problem and write "Find = ", "Where = ", and "Given = " with the appropriate information.
3. Write the Relating Equation and if needed, substitute another expression to get down to one variable.
4. Find the derivative of both sides of the equation with respect to $t$.
5. Substitute the "Given" and "When" and then solve for "Find".

---

1. The top of a 25-foot ladder is sliding down a vertical wall at a constant rate of 3 feet per minute. When the top of the ladder is 7 feet from the ground, what is the rate of change of the distance between the bottom of the ladder and the wall?



$\frac{dy}{dt} = -3$

$y = 7$

$x^2 + y^2 = 625$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$x\frac{dx}{dt} = -y\frac{dy}{dt}$

$\frac{dx}{dt} = \frac{-y\frac{dy}{dt}}{x}$

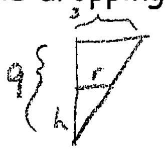
$x^2 + 7^2 = 625$

$x^2 = 576$

$x = 24$

$\frac{dx}{dt} = \frac{-7(-3)}{24} = \frac{21}{24} \approx \boxed{0.875 \frac{ft}{min}}$

2. An inverted cone has a height of 9 cm and a diameter of 6 cm. It is leaking water at the rate of 1 cm$^3$/min. Find the rate at which the water level is dropping when $h = 3$ cm. $\left(V = \frac{1}{3}\pi r^2 h\right)$



$\frac{h}{r} = \frac{9}{3}$

$h = 3r$

$\frac{1}{3}h = r$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$

$V = \frac{\pi}{27}h^3$

$\frac{dV}{dt} = \frac{3\pi}{27}h^2\frac{dh}{dt}$

$1 = \frac{3\pi}{27}(3)^2\frac{dh}{dt}$

$1 = \pi\frac{dh}{dt}$

$\boxed{\frac{1}{\pi} = \frac{dh}{dt}}$

# PTF #AB 19 – Position, Speed, Velocity, Acceleration

1. **Position Function**: the function that gives the position (relative to the origin) of an object as a function of time.
2. **Velocity (Instantaneous):** tells how fast something is going at that exact instant and in which direction (how fast position is changing.)
3. **Speed**: tells how fast an object is going (not the direction.)
4. **Acceleration**: tells how quickly the object picks up or loses speed (how fast the velocity is changing.)

Position Function: $s(t)$ or $x(t)$

Velocity Function: $v(t) = s'(t)$

Speed Function: $speed = |v(t)|$

Acceleration Function: $a(t) = v'(t) = s''(t)$

1. A particle moves along the $x$-axis so that at time $t$ (in seconds) its position is $x(t) = t^3 - 6t^2 + 9t + 11$ feet.

a) What is the velocity of the particle at $t = 0$? The acceleration at $t = 0$?

$V(t) = 3t^2 - 12t + 9$

$\boxed{v'(0) = 9}$

$a(t) = 6t - 12$

$\boxed{a(0) = -12}$

b) During what time intervals is the particle moving to the left? To the right?

$v(t) = 0$

$3(t^2 - 4t + 3) = 0$

$3(t-3)(t-1) = 0$

$t = 3 \quad t = 1$

\+ − +

1 3

Left: $(1, 3)$

Right: $(-\infty, 1)$

$(3, \infty)$

c) During what time intervals is the acceleration positive? Negative?

$a(t) = 0$

$6t - 12 = 0$

$t = 2$

− +

2

positive: $(2, \infty)$

negative: $(-\infty, 2)$

d) What is the average velocity on the interval $[1,3]$?

$\frac{x(3) - x(1)}{3-1} = \frac{11 - 15}{3-1} = \frac{-4}{2} = -2 \frac{ft}{s}$

e) What is the average acceleration on the interval $[3,6]$?

$\frac{1}{6-3}\int_3^6 6t - 12 dt = 15 \frac{ft^2}{s}$

f) What is the total distance traveled by the particle from $t = 0$ to $t = 5$?

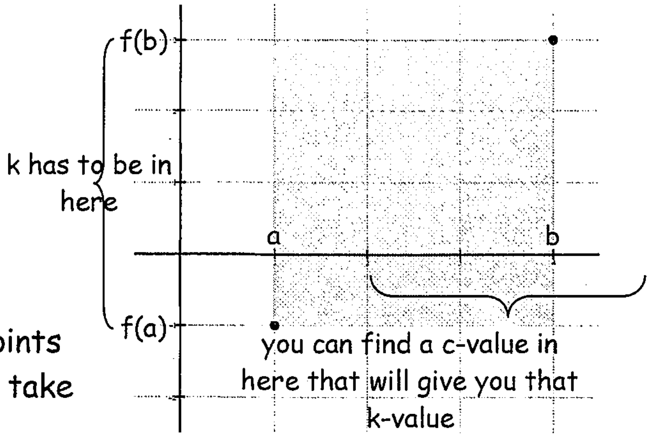
$\int_0^5 |3t^2 - 12t + 9| dt = 27.999 \approx 28$ feet

# PTF #AB 20 – Intermediate Value Theorem

If these three conditions are true for a function:

1. $f$ is continuous on the closed interval $[a,b]$
2. $f(a) \neq f(b)$
3. $k$ is any number between $f(a)$ and $f(b)$

Then there is at least one number $c$ in $[a,b]$ for which $f(c)=k$.



*As long as the function is continuous and the endpoints don't have the same y-value, then the function must take on every y-value between those of the endpoints.

---

1. Use the Intermediate Value Theorem to show that $f(x)=x^3+2x-1$ has a zero in the interval $[0,1]$.

$f(0)=-1$     $f(1)=2$

Since $f(x)$ is continuous and $f(0)=-1$ and $f(1)=2$, there exists a $c$, $f(c)=0$ by IVT.

2. Let $f(x)$ be a continuous function on the interval $-2 \leq x \leq 2$. Use the table of values below to determine which of the following statements must be true.

