Example 1: A graphing calculator may be used. 2012 Notes

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

$$W'(12) \approx \frac{W(15)-W(9)}{15-9} = \frac{67.9-61.8}{6} = 1.01\overline{6}$$

the water temp is increasing at a rate of ≈ 1.017°F/min at $t=12$ minutes

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

$$\int_0^{20} W'(t)\,dt = W(20)-W(0) = 71.0-55.0 = 16 \ \frac{\text{deg}}{\text{min}}\cdot\text{min}$$

the water has warmed (or temp has increased) by 16°F over the interval from $t=0$ to $t=20$ minutes

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

$$\frac{1}{20}\int_0^{20} W(t)\,dt \approx \frac{1}{20}\left[4(55)+5(57.1)+6(61.8)+5(67.9)\right]$$

$$= \frac{1}{20}(1215.8) = 60.79 \quad \frac{1}{\text{min}}(\text{deg}\cdot\text{min})$$

Over the interval [0,20]

60.79°F is an underestimate because $W(t)$ is strictly increasing and we are using a left Riemann Sum

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

$$W(25) = 71.0 + \int_{20}^{25} W'(t)\,dt$$

$$= 73.043°\text{F}$$

Example 2: No calculator is allowed. 2013 Notes

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

$$C'(3.5) \approx \frac{C(4)-C(3)}{4-3} = 12.8 - 11.2 = 1.6 \text{ ounces/min} \quad +1 \quad +1$$

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

C is differentiable on $[0,6]$ which implies C is continuous on $[0,6]$.

$$\frac{C(4)-C(2)}{4-2} = \frac{12.8-8.8}{2} = 2. \; +1$$

Therefore, by MVT +1 there is at least one time t, $2<t<4$ for which $C'(t)=2$.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

$$\frac{1}{6}\int_0^6 C(t)\,dt \approx \frac{1}{6}\left[2(5.3) + 2(11.2) + 2(13.8)\right] \quad +1$$

$$= \frac{1}{6}(60.6) = 10.1 \quad \frac{1}{\cancel{\text{min}}}(\text{ounces} \cdot \cancel{\text{min}}) \quad +1$$

10.1 ounces is the average amount of coffee in the cup, over the time interval $0 \le t \le 6$ minutes. +1

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

$$+1 \quad B'(t) = -16e^{-0.4t} \cdot (-0.4) = 6.4e^{-0.4t}$$

$$B'(5) = 6.4e^{-0.4(5)} = \frac{6.4}{e^2} \text{ ounces/min} \quad +1$$

Example 3: No calculator is allowed. 2014 Notes

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

$$\text{Avg Acc} = \frac{V_A(8) - V_A(2)}{8-2} = \frac{-120-100}{6} = -\frac{110}{3}\ \text{m/min}^2 \quad +1$$

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

V_A is differentiable $\Rightarrow$ V_A is continuous

+1 $V_A(8) = -120$; $V_A(5) = 40$. therefore, by

+1 IVT, there must be a time t, $5 < t < 8$, such that $V_A(t) = -100$.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

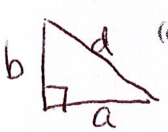
$$S_A(12) = S_A(2) + \int_2^{12} V_A(t)\,dt \quad +1$$

$$= 300 + \int_2^{12} V_A(t)\,dt$$

$$= 300 + \frac{1}{2}\left[3(140) + 3(-80) + 4(-270)\right] \quad +1$$

$$S_A(12) \approx 300 - 450 = -150 \quad +1$$

The position of Train A at $t = 12$ min is approx. 150 meters west of Origin Station



(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

Related Rate

$$A^2 + B^2 = d^2 \Rightarrow +1\ 2A\frac{dA}{dt} + 2B\cdot\frac{dB}{dt} = 2d\frac{dd}{dt} = A\cdot\frac{dA}{dt} + B\frac{dB}{dt} = d\cdot\frac{dd}{dt}$$

$A = 300$, $B = 400 \Rightarrow d = 500$

$V_B(2) = 5(2)^2 + 60(2) + 25 = 125$; $V_A(2) = 100$ +1

$$500\text{m}\cdot\frac{dd}{dt} = 300\text{m}\left(\frac{100\text{m}}{\text{min}}\right) + 400\text{m}\left(\frac{125\text{m}}{\text{m}}\right)$$

$$= \frac{8000\text{m}^2}{\text{m}} \cdot \frac{1}{500\text{m}} = 160\ \text{m/min} \quad +1$$

2015 #3 No Calculator

a) $v'(16) \approx \dfrac{240-200}{20-12} = 5 \text{ m/min}^2$ +1

b) $\displaystyle\int_0^{40} |v(t)|\,dt \approx 12\cdot|200| + 8\cdot|240| + 4\cdot|-220| + 16\cdot|150|$ +1

