LIM	AP CALCULUS AB	
3	Topic: 4.7	Using L'Hospital's Rule for Determining Limits of Indeterminate Functions

Learning Objective LIM-4.A: Determine limits of functions that result in indeterminate forms.

We return to where our study of calculus all started – limits.
Recall that we analyzed problems like $\lim\limits_{x\to 2}\frac{x-2}{x^2-4}$ and $\lim\limits_{x\to\infty}\frac{1-4x-5x^2}{3x^2-x-4}$.
Do you recall how to solve these?

The notion of "plugging in" the targeted value, c, for x (also known as our "green light" method), poses some big problems in the two limits above and for many more you will see.

Limit Problem	Result of directly substituting in "c" for x
$\lim\limits_{x\to 2}\frac{x-2}{x^2-4}$	$\frac{2-2}{4-4} \to \frac{0}{0}$
$\lim\limits_{x\to\infty}\frac{1-4x-5x^2}{3x^2-x-4}$	$\frac{-\infty}{\infty}$

These are called indeterminate forms. There are a variety of indeterminate forms, but the most commonly seen ones are:

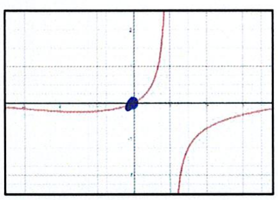
$$\frac{0}{0},\frac{\infty}{\infty},\frac{-\infty}{\infty},\frac{\infty}{-\infty},\frac{-\infty}{-\infty},$$

$$0\cdot(\infty),0\cdot(-\infty),\infty-\infty,1^{\infty},0^0,\text{ and }(\pm\infty)^0$$

The study of indeterminate forms and L'Hospital's Rule is new to the AP Calculus Exam for the year 2017.
In AP Calculus AB, we will focus primarily on forms involving $\frac{0}{0},\frac{\infty}{\infty},\frac{-\infty}{\infty},\frac{\infty}{-\infty},\frac{-\infty}{-\infty}$.

Notice how the two limits we were introduced to at the top of the page are rather easy to solve by factoring and cancelling.
Consider the following limit: $\lim\limits_{x\to 0}\frac{\cos x-1}{x^2-x}$
They cannot be solved using factoring (or any other algebraic technique for that matter.) Yet, the limit does exist as indicated by the graph.

$y=\frac{\cos x-1}{x^2-x}$



L'Hospital's Rule
If we have one of the two following cases: $\lim\limits_{x\to c}\frac{f(x)}{g(x)}$
where both $\lim\limits_{x\to c} f(x)=0$ and $\lim\limits_{x\to c} g(x)=0$ OR
where both $\lim\limits_{x\to c} f(x)\Rightarrow\pm\infty$ and $\lim\limits_{x\to c} g(x)\Rightarrow\pm\infty$
for any real number, c, or for c having the value of infinity or negative infinity,
then $\lim\limits_{x\to c}\frac{f(x)}{g(x)}=\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$.

It may be worth noting that the spelling of this mathematician's name varies according to which texts and manuscripts are read. Both L'Hospital and L'Hôpital are deemed correct.

MATHEMATICS & HISTORY

GUILLAUME L'HÔPITAL (1661-1704)
L'Hôpital's Rule is named after the French mathematician Guillaume François Antione de L'Hôpital.
L'Hopital is credited with writing the first text on differential calucus (in 1696) in which the rule publicly appeared. It was recently discovered that the rule and its proof were written in a letter from John Bernoulli to L'Hôpital. "...I acknowledge that I owe very much to the bright minds of the Bernoulli brothers...I have made free use of their discoveries...," said L'Hôpital.

It is **vitally important** that you **never** write $\lim\limits_{x\to c}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\lim\limits_{x\to c}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$. Neither $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are numeric values and thus cannot be equated to any other expression.

cos: -1, 1, 0 sin: 0, 0, -1

Example 1: **Find each of the following limits.**

a.) $\lim_{x\to0}\left(\frac{\cos x-1}{x^2-x}\right)$

$\lim_{x\to0}(\cos x-1)=0$; $\lim_{x\to0}(x^2-x)=0$

$$\lim_{x\to0}\frac{\cos x-1}{x^2-x}=\lim_{x\to0}\frac{-\sin x}{2x-1}=\frac{0}{-1}=\boxed{0}$$

b.) $\lim_{x\to-3}\left(\frac{x+3}{\sqrt{x^2-5}-2}\right)\to\frac{0}{0}$

$\lim_{x\to-3}(x+3)=0$; $\lim_{x\to-3}\left(\sqrt{x^2-5}-2\right)=0$

$$\lim_{x\to-3}\frac{x+3}{\sqrt{x^2-5}-2}=\lim_{x\to-3}\frac{1}{\frac{1\cdot2x}{2\sqrt{x^2-5}}}=\frac{1}{\frac{-6}{2\sqrt4}}=\frac{2\sqrt4}{-6}=\boxed{\frac{-2}{3}}$$

c.) $\lim_{x\to1}\left(\frac{\ln x-x+1}{e^x-ex}\right)$ $\to \ln1-1+1\to0$; $\to e^1-e\to0$

$\lim_{x\to1}(\ln x-x+1)=0$; $\lim_{x\to1}(e^x-ex)=0$

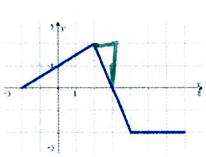
$$\lim_{x\to1}\frac{\ln x-x+1}{e^x-ex}=\lim_{x\to1}\frac{\frac1x-1}{e^x-e}\begin{matrix}\to0\\\to0\end{matrix}$$

$\lim_{x\to1}\left(\frac1x-1\right)=0$; $\lim_{x\to1}(e^x-e)=0$

$$=\lim_{x\to1}\frac{\frac{-1}{x^2}}{e^x}=\frac{-1}{e^1}=\boxed{\frac{-1}{e}}$$

Example 2: **Find each of the following limits.**

a.) The graph of $f(x)$ is shown below.



Find $\lim_{x\to3}\frac{f(x)}{x^2-9}$ $\to f(3)\to0$; $\to0$

$\lim_{x\to3}f(x)=0$; $\lim_{x\to3}(x^2-9)=0$

$$\lim_{x\to3}\frac{f(x)}{x^2-9}=\lim_{x\to3}\frac{f'(x)}{2x}=\frac{f'(3)}{6}=\frac{-2}{6}=\boxed{\frac{-1}{3}}$$

b.) The functions $f(x)$ and $g(x)$ are differentiable for all x. Use the following table to help find the limit.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	−2	0	3

Find $\lim_{x\to2}\frac{f(x)-4}{g(x)\cdot x^2}$ $\to0$; $\to0$

$\lim_{x\to2}(f(x)-4)=0$; $\lim_{x\to2}(g(x)\cdot x^2)=0$

$$\lim_{x\to2}\frac{f(x)-4}{g(x)\cdot x^2}=\lim_{x\to2}\frac{f'(x)}{g(x)\cdot2x+x^2\cdot g'(x)}$$

$$=\frac{f'(2)}{g(2)\cdot4+4\cdot g'(2)}=\frac{-2}{4\cdot3}=\boxed{\frac{-1}{6}}$$