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | -4 | 1 | 6 | 3 | -5 |

I. $f(x)$ takes on the value of 5 ✓

II. A zero of $f(x)$ is between -2 and -1 ✓

III. A zero of $f(x)$ is 6

(A) I only
(B) II only
(C) III only
[✓ (D)] I and II only
(E) I, II, and III

# PTF #AB 21 – Mean Value Theorem & Rolle's Theorem

**Mean Value Theorem**:

<u>What you need</u>: a function that is continuous and differentiable on a closed interval

<u>What you get</u>: $f'(c) = \frac{f(b)-f(a)}{b-a}$ where $c$ is an $x$-value in the given interval

*Verbally* it says: The instantaneous rate of change = average rate of change

*Graphically* it says: The tangent line is parallel to the secant line

**Rolle's Theorem** (special case of Mean Value Theorem):

<u>What you need</u>: a function that is continuous and differentiable on a closed interval AND the $y$-values at the endpoints to be equal

<u>What you get</u>: $f'(c) = 0$ where $c$ is an $x$-value in the given interval

*Verbally* it says: The derivative equals zero somewhere in the interval

*Graphically* it says: There is a horizontal tangent line (max or min)

---

1. Let $f$ be the function given by $f(x) = x^3 - 7x + 6$. Find the number $c$ that satisfies the conclusion of the Mean Value Theorem for $f$ on $[1,3]$.

$$\frac{f(3)-f(1)}{3-1} = \frac{12-0}{3-1} = \frac{12}{2} = 6$$

$$f'(x) = 3x^2 - 7$$

$$3x^2 - 7 = 6$$

$$3x^2 = 13$$

$$x^2 = \frac{13}{2}$$

$$x = \sqrt{\frac{13}{2}}, -\sqrt{\frac{13}{2}}$$

$$x \approx 2.5495$$

2. Determine if Rolle's Theorem applies. If so, find $c$. If not, tell why.

$f(x) = x^4 - 2x^2$ for $[-2,2]$

$f(2) = 8 \quad f(-2) = 8$

Since $f(2) = f(-2)$, Rolle's Theorem applies.

3. Let $f$ be a function that is differentiable on the interval $(1,10)$. If $f(2) = -5$, $f(5) = 5$, and $f(9) = -5$, which of the following must be true? Choose all that apply.

[✓ I.] $f$ has at least 2 zeros.

[✓ II.] The graph of $f$ has at least one horizontal tangent line.

[✓ III.] For some $c$, $2 < c < 5$, then $f(c) = 3$

# PTF #AB 22 – Extrema on an Interval

**Extrema**: the extreme values, i.e. the absolute maximums and minimums

**Extreme Value Theorem:** As long as $f$ is continuous on a *closed* interval, then $f$ will have both an absolute maximum and an absolute minimum.

**Finding Extrema on a *closed* interval:**

1. Find the critical numbers of the function in the specified interval.
2. Evaluate the function to find the $y$-values at all critical numbers and at each endpoint.
3. The smallest $y$-value is the absolute minimum and the largest $y$-value is the absolute maximum.

---

1. Find the absolute extrema of each function for the given interval:

a. $f(x)=x^2+1$ on $[1,2]$

$f'(x)=2x$

$2x=0$

$x=0$

$f(1)=2$

$f(2)=5$

$\boxed{\begin{array}{l}\text{abs max} = 5\\ \text{abs min} = 2\end{array}}$

b. $f(x)=x-2\cos(x)$ on $[0,2\pi]$

$f'(x)=1+2\sin(x)$

$1+2\sin(x)=0$

$2\sin x=-1$

$\sin x=-1/2$

$x=\frac{7\pi}{6}, \frac{11\pi}{6}$

| x | f(x) |
|---|---|
| 0 | -2 |
| $\frac{7\pi}{6}$ | 5.397 |
| $\frac{11\pi}{6}$ | 4.028 |
| $2\pi$ | $2\pi-2=4.283$ |

$\boxed{\begin{array}{l}\text{abs max} = 5\ 397\\ \text{abs min} = -2\end{array}}$

c. $f(x)=x+e^{2x}-1$ on $[0,3]$

$f'(x)=1+2e^x$

$1+2e^x=0$

$2e^x=-1$

$e^x=-1/2 \rightarrow$ no sol.

$f(0)=0$

$f(3)=2+e^6$

$\boxed{\begin{array}{l}\text{abs max} = 2+e^6\\ \text{abs min} = 0\end{array}}$

# PTF #AB 23 – Finding Increasing/Decreasing Intervals

1. Find the critical numbers.
2. Set up test intervals on a number line.
3. Find the sign of $f'(x)$ *(the derivative)* for each interval.
4. If $f'(x)$ is <u>positive</u> then $f(x)$ (the *original function*) is <u>increasing</u>.
   If $f'(x)$ is <u>negative</u> then $f(x)$ (the *original function*) is <u>decreasing</u>.

---

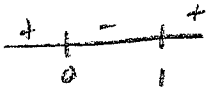
1. Find the intervals on which the function $f(x)=x^3-\frac{3}{2}x^2$ is increasing and decreasing. Justify.

$f'(x) = 3x^2 - 3x$

$3x^2-3x=0$

$3x(x-1)=0$

$x=0 \quad x=1$



Increasing: $(-\infty, 0) \cup (1, \infty)$

Decreasing: $(0, 1)$

2. Let $f$ be a function given by $f(x)=x^4+x^2-2$. On which intervals is $f$ increasing? Justify.

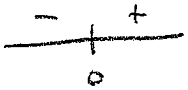
$f'(x) = 4x^3 + 2x$

$4x^3+2x=0$

$2x(2x^2+1)=0$

$x=0 \quad 2x^2+1=0$

$x^2=-\frac{1}{2} \rightarrow$ no sol.



Increasing for $x>0$

3. The derivative, $g'$, of a function is continuous and has two zeros. Selected values of $g'$ are given in the table below. If the domain of $g$ is the set of all real numbers, then $g$ is decreasing on which interval(s)? Increasing?

| $x$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $g'(x)$ | 2 | 3 | 0 | -3 | -2 | -1 | 0 | 3 | 2 |

Increasing for $(-4,-2) \cup (2,4)$

Decreasing for $(-2, 2)$

# PTF #AB 24 – Relative Maximums and Minimums

**First Derivative Test:**

1. If $f'(x)$ changes from + to -, then $x$ is a relative max.
2. If $f'(x)$ changes from - to +, then $x$ is a relative min.

**Second Derivative Test:**

1. If $f''(x)$ is neg (the function is ccd), then $x$ is a relative max.
2. If $f''(x)$ is pos (the function is ccu), then $x$ is a relative min.