$= 2400 + 1920 + 880 + 2400$

$= 7600$ meters

+1

+1 7600 meters is the total distance Johanna jogs over $0 \le t \le 40$ minutes

c) $B(t)$ is Bob's velocity

$B'(t)$ is Bob's acceleration

$B'(t) = 3t^2 - 12t$ +1

$B'(5) = 3(5)^2 - 12(5) = 15 \text{ m/min}^2$ +1

d) Avg vel. $= \dfrac{1}{10}\displaystyle\int_0^{10} B(t)\,dt$ +1

$=$

+1

$=$

+1 $= 350$ m/min

(use calculator for now)

2016 #1 Calculator

a) $R'(2) \approx \dfrac{950-1190}{3-1} = -120$ liters/hr^2 +1 +1

b) $\int_0^8 R(t)\,dt \approx (1)(1340) + (2)(1190) + 3(950) + 2(740)$ +1

$= 8050$ liters +1

this is an overestimate since $R(t)$ is decreasing and we are using a left Riemann Sum. +1

c) Total $\approx 50{,}000 + \int_0^8 w(t)\,dt - 8050$ +1

Initial + Amount IN − Amount Out

Skip for now

$= 50{,}000 + 7836.1953 - 8050$

Total $\approx$ +1 49,786 liters

d) $W(t) = R(t)$ or $W(t) - R(t) = 0$ +1

Skip for now

At $t=0$, $w(0) - R(0)$ is positive

$t=8$, $w(8) - R(8)$ is negative

therefore, by IVT +1 there must be atleast one time, t, $(0,8)$ for which $W(t) - R(t) = 0$, or $W(t) = R(t)$

2017 #1 Calculator

a) $\text{Volume} = \int_0^{10} A(h)\,dh$

$$\approx 2(50.3) + 3(14.4) + 5(6.5) \quad +1$$

$$= 176.3 \text{ ft}^3 \quad +1$$

b) This approximation is an overestimate +1 because $A(h)$ is strictly decreasing +1 and we are using a left Riemann Sum.

skip c) Unit 8

c) $+1 \int_0^{10} f(h)\,dh = 101.325338$

The volume is 101.325 ft^3 +1

d) $V(h) = \int_0^h f(x)\,dx \Rightarrow V'(h) = f(h)$ +1

$$\left.\frac{dV}{dt}\right|_{h=5} = \left.\left[\frac{dV}{dh}\cdot\frac{dh}{dt}\right]\right|_{h=5}$$

$$= \left.\left[f(h)\cdot\frac{dh}{dt}\right]\right|_{h=5} \quad +1$$

$$= f(5)\cdot 0.26$$

$$= \left[\frac{50.3}{e^{(0.2\cdot 5)}+5}\right]\cdot 0.26 = 1.694419$$

The volume of water is changing (increasing) at a rate of $1.694 \text{ ft}^3/\text{min}$. +1 when height is 5 feet

2018 #4 No Calculator

Tables

a) $H'(6) \approx \frac{11-6}{2} = \frac{5}{2}$ +1

The height of the tree is increasing at a rate 2.5 m/yr at t = 6 years. +1

b) $\frac{H(5)-H(3)}{5-3} = \frac{6-2}{2} = 2$ +1

+1 Since $H(t)$ is differentiable on $[3,5]$, $H(t)$ is continuous on $[3,5]$. By the MVT, there must exist atleast one value c on $(3,5)$, such that $H'(c)=2$.

c) Avg. height on $[2,10]$ is $\frac{\int_2^{10} H(t)\,dt}{10-2}$.

$$\frac{1}{8}\int_2^{10} H(t)\,dt \approx \frac{1}{8}\cdot\left(\frac{1}{2}\left[(1)(3.5) + 2(8) + 2(17) + 3(26)\right]\right) \text{ +1}$$

$$= \frac{1}{16}(131.5) \text{ +1}$$

The average height of the tree is $\frac{131.5}{16} = \frac{263}{32}$ meters over $t=2$ to $t=10$ years

2018 #4 d

$\boxed{\text{No Calc}}$

Hard

$$G(x) = \frac{100x}{1+x}$$

$$\frac{d}{dt}\left(G(x)\right) = \frac{(1+x)\cdot 100 \cdot \frac{dx}{dt} - 100x\left(\frac{dx}{dt}\right)}{(1+x)^2}$$ +2

$$\frac{dx}{dt} = 0.03 \text{ m/yr}$$

$$50 = \frac{100x}{1+x}$$

$$x = 1$$

$$\left.\frac{d}{dt}\left(G(x)\right)\right|_{x=1} = \frac{(2)(100)\cdot(0.03) - 100(1)(0.03)}{2^2}$$

$$= 0.03\,\frac{[200-100]}{4}$$

$$= \frac{3}{100}\cdot\frac{100}{4}$$

$$= \frac{3}{4} \text{ m/yr}$$ +1

$G(x) = \text{meters}$

$G'(x) = \frac{\text{m height}}{\text{m diameter}}$

$G'(x)\cdot\frac{dx}{dt} = \text{m/yr}$