* $x$ must be a critical number*

**To find the y-value or the max/min and to see if it is an absolute max/min:**

1. Take the $x$-values and plug them back in to the original equation.
2. Compare.

---

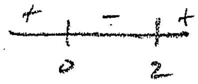
1. The function defined by $f(x)=x^3-3x^2$ for all real numbers has a relative maximum at $x$ = ? Justify.

$f'(x) = 3x^2 - 6x$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0 \quad x = 2$



Rel. max at $x=0$ since $f'(0)=0$, $f'(x)>0$ for $x<0$ and $f'(x)<0$ for $0<x<2$.

2. Find the relative maximum value for $f(x)=(x^2-3)e^x$. Justify.

$f'(x) = 2xe^x + (x^2-3)e^x$

$2xe^x + (x^2-3)e^x = 0$

$e^x(2x + x^2 - 3) = 0$

$2x + x^2 - 3 = 0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3 \quad x = 1$

Rel. max at $x=-3$ since $f'(-3)=0$, $f'(x)>0$ for $x<-3$ and $f'(x)<0$ for $-3<x<1$

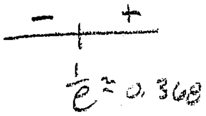
3. What is the minimum value of $f(x)=x\ln x$? Justify.

$f'(x) = \ln x + 1$

$\ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1}$



Since $f(x)$ is decreasing for $(0, \frac{1}{e})$ and increasing for $(\frac{1}{e}, \infty)$, $f(\frac{1}{e}) = -\frac{1}{e}$ is the minimum value of $f(x)$.

4. If $f$ has a critical number at $x=2$ and $f''(x)=3$, then what can you conclude about $f$ at $x=2$?

$f'(2) = 0$

$f''(x) > 0$

Then $f(x)$ has a rel. min at $x=2$

# PTF #AB 25 – Points of Inflection

**Points of Inflection:** Points on the original function where the concavity changes.

1. Find where $y''$ is *zero* or *undefined* - these are your possible points of inflection (PPOIs)
2. Must test intervals to find the actual POIs - they are only where the *second derivative* changes sign!

---

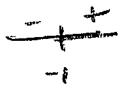
1. Write the equation of the line tangent to the curve $y = x^3 + 3x^2 + 2$ at its point of inflection.

$y' = 3x^2 + 6x$

$y'' = 6x + 6$

$6x + 6 = 0$

$x = -1$



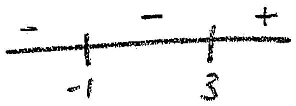
$y = (-1)^3 + 3(-1)^2 + 2 = -1 + 3 + 2 = 4$

$y'(-1) = 3 - 6 = -3$

$\boxed{y - 4 = -3(x+1)}$

2. Given $f''(x) = (x-3)(x+1)^2$, find the points of inflection of the graph of $y = f(x)$.

$f''(x) = 0$ at $x = 3$ $x = -1$



Point of Inflection at $\underline{x = 3}$

# PTF #AB 26 – Finding Concave Up/Concave Down Intervals

1. Find the PPOIs.
2. Set up test intervals on a number line.
3. Find the sign of $f''(x)$ *(the second derivative)* for each interval.
4. If $f''(x)$ is <u>positive</u> then $f(x)$ (the *original function*) is <u>concave up (ccu)</u>.
   If $f''(x)$ is <u>negative</u> then $f(x)$ (the *original function*) is <u>concave down (ccd)</u>.

---

1. Find the intervals on which the function $f(x) = 6\left(x^2+3\right)^{-1}$ is concave up or concave down. Justify.

$$f(x) = \frac{6}{x^2+3}$$

$$f'(x) = -\frac{6}{(x^2+3)^2}$$

$$f''(x) = \frac{12}{(x^2+3)^3} = 0$$

no solution

$$f''(1) = \frac{12}{(4)^3} > 0$$

f(x) is concave up for <u>all x</u>

2. Let $f$ be a function given by $f(x) = 3x^4 - 16x^3 + 24x^2 + 48$. On which intervals is $f$ concave down? Justify.

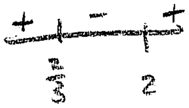
$$f'(x) = 12x^3 - 48x^2 + 48x$$

$$f''(x) = 36x^2 - 96x + 48$$

$$12(3x^2 - 8x + 4) = 0$$

$$12(3x-2)(x-2) = 0$$

$$x = \tfrac{2}{3} \qquad x = 2$$



Concave down for $(\tfrac{2}{3}, 2)$. since $f''(x) < 0$ for $\tfrac{2}{3} < x < 2$.

# PTF #AB 27 – "U-Substitution" Rule

1. Let $u =$ inner function.
2. Find $du$, then solve for $dx$.
3. Substitute $u$ & $du$ into the integrand (it should know fit one of the integration rules).
4. Integrate.
5. Substitute the inner function back for $u$.

---

1. Integrate $\int \left[ 9\left(x^2+3x+5\right)^8 (2x+3) \right] dx$

$u = x^2+3x+5$

$du = (2x+3)\,dx$

$\frac{1}{2x+3}\,du = dx$

$$\int 9u^8 \cancel{(2x+3)} \frac{1}{\cancel{2x+3}}\,du = u^9 + C$$

$$= \boxed{(x^2+3x+5)^9 + C}$$

2. Integrate $\int \left(\sin^2 3x \cos 3x\right) dx$

$u = \sin 3x$

$du = 3\cos 3x\,dx$

$\frac{1}{3\cos 3x}\,du = dx$

$$\int u^2 \cancel{\cos 3x} \cdot \frac{1}{3\,\cancel{\cos 3x}}\,du = \frac{u^3}{9} + C$$

$$= \boxed{\frac{\sin^3(3x)}{9} + C}$$

3. Integrate $\int e^{3x+1} dx$

$$\int e^{3x+1}\,dx = \boxed{\frac{1}{3} e^{3x+1} + C}$$

$$* \int e^{kx}\,dx = \frac{1}{k} e^{kx} + C$$

4. Integrate $\int \frac{e^{\tan x}}{\cos^2 x} dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$\cos^2 x\,du = dx$

$$\int \frac{e^u}{\cancel{\cos^2 x}} \cdot \cancel{\cos^2 x}\,du = \int e^u\,du = e^u + C$$

$$= \boxed{e^{\tan x} + C}$$

5. Integrate $\int \frac{e^x}{1+e^x} dx$

$u = 1+e^x$

$du = e^x\,dx$

$\frac{1}{e^x}\,du = dx$

$$\int \frac{\cancel{e^x}}{u} \cdot \frac{1}{\cancel{e^x}}\,du = \ln|u| + C$$

$$= \boxed{\ln|1+e^x| + C}$$

6. Using the substitution $u = 2x+1$, $\int_0^2 \left(\sqrt{2x+1}\right) dx$ is equal to

(A) $\frac{1}{2}\int_{-1/2}^{1/2} \sqrt{u}\,du$

(B) $\frac{1}{2}\int_0^2 \sqrt{u}\,du$

[✓ (C)] $\frac{1}{2}\int_1^5 \sqrt{u}\,du$

(D) $\int_0^2 \sqrt{u}\,du$

(E) $\int_1^5 \sqrt{u}\,du$

$u = 2x+1$

$du = 2\,dx$

$\frac{1}{2}\,du = dx$

$u(2) = 5$

$u(0) = 1$

# PTF #AB 28 – Approximating Area

Finding a Left or Right Riemann Sum or Trapezoidal Sum:

1. Divide the interval into the appropriate subintervals.
2. Find the y-value of the function at each subinterval.
3. Use the formula for a rectangle ($bh$) or trapezoid ($\frac{1}{2}b(h_1+h_2)$) to find the area of each individual piece.
4. You must show work to earn credit on these!
5. Always justify a left or right Riemann sum as an over or under approximation using the fact that the function is increasing or decreasing.

| | Left Sum | Right Sum |
|---|---|---|
| Increasing curve | Under approx. | Over approx. |
| Decreasing curve | Over approx. | Under approx. |

1. Use a left Riemann Sum with 4 equal subdivisions to approximate $\int_0^4 x^2 dx$.

$$\int_0^4 x^2 dx = (1)(0) + (1)(1) + (1)(4) + (1)(9)$$

$$= 0 + 1 + 4 + 9$$

$$= \boxed{14}$$

a. Is this approximation an over or underestimate? Justify.

under estimate since $f(x) = x^2$ is increasing for $0 < x < 4$

2. Values of a continuous function $f(x)$ are given below. Use a trapezoidal sum with four subintervals of equal length to approximate $\int_1^{2.2} f(x)$

| $x$ | 1 | 1.3 | 1.6 | 1.9 | 2.2 |
|---|---|---|---|---|---|
| $f(x)$ | 6.0 | 5.1 | 4.3 | 2.0 | 0.3 |

$$\int_1^{2.2} f(x)\,dx \approx \frac{1}{2}(0.3)(11.1) + \frac{1}{2}(0.3)(9.4) + \frac{1}{2}(0.3)(6.3) + \frac{1}{2}(0.3)(2.03)$$

$$= 1.665 + 1.41 + 0.945 + 0.3045$$

$$= \boxed{4.3245}$$

# PTF #AB 29 – Fundamental Theorem of Calculus

If $f$ is a continuous function on $[a,b]$ and $F$ is an antiderivative of $f$ on $[a,b]$, then

$$\int_a^b (f(x))\,dx = F(b)-F(a)$$

Graphically this means the **signed area** bounded by $x=a$, $x=b$, $y=f(x)$, and the $x$-axis.

---

1. Evaluate: $\int_1^2 \left(4x^3+6x\right)dx$

$\int_1^2 4x^3+6x\,dx = x^4+3x^2\Big|_1^2$

$= (16+12)-(1+3)$

$= \boxed{24}$

2. Evaluate: $\int_0^{\pi/4} (\sin x)\,dx$

$\int_0^{\pi/4} \sin x\,dx = -\cos x\Big|_0^{\pi/4}$

$= -\frac{\sqrt{2}}{2}+1$

$= \boxed{\frac{-\sqrt{2}-2}{2}}$

3. Evaluate: $\int_0^1 e^{-4x}\,dx$

$\int_0^1 e^{-4x}\,dx = -\frac{1}{4}e^{-4x}\Big|_0^1$

$= -\frac{1}{4}e^{-4}+\frac{1}{4}e^{0}$

$= \boxed{-\frac{1}{4}e^{-4}+\frac{1}{4}}$

4. Evaluate: $\int_{\ln 2}^3 \left(5e^x\right)dx$

$\int_{\ln 2}^3 5e^x\,dx = 5e^x\Big|_{\ln 2}^3$

$= 5e^3 - 5e^{\ln 2}$

$= 5e^3 - 5(2)$

$= \boxed{5e^3-10}$

5. Evaluate: $\int_1^2 \left(\frac{x-4}{x^2}\right)dx$

$\int_1^2 \frac{x-4}{x^2}\,dx = \int_1^2 \frac{1}{x}-\frac{4}{x^2}\,dx$

$= \ln|x| + \frac{4}{x}\Big|_1^2$

$= (\ln 2 + 2) - (\ln 1 + 4)$

$= \boxed{\ln 2 - 2}$

6. What are all the values of $k$ for which $\int_{-3}^k x^2\,dx = 0$?

$\int_{-3}^k x^2\,dx = \frac{1}{3}x^3\Big|_{-3}^k = \frac{1}{3}k^3 + \frac{1}{3}(27)$

$\frac{1}{3}k^3 + 9 = 0$

$\frac{1}{3}k^3 = -9$

$k^3 = -27$

$\boxed{k=-3}$

# PTF #AB 30 – Properties of Definite Integrals

1. If $f$ is defined at $x=a$, then $\int_a^a (f(x))\,dx=0$
2. If $f$ is integrable on $[a,b]$, then $\int_b^a (f(x))\,dx=-\int_a^b (f(x))\,dx$
3. If $f$ is integrable, then $\int_a^b (f(x))\,dx=\int_a^c (f(x))\,dx+\int_c^b (f(x))\,dx$

1. If $\int_1^{10} (f(x))\,dx=4$ and $\int_{10}^{3} (f(x))\,dx=7$, then $\int_1^3 (f(x))\,dx=?$

$$\int_1^3 f(x)dx + \int_3^{10} f(x)\,dx = \int_1^{10} f(x)\,dx$$

$$\int_1^3 F(x)\,dx + (-7) = 4$$

$$\int_1^3 F(x)\,dx = \boxed{11}$$

2. Which, if any, of the following are *false*?

I. $\int_a^b (f(x)+g(x))\,dx=\int_a^b (f(x))\,dx+\int_a^b (g(x))\,dx$

[✓ II.] $\int_a^b (f(x)g(x))\,dx=\left(\int_a^b (f(x))\,dx\right)\left(\int_a^b (g(x))\,dx\right)$

III. $\int_a^b (cf(x))\,dx=c\int_a^b (f(x))\,dx$

# PTF #AB 31 – Average Value of a Function

If $f$ is integrable on $[a,b]$, then the average value from the interval is

$$\frac{1}{b-a}\int_a^b (f(x))\,dx$$

To find where this height occurs in the interval:

1. Set $f(x)=$ answer (average value).
2. Solve for $x$.
3. Check to see if the $x-$value in the given interval.

---

1. Find the average value of $f(x)=\sin x$ over $[0,\pi]$.

$$\frac{1}{\pi-0}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi}$$

$$= -\frac{1}{\pi}(-1) + \frac{1}{\pi}(1)$$

$$= \frac{1}{\pi} + \frac{1}{\pi}$$

$$= \boxed{\frac{2}{\pi}}$$

2. Find the average value of $y=x^2\sqrt{x^3+1}$ on the interval $[0,2]$, then find where this value occurs in the interval.

$$\frac{1}{2-0}\int_0^2 x^2\sqrt{x^3+1}\,dx = \frac{1}{2}\int_0^2 x^2\sqrt{x^3+1}\,dx$$

$u = x^3+1$ $\quad u(2)=9$

$du = 3x^2\,dx$ $\quad u(0)=1$

$\frac{1}{3x^2}du = dx$

$$\frac{1}{2}\int_1^9 \cancel{x^2}\sqrt{u}\cdot\frac{1}{3\cancel{x^2}}du = \frac{1}{2}\int_0^9 \frac{1}{3}\sqrt{u}\,du =$$

$$\frac{1}{6}\int_1^9 \sqrt{u}\,du = \frac{1}{6}\cdot\frac{2}{3}u^{3/2}\Big|_1^9 = \frac{1}{9}(27) - \frac{1}{9}(1)$$

$$= 3 - \frac{1}{9}$$

$$= \underline{\frac{26}{9}}$$

$$\frac{26}{9} = x^2\sqrt{x^3+1}$$

$\underline{x = 1.281}$ (calculator)

3. The function

$$f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$$

is used to model the velocity of a plane in miles per minute. According to this model, what is the average velocity of the plane for $0 \le t \le 40$? (*calculator*)

$$\bar{V} = \frac{1}{40-0}\int_0^{40} 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$$

$\bar{V} = 5.916\ \frac{\text{miles}}{\text{minute}}$

# PTF #AB 32 – 2nd Fundamental Theorem of Calculus

To find the derivative of an integral:

$$\frac{d}{dx}\left[\int_a^x (f(t))\,dt\right] = f(x)\bullet dx$$

*Remember that $a$ must be a constant. If it is not, then you must use your properties of integrals to make it a constant.

---

1. For $F(x)=\int_2^x \sqrt{1+t^2}\,dt$, find

(a) $F(2)$ $=\int_2^2 \sqrt{1+t^2}\,dt = \boxed{0}$

(b) $F'(3)$

$F'(x) = \sqrt{1+x^2}$

$F'(3) = \sqrt{1+9} = \boxed{\sqrt{10}}$

2. Evaluate: $\frac{d}{dx}\int_{2x}^{3}(e^t+3)\,dt$

$\frac{d}{dx}\int_{2x}^{3} e^t + 3\,dt = \frac{d}{dx}\int_3^{2x} -(e^t+3)\,dt$

$= -(e^{2x}+3)\cdot\frac{d}{dx}(2x)$

$= \boxed{-2e^{2x}-6}$

3. Find $F'(x)$ if $F(x)=\int_x^{x^3}(\sec^2 t)\,dt$

$\int_x^{x^3} \sec^2 t\,dt = \int_x^0 \sec^2 t\,dt + \int_0^{x^3} \sec^2 t\,dt$

$= -\int_0^x \sec^2 t\,dt + \int_0^{x^3} \sec^2 t\,dt$

$F'(x) = -\sec^2 x + \sec^2(x^3)\cdot 3x^2$

$F'(x) = \boxed{-\sec^2 x + 3x^2 \sec^2(x^3)}$

4. Given $f(x)=\int_0^{3x}(4-2t)\,dt$ and $g(x)=f(e^x)$, find $*\ g(x) = \int_0^{3e^x} (4-2t)\,dt$

(a) $f'(-1)$

$f'(x) = (4-2(3x))\cdot 3$

$f'(-1) = (4-2(-3))\cdot 3 = \boxed{30}$

(b) $g(x)$ in terms of an integral

$g(x) = \int_0^{3e^x} (4-2t)\,dt$

(c) $g'(x)$

$g'(x) = (4-2(3e^x))\cdot 3e^x$

$= \boxed{12e^x - 18e^{2x}}$

(d) $g'(0)$

$g'(0) = 12e^0 - 18e^0 = \boxed{-6}$

(e) Write the equation for the tangent line to $g(x)$ at $x=0$

$g(0) = \int_0^3 4-2t\,dt = 4t - t^2\Big|_0^3 = 12-9 = 3$

$\boxed{y-3 = -6(x-0)}$

# PTF #AB 33 – Extensions of FTC

1. FTC as Accumulation ("Integrate removes the rate!"):
    a. Change in Population: $\int_a^b (P'(t))\,dt = P(b) - P(a)$ (gives total population *added* between time a and b)
    b. Change in Amount: $\int_a^b (R'(t))\,dt = R(b) - R(a)$ (gives total amount *added* of water, sand, traffic, etc. between time a and b)
2. FTC as Final Position ("Integrate to find the end!"):
    - Particle Position: $S(b) = S(a) + \int_a^b (v(t))\,dt$ (gives particle position at a certain time, b)
    - Total Amount: $R(b) = R(a) + \int_a^b (R'(t))\,dt$ (gives total amount of water, sand, traffic, etc. at a given time, b)

---

1. A particle moves along the $y$-axis so that $v(t) = t\sin(t^2)$ for $t \geq 0$. Given that $s(t)$ is the position of the particle and that $s(0) = 3$, find $s(2)$.

$\int_0^2 v(t)\,dt = s(2) - s(0)$

$s(2) = s(0) + \int_0^2 v(t)\,dt$

$s(2) = 3 + \int_0^2 t\sin(t^2)\,dt$

$u = t^2$ $\quad u(2) = 4$

$du = 2t\,dt$ $\quad u(0) = 0$

$\frac{1}{2t}\,du = dt$

$3 + \int_0^4 \cancel{t}\sin u \cdot \frac{1}{2\cancel{t}}\,du$

$3 + -\frac{1}{2}\cos u \Big|_0^4$

$3 + \left(-\frac{1}{2}\cos 4 + \frac{1}{2}\cos 0\right)$

$3 - \frac{1}{2}\cos 4 + \frac{1}{2} = \boxed{3.827}$

2. A metal A metal of length 8 cm is heated at one end. The function $T'(x) = 2x + 3$ gives the temperature, in °C, of the wire $x$ cm from the heated end. Find $\int_0^8 (T'(x))\,dx$ and indicate units of measure. Explain the meaning of the temperature of the wire.

$\int_0^8 2x+3\,dx = x^2 + 3x \Big|_0^8 = 64 + 24 = \underline{88}$

8 cm from the heated end of the rod is 88°C

# PTF #AB 34 – Accumulating Rates

- Identify the rate going in and the rate going out.
- To find a max or min point, set the two rates equal to each other and solve.
- To find the total amount

$$\text{Total} = \text{Initial Amt} + \int_a^b \textit{Rate Added} - \int_a^b \textit{Rate Removed}$$

- Remember to think of different blocks of time for piece-wise functions. Try to visualize what is happening in the situation before you try to put the math to work.

---

A factory produces bicycles at a rate of $p(w) = 95 + 0.1w^2 - w$ bikes per week for $0 \le w \le 25$. They can ship bicycles out at a rate of $s(w) = \begin{cases} 90 & 0 \le w < 3 \\ 95 & 3 \le w \le 25 \end{cases}$ bikes/week.

1. How many bicycles are produced in the first 2 weeks?

$\int_0^2 95 + 0.1w^2 - w \, dw = 188.267$

188 bicycles

2. How many bicycles are in the warehouse at the end of week 3?

$\int_0^3 95 + 0.1w^2 - w \, dw - \int_0^3 90 \, dw = 11.4$

11 bicycles

3. Find when the number of bicycles in the warehouse is at a minimum.

$b(t) = \int_0^t 95 + 0.1w^2 - w \, dw - \int_0^t 90 \, dw$

$b'(t) = 5 + 0.1t^2 - t = 0$

$\underline{\text{at } t = 0}$

4. The factory needs to stop production if the number of bicycles stored in the warehouse reaches 20 or more. Does the factory need to stop production at any time during the first 25 weeks? If so, when?

$b(t) = \int_0^t 5 + 0.1w^2 - w \, dw$

$b(t) = 5t + \frac{0.1t^3}{3} - \frac{t^2}{2}$

$t = 6.304$ weeks

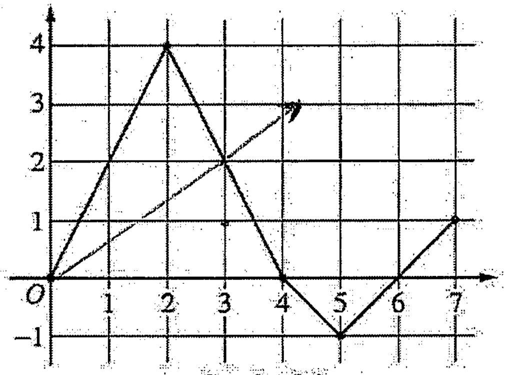
# PTF #AB 35 – Functions Defined by Integrals

$$F(x)=\int_a^x (f(t))\,dt$$

- $F'(x)=f(t)$ (The function in the integrand is the derivative equation!)
- These problems work just like curve sketching problems - you are looking at a derivative graph so answer accordingly.
- To evaluate $F(b)$, find the area under the curve from where it tells you to start (a) to the number given (b).

---

Let $f$ be a function defined on the closed interval $[0,7]$. The graph of $f$, consisting of four line segments, is shown below. Let $g$ be the function given by $g(x)=\int_2^x f(t)\,dt$.



Graph of $f$

1. Find $g(3)$, $g'(3)$, and $g''(3)$.

$g(3)=\int_2^3 f(t)dt=3$

$g'(3)=f(3)=2$

$g''(3)=f'(3)=-2$

2. Find the average rate of change of $g$ on the interval $0\le x\le 3$.

$\frac{g(3)-g(0)}{3-0} = \frac{3-\int_2^0 f(t)dt}{3} = \frac{3-(-4)}{3}$

$=\frac{7}{3}$

3. Find the $x$-coordinate of each point of inflection of the graph of $g$ on the interval $0<x<7$. Justify your answer.

$g''(x)=f'(x)$

at $t=2$ and $t=5$

Since $g''(x)-f'(x)$ and the sign of $f'(x)$ changes at $t=2$ and $t=5$

4. Let $h(x)=\int_2^x f(t)dt-\frac{1}{3}x^2$. Find all critical values for $h(x)$ and classify them as a minimum, maximum or neither.

$h(x)=\int_2^x f(t)dt \quad -\frac{1}{3}x^2$

$h'(x)=f(x)-\frac{2}{3}x=0$

$f(x)=\frac{2}{3}x$ (look for intersections)

at $t=3$ there is a max since $f(x)-\frac{2}{3}x$ changes from positive to negative at $t=3$

# PTF #AB 36 – Solving Differential Equations

1. Separate the variables (usually worth 1 point on a free response question).
2. Integrate both sides, putting "C" on the side with the dependent variable (found on the bottom of the differential). (If there is no "C", you lose all points for this part on a free response question.)
3. If there is an initial condition, get to a point where it is easy to substitute in the initial condition and then solve for "C".
4. Use the "C" you found and then continue to solve for $f(x)$ (if needed.)

---

1. Find a solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$ satisfying $f(0) = \frac{1}{2}$.

$\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$

$e^{2y}\,dy = 3x^2\,dx$

$\int e^{2y}\,dy = \int 3x^2\,dx$

$\frac{1}{2}e^{2y} = x^3 + C$

$\frac{1}{2}e^1 = C$

$\frac{1}{2}e^{2y} = x^3 + \frac{1}{2}e$

$e^{2y} = 2x^3 + e$

$2y = \ln(2x^3 + e)$

$\boxed{y = \frac{1}{2}\ln(2x^3 + e)}$

2. If $\frac{dy}{dx} = 2y^2$ and if $y = -1$ when $x = 1$, then when $x = 2$, $y = ?$

$\frac{dy}{dx} = 2y^2$

$\frac{1}{y^2}\,dy = 2\,dx$

$-\frac{1}{y} = 2x + C$

$1 = 2 + C$

$-1 = C$

$-\frac{1}{y} = 2x - 1$

$-\frac{1}{y} = 4 - 1$

$-\frac{1}{y} = 3$

$\boxed{y = -\frac{1}{3}}$

3. Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with initial condition $f(3) = \frac{1}{4}$.

$\frac{dy}{dx} = y^2(6 - 2x)$

$\frac{1}{y^2}\,dy = 6 - 2x\,dx$

$-\frac{1}{y} = 6x - x^2 + C$

$-4 = 18 - 9 + C$

$-4 = 9 + C$

$-13 = C$

$-\frac{1}{y} = 6x - x^2 - 13$

$\boxed{y = \frac{1}{-6x + x^2 + 13}}$

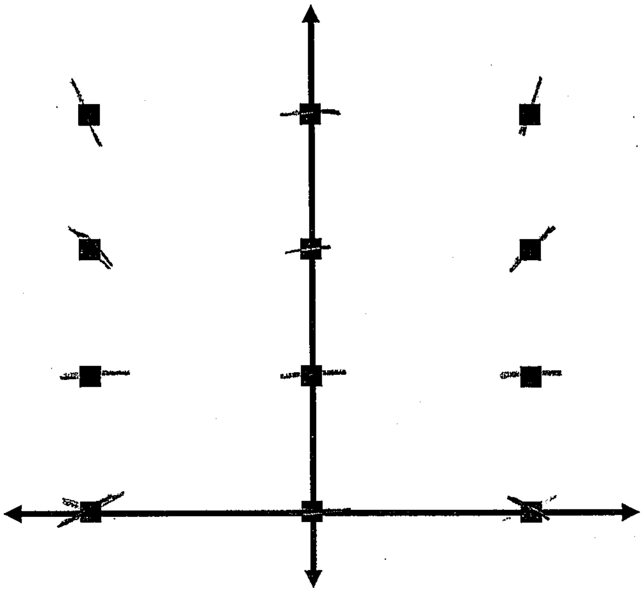
# PTF #AB 37 – Slope Fields

1. Substitute ordered pairs into the derivative to compute slope values at those points.
2. Construct short line segments on the dots to approximate the slope values.
3. For a particular solution, sketch in the curve using the initial condition and guided by the tangent lines.

---

1. Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$.

   a. On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.



   b. Describe all points in the $xy-$plane for which the slopes are positive.

$x^2(y-1) > 0$ when $x>0$ and $y>1$

and $x<0$ and $y<1$

   c. Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Draw in the solution.

$$\frac{dy}{dx} = x^2(y-1)$$

$$\frac{1}{y-1}\,dy = x^2\,dx$$

$$\int \frac{1}{y-1}\,dy = \int x^2\,dx$$

$$\ln|y-1| = \frac{x^3}{3} + C$$

$$\ln(3-1) = 0 + C$$

$$\ln 2 = C$$

$$\ln|y-1| = \frac{x^3}{3} + \ln 2$$

$$y-1 = e^{\frac{x^3}{3} + \ln 2}$$

$$y-1 = 2e^{\frac{x^3}{3}}$$

$$\boxed{y = 2e^{x^3/3} + 1}$$

# PTF #AB 38 – Exponential Growth & Decay

- **Direct Variation** is denoted by $y = kx$.
- **Inverse Variation** is denoted by $y = \frac{k}{x}$.

*$k$ is called the constant of variation and must be found in each problem by using the initial conditions.

- If $y$ is a differentiable function of $t$ such that $\frac{dy}{dt} = ky$, then $y = Ce^{kt}$.

---

1. If $\frac{dy}{dt} = ky$ and $k$ is a non-zero constant, then $y$ could be

(A) $2e^{kty}$ [✓ (B)] $2e^{kt}$ (C) $e^{kt} + 3$

(D) $kty + 5$ (E) $\frac{1}{2}ky^2 + \frac{1}{2}$

2. The number of bacteria in a culture is growing at a rate of $3000e^{2t/5}$ per unit of time $t$. At $t = 0$, the number of bacteria present was 7,500. find the number present at $t = 5$.

$\frac{dy}{dt} = 3000e^{2t/5}$ $y(0) = 7,500$

$y(5) = y(0) + \int_0^5 \frac{dy}{dt}\,dt$

$y(5) = 7500 + \int_0^5 3000e^{2t/5}\,dt$

$= 7500 + \frac{5}{2} 3000 e^{2t/5} \Big|_0^5$

$= 7500 + 7500 e^{2t/5} \Big|_0^5$

$= 7500 + 7500e^2 - 7500$

$= 7500e^2 = 55417.9207$

$= 55{,}417$ bacteria

# PTF #AB 39 – Particle Motion Summary

- Position Function: $s(t)$ or $x(t)$
- Velocity Function: $v(t)=s'(t)$
- Acceleration Function: $a(t)=v'(t)=s''(t)$
- Displacement: $\int_a^b v(t)\,dt$
- Total Distance: $\int_a^b |v(t)|\,dt$
- Position of the Particle at time $t=b$: $s(b)=s(a)+\int_a^b v(t)\,dt$

---

A particle moves along the $x$-axis with velocity at time $t\geq 0$ given by $v(t)=-1+e^{1-t}$. At time $t=0$, $s=2$.

1. Find $a(3)$, $v(3)$ and $s(3)$.

$a(t)=-e^{1-t}$

$a(3)=-e^{1-3}=-\frac{1}{e^2}=-0.135$

$v(3)=-1+e^{1-3}=-0.865$

$S(3)=S(0)+\int_0^3 -1+e^{1-t}dt$

$=2+\int_0^3 -1+e^{1-t}dt$

$=1.583$

2. Is the speed of the particle increasing at time $t=3$? Give a reason for your answer.

Since $v(3)<0$ and $a(3)<0$, speed is increasing

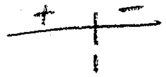
3. Find all values of $t$ for which the particle changes direction. Justify your answer.

$v(t)=0$

$-1+e^{1-t}=0$

$e^{1-t}=1$

$t=1$



Since $v(t)$ changes sign at $t=1$, the particle changes direction at $t=1$

4. Find the displacement and total distance of the particle over the time interval $0\leq t\leq 3$.

$\int_0^3 -1+e^{1-t}dt=-0.417$

$\int_0^3 |-1+e^{1-t}|dt=1.854$

Displacement $=-0.417$

Distance $=1.854$

# PTF #AB 40 – Area Between 2 Curves

If $f$ and $g$ are continuous on $[a,b]$ and $g(x) \le f(x)$ bounded by the vertical lines $x=a$ and $x=b$, then the area between the curves is found by

$$A = \int_a^b (f(x) - g(x))\,dx$$

To find the area of a region:

1. Sketch or draw the graphs.
2. Determine whether you need $dx$ or $dy$ (going vertically or horizontally)
3. Find the limits from the boundaries, axes or intersections.
4. Set up the integral by $Top - Bottom$ if $dx$ or $Right - Left$ if $dy$.
5. Integrate and evaluate the integral.

---

1. Find the area of the region in the first quadrant that is enclosed by the graphs of $y = x^3 + 8$ and $y = x + 8$.

$$\int_0^1 (x+8) - (x^3+8)\,dx = \underline{\frac{1}{4}}$$

2. The area of the region bounded by the lines $x = 0$, $x = 2$ and $y = 0$ and the curve $y = e^{x/2}$ is ____.

$$\int_0^2 e^{x/2}\,dx = 2e^{x/2}\Big|_0^2 = \underline{2e^1 - 2}$$

3. Find the area of R, the region in the first quadrant enclosed by the graphs of $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. (*calculator*)

$$\int_0^{1.136} (1 + \sin 2x) - e^{x/2}\,dx = \underline{0.429}$$

# PTF #AB 41 – Volumes of Slabs (Cross Sections)

**Volume** = $\int Area$

**Volume of Slabs (Cross Sections):**

- If the solid does NOT revolve around an axis, but instead has cross sections of a certain shape.
- $V = \int_a^b (A(x))\,dx$ (perpendicular to the $x-$axis) or $V = \int_c^d (A(y))\,dy$ (perpendicular to the $y-$axis)
- $A(x)$ represents the area of the cross section

Equilateral Triangle: $A = \frac{\sqrt{3}}{4}s^2$

Rectangle: $A = s(\text{height})$

Isos. Rt. Tri (on hyp.) : $A = \frac{1}{4}s^2$

Semicircle: $A = \frac{\pi}{8}s^2$

Square: $A = s^2$

Isos. Rt. Tri (on leg): $A = \frac{1}{2}s^2$

---

1. Let $R$ be the region in the first quadrant under $y = \frac{1}{\sqrt{x}}$ for $4 \le x \le 9$.

   Find the volume of the solid whose base is the region $R$ and whose cross sections cut by planes $\perp$ to the $x$-axis are squares.

$$V = \int_4^9 \left(\frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \frac{1}{x}\,dx = \ln|x| \Big|_4^9$$

$$= \ln 9 - \ln 4$$

$$= \underline{0.811}$$

2. Find the volume of the sold whose base is enclosed by $x^2 + y^2 = 1$ and whose cross sections taken perpendicular to the base are semicircles.

$x^2+y^2=1$

$y^2=1-x^2$

$y=\sqrt{1-x^2}$

$y=-\sqrt{1-x^2}$

$$V = 2\int_{-1}^{1} \frac{\pi}{4}\left(\sqrt{1-x^2}\right)^2 dx$$

$$= 2.094$$

3. Find the volume of a solid whose base is the circle $(x+1)^2 + y^2 = 9$ and whose cross-sections have area formula given by $A(x) = \sin(\pi x) - 2x$.

$$\int_{-4}^{2} \sin(\pi x) - 2x\,dx = 12$$

# PTF #AB 42 – Volumes of Rotations (Discs & Washers)

**Volume of Disks:**

- If the solid revolves around a horizontal/vertical axis and is flush up against the line of rotation.
- $V=\pi\int_a^b \left(r^2\right)dx$ (horizontal axis) or $V=\pi\int_c^d \left(r^2\right)dy$ (vertical axis)
- $r$ is the length of chord from curve to axis of rotation

**Volume of Washer:**

- If the solid revolves around a horizontal/vertical axis and is NOT flush up against the line of rotation.
- $V=\pi\int_a^b \left(R^2-r^2\right)dx$ (horizontal axis) or $V=\pi\int_c^d \left(R^2-r^2\right)dy$ (vertical axis)
- $R$ is the length of chord from farthest away curve to axis of rotation
- $r$ is the length of chord from closest in curve to axis of rotation

1. Find the volume of the solid generated by the graph bounded by $y=x^2$ and the line $y=4$ when it is revolved about the $x$-axis. (*calculator*)

$$\pi\int_{-2}^{2} (4)^2-(x^2)^2\,dx = 160.850$$

2. The region enclosed by the $x$-axis, the line $x=3$, and the curve $y=\sqrt{x}$ is rotated about the $x$-axis. What is the volume of the solid generated?

$$\pi\int_0^3 (\sqrt{x})^2\,dx = 14.137$$

3. Find the volume of the solid generated by revolving $x=\sqrt{1+y}$ with $y=3$ and $x=0$ about the $y$-axis.

$$\pi\int_{-1}^{3} \left(\sqrt{1+y}\right)^2 dy = 25.133$$